

**A GENERALIZATION OF A CERTAIN SUBCLASS
OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS**

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ABSTRACT. The object of the present paper is to derive several interesting properties of the class $R(\alpha, \beta, n)$ consisting of analytic and univalent functions with negative coefficients. Coefficient inequalities, distortion theorems and closure theorems of functions in the class $R(\alpha, \beta, n)$ are determined. Also radii of starlikeness and convexity are determined. We also obtain several interesting results for the modified Hadamard product of functions belonging to the class $R(\alpha, \beta, n)$.

1. Introduction

Let S denote the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic and univalent in the unit disk $U = \{z : |z| < 1\}$. For a function $f(z)$ in the class S , we define

$$(1.2) \quad D^0 f(z) = f(z),$$

$$(1.3) \quad D^1 f(z) = Df(z) = zf'(z),$$

and

$$(1.4) \quad D^{n+1} f(z) = D(D^n f(z)) \quad (n \in \mathbb{N} = \{1, 2, 3, \dots\}).$$

The differential operator D^n was introduced by Salagean [4]. With the help of the differential operator D^n , we say that a function $f(z)$ belonging to the class S is in the class $S_n(\beta)$ if and only if

$$(1.5) \quad \operatorname{Re} \left\{ \frac{D^{n+1} f(z)}{D^n f(z)} \right\} > \beta \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$$

for some $\beta(0 \leq \beta < 1)$ and for all $z \in U$. The class $S_n(\beta)$ was introduced by Salagean [4].

We note that, by specializing n , we obtain the following subclasses of the class S :

- (1) $S_0(\beta) = S^*(\beta)$, where the class $S^*(\beta)$ is called as the class of starlike functions of order β ,

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(2) $S_1(\beta) = C(\beta)$, where the class $C(\beta)$ is called as the class of convex functions of order β .

Let T be the subclass of the class S consisting of functions of the form

$$(1.6) \quad f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0).$$

Also, we define the class $T(n, \beta)$ by

$$(1.7) \quad T(n, \beta) = S_n(\beta) \cap T.$$

The class $T(n, \beta)$ was studied by Hur and Oh [3].

Further, let $R(\alpha, \beta, n)$ denote the subclass of the class T consisting of functions $f(z)$ which satisfy

$$(1.8) \quad \operatorname{Re} \left\{ (1 - \alpha)(D^n f(z))' + \alpha(D^{n+1} f(z))' \right\} > \beta$$

for some $\alpha(\alpha \geq 0)$, $\beta(0 \leq \beta < 1)$, $n \in \mathbb{N}_0$, and for all $z \in U$.

We note that, by specializing the parameters α, β and n , we obtain the following subclasses studied by various authors:

(3) $R(\alpha, \beta, 0) = R(\alpha, \beta)$ (Altintas [2]),

(4) $R(0, \beta, 0) = T^{**}(\beta)$ (Sarangi and Uralegaddi [5] and Al-Amiri [1]).

Furthermore, by specializing the parameters α, β and n , we obtain the following new subclasses:

(5) $R(\alpha, \beta, 1) = K(\alpha, \beta)$, where the class $K(\alpha, \beta)$ represents the class of functions $f(z) \in T$ satisfying the condition

$$(1.9) \quad \operatorname{Re} \left\{ (1 - \alpha)(zf'(z))' + \alpha(z(zf'(z))')' \right\} > \beta,$$

where $\alpha \geq 0, 0 \leq \beta < 1$ and $z \in U$.

(6) $R(0, \beta, 1) = K(\beta)$, where the class $K(\beta)$ represents the class of functions $f(z) \in T$ satisfying the condition

$$(1.10) \quad \operatorname{Re} \left\{ (zf'(z))' \right\} > \beta \quad (0 \leq \beta < 1, z \in U).$$

(7) The class $R(0, \beta, n)$ represents the class of functions $f(z) \in T$ satisfying the condition

$$(1.11) \quad \operatorname{Re} \left\{ (D^n f(z))' \right\} > \beta \quad (0 \leq \beta < 1, n \in \mathbb{N}_0, z \in U).$$

(8) The class $R(1, \beta, n)$ represents the class of functions $f(z) \in T$ satisfying the condition

$$(1.12) \quad \operatorname{Re} \left\{ (D^{n+1} f(z))' \right\} > \beta \quad (0 \leq \beta < 1, n \in \mathbb{N}_0, z \in U).$$

For the class $R(\alpha, \beta, n)$, we give several interesting properties including coefficient inequalities, distortion theorems and closure theorems. Also radii of starlikeness and convexity are determined. We also obtain several interesting results for the modified Hadamard product of functions belonging to the class $R(\alpha, \beta, n)$.

2. Basic properties of the class $R(\alpha, \beta, n)$

We begin by recalling the following result:

Lemma 1 [3]. A function $f(z)$ defined by (1.6) is in the class $T(n, \beta)$ if and only if

$$(2.1) \quad \sum_{k=2}^{\infty} k^n(k - \beta)a_k \leq 1 - \beta.$$

The result is sharp.

Theorem 1. Let the function $f(z)$ be defined by (1.6). Then $f(z) \in R(\alpha, \beta, n)$ if and only if

$$(2.2) \quad \sum_{k=2}^{\infty} k^{n+1}(1 + (k - 1)\alpha)a_k \leq 1 - \beta.$$

The result is sharp.

Proof. Assume that the inequality (2.2) holds true. Then we find that

$$(2.3) \quad \begin{aligned} & |(1 - \alpha)(D^n f(z))' + \alpha(D^{n+1} f(z))' - 1| \\ &= \left| - \sum_{k=2}^{\infty} k^{n+1}(1 + (k - 1)\alpha)a_k z^{k-1} \right| \\ &\leq \sum_{k=2}^{\infty} k^{n+1}(1 + (k - 1)\alpha)a_k \\ &\leq 1 - \beta. \end{aligned}$$

This implies that the values of the function

$$(2.4) \quad \Phi(z) = (1 - \alpha)(D^n f(z))' + \alpha(D^{n+1} f(z))' \quad (z \in U)$$

lie in the circle centered at 1 whose radius is $1 - \beta$, that is, that $f(z) \in R(\alpha, \beta, n)$.

Conversely, suppose that the function $f(z)$ defined by (1.6) is in the class $R(\alpha, \beta, n)$. Then, we note that

$$(2.5) \quad \begin{aligned} \operatorname{Re} \Phi(z) &= \operatorname{Re} \{ (1 - \alpha)(D^n f(z))' + \alpha(D^{n+1} f(z))' \} \\ &= \operatorname{Re} \left\{ 1 - \sum_{k=2}^{\infty} k^{n+1}(1 + (k - 1)\alpha)a_k z^{k-1} \right\} \\ &> \beta \quad (|z| < 1). \end{aligned}$$

Choosing values of z on the real axis such that the function $\Phi(z)$ is real and let $z \rightarrow 1 - 0$ through real values, we obtain

$$(2.6) \quad 1 - \sum_{k=2}^{\infty} k^{n+1}(1 + (k - 1)\alpha)a_k \geq \beta.$$

This implies that the inequality (2.2) holds true.

Finally the function $f_k(z)$ given by

$$(2.7) \quad f_k(z) = z - \frac{1 - \beta}{k^{n+1}(1 + (k - 1)\alpha)} z^k \quad (k \geq 2)$$

is an extremal function for the theorem. \square

Corollary 1. *Let the function $f(z)$ defined by (1.6) be in the class $R(\alpha, \beta, n)$. Then we have*

$$(2.8) \quad a_k \leq \frac{1 - \beta}{k^{n+1}(1 + (k-1)\alpha)} \quad (k \geq 2).$$

The equality in (2.8) is attained for the function $f_k(z)$ defined by (2.7).

We now prove three interesting properties of the class $R(\alpha, \beta, n)$, contained in Theorems 2, 3 and 4 below.

Theorem 2. *Let $0 \leq \beta < 1$ and $0 \leq \alpha \leq 1 - \beta$. Then we have*

$$(2.9) \quad T(n+1, 0) = R(1, 0, n) \quad (n \in \mathbb{N}_0),$$

and

$$(2.10) \quad T(n+1, \beta) \not\subseteq R(\alpha, \beta, n) \quad (n \in \mathbb{N}_0),$$

except for $(\alpha, \beta) = (1, 0)$.

Proof. It follows from the hypothesis $0 \leq \beta \leq 1 - \alpha$ that

$$(2.11) \quad (k - \beta) - (1 + (k-1)\alpha) = (k-1)(1-\alpha) - \beta \geq (k-2)\beta \geq 0 \quad (k \geq 2).$$

Thus, by means of Lemma 1, we have

$$(2.12) \quad \sum_{k=2}^{\infty} k^{n+1}(1 + (k-1)\alpha)a_k \leq \sum_{k=2}^{\infty} k^{n+1}(k - \beta)a_k \leq 1 - \beta,$$

for functions $f(z) \in T(n+1, \beta)$, which (in view of Theorem 1) proves that $T(n+1, \beta) \subset R(\alpha, \beta, n)$. Since $f_k(z) \in \{R(\alpha, \beta, n) - T(n+1, \beta)\}$ for the function $f_k(z), k \geq 3$, defined by (2.7), we have the assertion of the theorem. \square

Theorem 3. *Let $0 \leq \alpha_1 < \alpha_2$ and $0 \leq \beta < 1$. Then we have*

$$(2.13) \quad R(\alpha_2, \beta, n) \not\subseteq R(\alpha_1, \beta, n) \quad (n \in \mathbb{N}_0).$$

Proof. It follows from Theorem 1 that

$$(2.14) \quad \sum_{k=2}^{\infty} k^{n+1}(1 + (k-1)\alpha_1)a_k \leq \sum_{k=2}^{\infty} k^{n+1}(1 + (k-1)\alpha_2)a_k \leq 1 - \beta$$

for functions $f(z) \in R(\alpha_2, \beta, n)$, and that $f_{k,1}(z) \in \{R(\alpha_1, \beta, n) - R(\alpha_2, \beta, n)\}$ for the function

$$(2.15) \quad f_{k,1}(z) = z - \frac{1 - \beta}{k^{n+1}(1 + (k-1)\alpha_1)} z^k \quad (k \geq 2). \quad \square$$

Theorem 4.

$$(2.16) \quad R(\alpha, \beta, n + 1) \subsetneq R(\alpha, \beta, n).$$

The proof follows from Theorem 1 and from that $f_k(z) \in \{R(\alpha, \beta, n) - R(\alpha, \beta, n + 1)\}$ for the function $f_k(z)$ defined by (2.7).

3. Distortion theorems

Theorem 5. *Let the function $f(z)$ defined by (1.6) be in the class $R(\alpha, \beta, n)$. Then we have for $|z| < 1$,*

$$(3.1) \quad |z| - \frac{1 - \beta}{2^{n+1}(1 + \alpha)}|z|^2 \leq |f(z)| \leq |z| + \frac{1 - \beta}{2^{n+1}(1 + \alpha)}|z|^2.$$

Furthermore, we have for $\alpha \geq 1$

$$(3.2) \quad 1 - \frac{1 - \beta}{2^n(1 + \alpha)}|z| \leq |f'(z)| \leq 1 + \frac{1 - \beta}{2^n(1 + \alpha)}|z|.$$

The bounds in (3.1) and (3.2) are attained for the function

$$(3.3) \quad f_2(z) = z - \frac{1 - \beta}{2^{n+1}(1 + \alpha)}z^2 \quad (z = \pm|z|).$$

Proof. From Theorem 1, we have

$$(3.4) \quad 2^{n+1}(1 + \alpha) \sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty} k^{n+1}(1 + (k - 1)\alpha)a_k \leq 1 - \beta.$$

This implies that

$$(3.5) \quad \sum_{k=2}^{\infty} a_k \leq \frac{1 - \beta}{2^{n+1}(1 + \alpha)}.$$

Thus we have

$$(3.6) \quad |f(z)| \leq |z| + \sum_{k=2}^{\infty} a_k |z|^k \leq |z| + |z|^2 \sum_{k=2}^{\infty} a_k \leq |z| + \frac{1 - \beta}{2^{n+1}(1 + \alpha)}|z|^2$$

and

$$(3.7) \quad |f(z)| \geq |z| - \sum_{k=2}^{\infty} a_k |z|^k \geq |z| - |z|^2 \sum_{k=2}^{\infty} a_k \geq |z| - \frac{1 - \beta}{2^{n+1}(1 + \alpha)}|z|^2.$$

Hence the inequalities (3.1) follows. Further we have

$$(3.8) \quad |f'(z)| \leq 1 + \sum_{k=2}^{\infty} k a_k |z|^{k-1} \leq 1 + |z| \sum_{k=2}^{\infty} k a_k$$

and

$$(3.9) \quad |f'(z)| \geq 1 - \sum_{k=2}^{\infty} k a_k |z|^{k-1} \geq 1 - |z| \sum_{k=2}^{\infty} k a_k.$$

From Theorem 1 and inequality (3.5), it follows that

$$(3.10) \quad \sum_{k=2}^{\infty} k a_k - \frac{\alpha-1}{\alpha} \sum_{k=2}^{\infty} a_k \leq \frac{1}{\alpha} \sum_{k=2}^{\infty} (1+(k-1)\alpha) \left(\frac{k}{2}\right)^{n+1} a_k \leq \frac{1-\beta}{\alpha 2^{n+1}}$$

and

$$(3.11) \quad \sum_{k=2}^{\infty} k a_k \leq \frac{1-\beta}{2^{n+1}\alpha} + \frac{\alpha-1}{\alpha} \sum_{k=2}^{\infty} a_k \leq \frac{1-\beta}{2^n(1+\alpha)}.$$

Hence inequality (3.2) follows also. \square

Corollary 2. *Let the function $f(z)$ defined by (1.6) be in the class $R(\alpha, \beta, n)$. Then the unit disk U is mapped by the function $f(z)$ onto a domain that contains the disk $|w| < 1 - \frac{1-\beta}{2^{n+1}(1+\alpha)}$. The result is sharp with extremal function $f_2(z)$ defined by (3.3).*

Theorem 6. *Let the function $f(z)$ defined by (1.6) be in the class $R(\alpha, \beta, n)$. Then we have for $|z| < 1$,*

$$(3.12) \quad |z| - \frac{1-\beta}{2^{n+1-j}(1+\alpha)} |z|^2 \leq |D^j f(z)| \leq |z| + \frac{1-\beta}{2^{n+1-j}(1+\alpha)} |z|^2,$$

where $0 \leq j \leq n$. The bounds are attained for the function $f_2(z)$ defined by (3.3).

Proof. Note that $f(z) \in R(\alpha, \beta, n)$ if and only if $D^j f(z) \in R(\alpha, \beta, n-j)$, and that

$$(3.13) \quad D^j f(z) = z - \sum_{k=2}^{\infty} k^j a_k z^k.$$

Using Theorem 1, we know that

$$(3.14) \quad 2^{n+1-j}(1+\alpha) \sum_{k=2}^{\infty} k^j a_k \leq \sum_{k=2}^{\infty} k^{n+1}(1+(k-1)\alpha) a_k \leq 1-\beta,$$

that is,

$$(3.15) \quad \sum_{k=2}^{\infty} k^j a_k \leq \frac{1-\beta}{2^{n+1-j}(1+\alpha)}.$$

It follows from (3.13) and (3.15) that

$$(3.16) \quad |D^j f(z)| \geq |z| - |z|^2 \sum_{k=2}^{\infty} k^j a_k \geq |z| - \frac{1-\beta}{2^{n+1-j}(1+\alpha)} |z|^2$$

and

$$(3.17) \quad |D^j f(z)| \leq |z| + |z|^2 \sum_{k=2}^{\infty} k^j a_k \leq |z| + \frac{1-\beta}{2^{n+1-j}(1+\alpha)} |z|^2.$$

Finally, we note that the equalities in (3.12) are attained for the function $f_2(z)$ defined by (3.3) at $z = \pm|z|$. This completes the proof of Theorem 6. \square

Remarks.

- (1) Taking $j = 0$ in Theorem 6, we can easily obtain (3.1).
- (2) Taking $j = 1$ in Theorem 6, we can easily obtain (3.2).

4. Convex linear combinations

In this section, we shall prove that the class $R(\alpha, \beta, n)$ is closed under convex linear combinations.

Theorem 7. *The class $R(\alpha, \beta, n)$ is a convex set.*

Proof. Let the functions

$$(4.1) \quad g_\nu(z) = z - \sum_{k=2}^{\infty} a_{\nu,k} z^k \quad (a_{\nu,k} \geq 0, \nu = 1, 2)$$

be in the class $R(\alpha, \beta, n)$. It is sufficient to show that the function $g(z)$ defined by

$$(4.2) \quad g(z) = \lambda g_1(z) + (1 - \lambda)g_2(z) \quad (0 \leq \lambda \leq 1)$$

is in the class $R(\alpha, \beta, n)$. Since

$$(4.3) \quad g(z) = z - \sum_{k=2}^{\infty} \{\lambda a_{1,k} + (1 - \lambda)a_{2,k}\} z^k,$$

with the aid of Theorem 1, we have

$$(4.4) \quad \begin{aligned} & \sum_{k=2}^{\infty} k^{n+1}(1 + (k - 1)\alpha) \{\lambda a_{1,k} + (1 - \lambda)a_{2,k}\} \\ & \leq \lambda(1 - \beta) + (1 - \lambda)(1 - \beta) = 1 - \beta, \end{aligned}$$

which implies that $g(z) \in R(\alpha, \beta, n)$. Hence the class $R(\alpha, \beta, n)$ is convex. \square

Theorem 8. *Let $f_1(z) = z$ and let $f_k(z)$ ($k \geq 2$) be defined by (2.7). Then $f(z)$ is in the class $R(\alpha, \beta, n)$ if and only if it can be expressed in the form*

$$(4.5) \quad f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z),$$

where $\lambda_k \geq 0$ ($k \in \mathbb{N}$) and $\sum_{k=1}^{\infty} \lambda_k = 1$.

Proof. Assume that

$$(4.6) \quad f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) = z - \sum_{k=2}^{\infty} \frac{1 - \beta}{k^{n+1}(1 + (k - 1)\alpha)} \lambda_k z^k$$

for $\lambda_k \geq 0$ ($k \in \mathbb{N}$) and $\sum_{k=1}^{\infty} \lambda_k = 1$. Then it follows that

$$(4.7) \quad \begin{aligned} & \sum_{k=2}^{\infty} \frac{k^{n+1}(1 + (k - 1)\alpha)}{1 - \beta} \cdot \frac{1 - \beta}{k^{n+1}(1 + (k - 1)\alpha)} \lambda_k \\ & = \sum_{k=2}^{\infty} \lambda_k = 1 - \lambda_1 \leq 1. \end{aligned}$$

So by Theorem 1, $f(z) \in R(\alpha, \beta, n)$.

Conversely, assume that the function $f(z)$ defined by (1.6) belongs to the class $R(\alpha, \beta, n)$. Then, by Corollary 1, we have inequality (2.8). Setting

$$(4.8) \quad \lambda_k = \frac{k^{n+1}(1+(k-1)\alpha)}{1-\beta} a_k \geq 0 \quad (k \geq 2, n \in \mathbb{N}_0)$$

and

$$(4.9) \quad \lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k,$$

we have $\lambda_1 \geq 0$ by Theorem 1, and (4.5). And, we can see that the function $f(z)$ can be expressed in the form (4.5). This completes the proof of Theorem 8. \square

5. Modified Hadamard Products

Let the functions $g_\nu(z)$ ($\nu = 1, 2$) be defined by (4.1). The modified Hadamard product of $g_1(z)$ and $g_2(z)$ is defined by

$$(5.1) \quad g_1 * g_2(z) = z - \sum_{k=2}^{\infty} a_{1,k} a_{2,k} z^k.$$

Theorem 9. *Let the functions $g_\nu(z)$ ($\nu = 1, 2$) defined by (4.1) be in the class $R(\alpha, \beta, n)$. Then $g_1 * g_2(z)$ belongs to the class $R(\alpha, \gamma(\alpha, \beta, n), n)$, where*

$$(5.2) \quad \gamma(\alpha, \beta, n) = 1 - \frac{(1-\beta)^2}{2^{n+1}(1+\alpha)}.$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [6], we need to find the largest $\gamma = \gamma(\alpha, \beta, n)$ such that

$$(5.3) \quad \sum_{k=2}^{\infty} \frac{k^{n+1}(1+(k-1)\alpha)}{1-\gamma} a_{1,k} a_{2,k} \leq 1.$$

Since

$$(5.4) \quad \sum_{k=2}^{\infty} \frac{k^{n+1}(1+(k-1)\alpha)}{1-\beta} a_{\nu,k} \leq 1 \quad (\nu = 1, 2),$$

by the Cauchy-Schwarz inequality, we have

$$(5.5) \quad \sum_{k=2}^{\infty} \frac{k^{n+1}(1+(k-1)\alpha)}{1-\beta} \sqrt{a_{1,k} a_{2,k}} \leq 1.$$

Thus it is sufficient to show that

$$(5.6) \quad \begin{aligned} & \frac{k^{n+1}(1+(k-1)\alpha)}{1-\gamma} a_{1,k} a_{2,k} \\ & \leq \frac{k^{n+1}(1+(k-1)\alpha)}{1-\beta} \sqrt{a_{1,k} a_{2,k}} \quad (k \geq 2), \end{aligned}$$

that is, that

$$(5.7) \quad \sqrt{a_{1,k}a_{2,k}} \leq \frac{1-\gamma}{1-\beta}.$$

Note that

$$(5.8) \quad \sqrt{a_{1,k}a_{2,k}} \leq \frac{1-\beta}{k^{n+1}(1+(k-1)\alpha)} \quad (k \geq 2).$$

Consequently, we need only to prove that

$$(5.9) \quad \frac{1-\beta}{k^{n+1}(1+(k-1)\alpha)} \leq \frac{1-\gamma}{1-\beta} \quad (k \geq 2),$$

or equivalently, that

$$(5.10) \quad \gamma \leq 1 - \frac{(1-\beta)^2}{k^{n+1}(1+(k-1)\alpha)} \quad (k \geq 2).$$

Since

$$(5.11) \quad \gamma(k) = 1 - \frac{(1-\beta)^2}{k^{n+1}(1+(k-1)\alpha)}$$

is an increasing function of k ($k \geq 2$), letting $k = 2$ in (5.11), we obtain

$$(5.12) \quad \gamma \leq \gamma(2) = 1 - \frac{(1-\beta)^2}{2^{n+1}(1+\alpha)},$$

which completes the proof of Theorem 9.

Finally, by taking the functions $g_\nu(z)$ ($\nu = 1, 2$) defined by

$$(5.13) \quad g_\nu(z) = z - \frac{1-\beta}{2^{n+1}(1+\alpha)}z^2,$$

we can see that the result is sharp. \square

Theorem 10. *Let the function $g_1(z)$ defined by (4.1) be in the class $R(\alpha, \beta, n)$ and the function $g_2(z)$ defined by (4.1) be in the class $R(\alpha, \tau, n)$. Then $g_1 * g_2(z) \in R(\alpha, \zeta(\alpha, \beta, \tau, n), n)$, where*

$$(5.14) \quad \zeta(\alpha, \beta, \tau, n) = 1 - \frac{(1-\beta)(1-\tau)}{2^{n+1}(1+\alpha)}.$$

The result is sharp.

Proof. Proceeding as in the proof of Theorem 9, we obtain

$$(5.15) \quad \zeta \leq \zeta(k) = 1 - \frac{(1-\beta)(1-\tau)}{k^{n+1}(1+(k-1)\alpha)} \quad (k \geq 2).$$

Since the function $\zeta(k)$ is an increasing function of k ($k \geq 2$), letting $k = 2$ in (5.15) we obtain

$$(5.16) \quad \zeta \leq \zeta(2) = 1 - \frac{(1-\beta)(1-\tau)}{2^{n+1}(1+\alpha)},$$

which evidently proves Theorem 10.

Finally the result is best possible for the functions

$$(5.17) \quad g_1(z) = z - \frac{1-\beta}{2^{n+1}(1+\alpha)}z^2 \quad \text{and} \quad g_2(z) = z - \frac{1-\tau}{2^{n+1}(1+\alpha)}z^2. \quad \square$$

Corollary 3. Let the functions $g_\nu(z)$ ($\nu = 1, 2, 3$) defined by (4.1) be in the class $R(\alpha, \beta, n)$. Then $g_1 * g_2 * g_3(z) \in R(\alpha, \eta(\alpha, \beta, n), n)$, where

$$(5.18) \quad \eta(\alpha, \beta, n) = 1 - \frac{(1 - \beta)^3}{2^{2(n+1)}(1 + \alpha)^2}.$$

The result is best possible for the functions

$$(5.19) \quad g_\nu(z) = z - \frac{1 - \beta}{2^{n+1}(1 + \alpha)} z^2 \quad (\nu = 1, 2, 3).$$

Proof. From Theorem 9, we have $g_1 * g_2(z) \in R(\alpha, \gamma(\alpha, \beta, n), n)$, where γ is defined by (5.2). We now use Theorem 10 and obtain $g_1 * g_2 * g_3(z) \in R(\alpha, \eta(\alpha, \beta, n), n)$, where

$$(5.20) \quad \eta(\alpha, \beta, n) = 1 - \frac{(1 - \beta)(1 - \gamma)}{2^{n+1}(1 + \alpha)} = 1 - \frac{(1 - \beta)^3}{2^{2(n+1)}(1 + \alpha)^2}.$$

This completes the proof of Corollary 3. \square

Theorem 11. Let the functions $g_\nu(z)$ ($\nu = 1, 2$) defined by (4.1) be in the class $R(\alpha, \beta, n)$. Then the function

$$(5.21) \quad h(z) = z - \sum_{k=2}^{\infty} \{(a_{1,k})^2 + (a_{2,k})^2\} z^k$$

belongs to the class $R(\alpha, \delta(\alpha, \beta, n), n)$, where

$$(5.22) \quad \delta(\alpha, \beta, n) = 1 - \frac{(1 - \beta)^2}{2^n(1 + \alpha)}.$$

The result is sharp for the functions $g_\nu(z)$ ($\nu = 1, 2$) defined by (5.13).

Proof. By virtue of Theorem 1, we obtain

$$(5.23) \quad \begin{aligned} & \sum_{k=2}^{\infty} \left\{ \frac{k^{n+1}(1 + (k-1)\alpha)}{1 - \beta} \right\}^2 (a_{\nu,k})^2 \\ & \leq \left\{ \sum_{k=2}^{\infty} \frac{k^{n+1}(1 + (k-1)\alpha)}{1 - \beta} a_{\nu,k} \right\}^2 \leq 1 \quad (\nu = 1, 2). \end{aligned}$$

It follows from (5.23) that

$$(5.24) \quad \sum_{k=2}^{\infty} \frac{1}{2} \left\{ \frac{k^{n+1}(1 + (k-1)\alpha)}{1 - \beta} \right\}^2 \{(a_{1,k})^2 + (a_{2,k})^2\} \leq 1.$$

Therefore, we need to find the largest $\delta = \delta(\alpha, \beta, n)$ such that

$$(5.25) \quad \frac{k^{n+1}(1 + (k-1)\alpha)}{1 - \delta} \leq \frac{1}{2} \left\{ \frac{k^{n+1}(1 + (k-1)\alpha)}{1 - \beta} \right\}^2 \quad (k \geq 2),$$

that is,

$$(5.26) \quad \delta \leq 1 - \frac{2(1 - \beta)^2}{k^{n+1}(1 + (k - 1)\alpha)} \quad (k \geq 2).$$

Since

$$(5.27) \quad \delta(k) = 1 - \frac{2(1 - \beta)^2}{k^{n+1}(1 + (k - 1)\alpha)}$$

is an increasing function of $k(k \geq 2)$, we readily have

$$(5.28) \quad \delta \leq \delta(2) = 1 - \frac{(1 - \beta)^2}{2^n(1 + \alpha)}$$

and Theorem 11 follows at once. \square

6. Radii of starlikeness and convexity

Theorem 12. *Let the function $f(z)$ defined by (1.6) be in the class $R(\alpha, \beta, n)$. Then the function $f(z)$ is starlike of order β . If $\beta < \rho < 1$, then the function $f(z)$ is starlike of order ρ in $|z| < r_1(\alpha, \beta, \rho, n)$, where*

$$(6.1) \quad r_1(\alpha, \beta, \rho, n) = \inf \left\{ \left[\frac{(1 - \rho)k^{n+1}(1 + (k - 1)\alpha)}{(1 - \beta)(k - \rho)} \right]^{\frac{1}{k-1}} : k \geq 2, a_k > 0 \right\} (\leq 1).$$

The result is trivial for the function $f_1(z) = z$. The result for the radius is sharp with the extremal function $f_k(z)$ defined by (2.7).

Proof. It is sufficient to show that

$$(6.2) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho \quad (|z| < r_1(\alpha, \beta, \rho, n)).$$

We have

$$(6.3) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{k=2}^{\infty} (k - 1)a_k |z|^{k-1}}{1 - \sum_{k=2}^{\infty} a_k |z|^{k-1}}.$$

Thus $\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho$ if

$$(6.4) \quad \sum_{k=2}^{\infty} \frac{k - \rho}{1 - \rho} a_k |z|^{k-1} \leq 1.$$

Hence, by using (2.2), (6.4) holds true if

$$(6.5) \quad \frac{k - \rho}{1 - \rho} |z|^{k-1} \leq \frac{k^{n+1}(1 + (k - 1)\alpha)}{1 - \beta} \quad (k \geq 2),$$

or if

$$(6.6) \quad |z| \leq \left\{ \frac{(1 - \rho)k^{n+1}(1 + (k - 1)\alpha)}{(1 - \beta)(k - \rho)} \right\}^{\frac{1}{k-1}} \quad (k \geq 2).$$

Since $r_1(\alpha, \beta, \beta, n) = 1$, $f(z) \in S^*(\beta)$. The last part of the theorem follows easily from (6.6). \square

Theorem 13. Let the function $f(z)$ defined by (1.6) be in the class $R(\alpha, \beta, n)$. If $n \in \mathbb{N}$, then the function $f(z)$ is convex of order β and the function $f(z)$ is convex of order ρ ($\beta \leq \rho < 1$) in $|z| < r_2(\alpha, \beta, \rho, n)$, where

$$(6.7) \quad r_2(\alpha, \beta, \rho, n) = \inf \left\{ \left[\frac{(1-\rho)k^n(1+(k-1)\alpha)}{(1-\beta)(k-\rho)} \right]^{\frac{1}{k-1}} : k \geq 2, a_k > 0 \right\} (\leq 1).$$

If $n = 0$ and $0 < \rho \leq 1 - \frac{1-\beta}{\alpha}$ (< 1), then the function $f(z)$ is convex of order ρ . If $n = 0$ and $0 < 1 - \frac{1-\beta}{\alpha} < \rho < 1$, then the function $f(z)$ is convex of order ρ in $|z| < r_2(\alpha, \beta, \rho, 0)$. The results are trivial for the function $f_1(z) = z$. The results for the radii are sharp with extremal function $f(z)$ defined by (2.7).

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