

## ON EXTENDED BANACH-MAZUR GAMES ON BOOLEAN ALGEBRAS

MAKOTO TAKAHASHI

Received October 24, 1997

ABSTRACT. We extend the Banach-Mazur game on Boolean algebras so that at each stage player I can play simultaneously many elements. We introduce two games  $SG(B)$  and  $DG(B)$ . We show that  $SG(B)$  is determined for all Boolean algebras and  $DG(B)$  is determined for several Boolean algebras for which ordinary Banach-Mazur game is undetermined.

**1. Introduction.** Transfinite games on Boolean algebras have been an object of research in set theory. In [6] Jech investigated the Banach-Mazur game  $\mathcal{G}(B)$  (see §3) of length  $\omega$  played on a Boolean algebra  $B$  between two players I and II where I moves first. He proved that I does not have a winning strategy in  $\mathcal{G}(B)$  if and only if  $B$  is  $(\omega, \infty)$ -distributive. He also showed that  $\mathcal{G}$  is not determined; i.e., there exists a Boolean algebra  $B$  such that in  $\mathcal{G}(B)$  both players do not have a winning strategy.

Foreman[2] and Levinski[12] extended  $\mathcal{G}$  to the games of length greater than  $\omega$  those are the games of variable length between  $\omega$  and  $\lambda$  (say of potential length  $\lambda$ ). Both games differ in the winning conditions. The length of games depend on the moves the two players make. Levinski's game of potential length  $\omega_1$  on  $B$ ,  $G(B, \omega_1)$ , is easier for I than  $\mathcal{G}(B)$ ; i.e., if I has a winning strategy in  $\mathcal{G}(B)$ , then I has a winning strategy in  $G(B, \omega_1)$ . II has no advantage in  $G(B, \omega_1)$ . Actually, by using the observation in [7] about Foreman's Theorem [2], it is easy to see that II has a winning strategy in  $G(B, \omega_1)$  if and only if he has one in  $\mathcal{G}(B)$ . We say that a game is *BM-like* if it satisfies the same properties for I and II as above. We do not know whether there exists in ZFC a Boolean algebra  $B$  for which  $G(B, \omega_1)$  is not determined. Under  $V = L$ , there exists an  $\omega_2$ -Suslin tree  $T$  which is not essentially  $\sigma$ -closed (see Example 4 below). Then  $G(B(T), \omega_1)$  is not determined where  $B(T)$  is a completion of  $T$ . We will show that  $\neg CH +$  nondeterminacy of  $G(*, \omega_1)$  is consistent to ZFC. Nondeterminacy of  $G(*, \omega_1)$  follows from PFA also.

In this paper, we will introduce two BM-like games  $SG(B)$  and  $DG(B)$ .  $\mathcal{G}$  is extended to  $G(*, \omega_1)$  by changing the length of the game. We extend  $\mathcal{G}$  to  $SG(B)$  and  $DG(B)$  so that at each stage I can play simultaneously many elements. A play of  $SG(B)$  proceeds most advantageous to I in BM-like games. So it is easy to see by the definition that  $SG(B)$  is determined for all Boolean algebras. In  $DG(B)$ , I's play is restricted in comparison with his play in  $SG(B)$ , but  $DG(B)$  is also easier for I than  $\mathcal{G}(B)$ . We will show that I has a winning strategy for several Boolean algebras for which  $\mathcal{G}$  is undetermined. However, it remains open whether  $DG(B)$  is determined for all Boolean algebras.

**2. Preliminaries.** The reader is assumed to be familiar with the basic facts about Boolean algebras found in [10]. For basic facts about set theory, we refer to [5, 11]. We use the letters  $\kappa, \lambda, \mu$  for cardinals; the letters  $\alpha, \beta, \gamma, \delta, \eta, \nu$  for ordinals; the letters  $A, B$  for infinite Boolean algebras. For a function  $f$ , we denote the domain of  $f$  by  $\text{Dom}(f)$  and the image

of  $f$  by  $\text{Im}(f)$ . We denote the finite Boolean operations by  $\wedge_B, \vee_B, \sim_B$ , the infinitary joins and meets by  $\bigvee_B$  and  $\bigwedge_B$ , the least element by  $0_B$  and the greatest element by  $1_B$ .  $\leq_B$  is the canonical ordering of  $B$ . We shall omit subscripts if there is no confusion. Let  $B^+$  denote  $\{b \in B \mid b > 0\}$ . For each  $b \in B^+$ ,  $B|b$  is the algebra restricted to  $b$ . The cardinality of a set  $S$  is denoted by  $|S|$ . A separative partially ordered set  $(P, \leq)$  determines uniquely (up to isomorphism) a complete Boolean algebra  $B(P)$  such that  $(P, \leq)$  is isomorphic to a dense subset of  $B(P)$ . It is called a completion of  $P$ . In particular, every Boolean algebra  $B$  has the unique (up to isomorphism) completion. usually construct a partially ordered set, and the Boolean algebra is the completion of it. A complete Boolean algebra  $B$  is  $(\kappa, \lambda)$ -distributive iff it satisfies the following law:

$$\bigwedge_{\alpha < \kappa} \bigvee_{\beta < \lambda} a_{\alpha, \beta} = \bigvee_{f: \kappa \rightarrow \lambda} \bigwedge_{\alpha < \kappa} a_{\alpha, f(\alpha)}$$

A partially ordered set is  $\kappa$ -Baire iff its completion is  $(\kappa, \infty)$ -distributive. We say  $\sigma$ -Baire instead of  $\omega$ -Baire. A partially ordered set  $P$  is said to be  $\sigma$ -closed if every descending  $\omega$ -sequence of  $P$  has a lower bound in  $P$ . Two elements  $a, b$  of  $B^+$  are disjoint if  $a \wedge b = 0$ . A partition of  $B$  is a maximal pairwise disjoint family of  $B$ . The set of all partitions of  $B$  is denoted by  $\text{Part}(B)$ . A complete Boolean algebra  $B$  satisfies the condition  $\text{P}(\kappa, \lambda, \mu)$  if for each  $a \in B^+$  and any partitions  $P_\alpha \in \text{Part}(B|a)$  ( $\alpha < \kappa$ ) with  $|P_\alpha| \leq \lambda$ , there exists a function  $f \in \prod_{\alpha < \kappa} P_\alpha$  such that  $\bigwedge_{\alpha \in X} f(\alpha) > 0$  for every  $X \subset \kappa$  with  $|X| < \mu$ . Thus,  $\text{P}(\kappa, \kappa, \omega)$  is the  $\kappa$ -representability;  $\text{P}(\kappa, \lambda, \kappa^+)$  is the  $(\kappa, \lambda)$ -distributivity.

A tree is a partially ordered set  $(T, >_T)$ , such that for each  $t \in T$ , the set  $\{s \in T \mid s <_T t\}$  of predecessors of  $t$  is well-ordered by  $<_T$ . The order type of  $\{s \in T \mid s <_T t\}$  is denoted by  $ht(t, T)$ . The height of  $T$  is the least  $\alpha$  such that  $Lev_n(T) = \{t \in T \mid ht(t, T) = \alpha\} = \emptyset$  and is denoted by  $ht(T)$ . For  $t \in T$ , we denote  $s <_T t$  with  $ht(s, T) = \alpha$  by  $t|_\alpha$ . We say that a tree  $T$  is normal if for each  $t \in T$  there is  $s > t$  at each greater level less than  $ht(T)$  and every elements on limit levels are uniquely determined by their sets of predecessors. If  $T$  is a tree and  $X \subseteq ht(T)$ , then  $T \upharpoonright X = \{t \in T \mid ht(t, T) \in X\}$ .

**3. Games and Examples.** Let us recall game  $\mathcal{G}(B)$  played on a Boolean algebra  $B$  which was first studied by Jech [6]. Players I and II alternately play a descending sequence of elements of  $B$ .

$$\begin{array}{c|cccc} \text{I} & w_0 & w_1 & w_2 & \cdots \\ \hline \text{II} & b_0 & b_1 & b_2 & \cdots \\ \hline & w_0 \geq b_0 & \geq w_1 & \geq b_1 & \geq w_2 \geq b_2 \geq \cdots \end{array}$$

II wins the game iff  $\exists b > 0 \forall i \in \omega [b_i \geq b]$ .

If  $P$  is a separative partially ordered set, we can define the game  $G(P)$  played on  $P$  in the obvious way. Jech[6] showed

**Theorem A** ([6]). (1) *I (II) has a winning strategy in  $G(P)$  iff I (II) has a winning strategy in  $G(B(P))$ .*

(2) *I does not have a winning strategy in  $\mathcal{G}(B)$  iff  $B$  is  $\sigma$ -Baire.*

A Boolean algebra  $B$  is called *game-closed* if II has a winning strategy in  $\mathcal{G}(B)$ . If  $B$  has a dense  $\sigma$ -closed subset, then  $B$  is game-closed. In [2], Foreman proved that the converse holds when  $B$  has a dense subset of size  $\omega_1$ . Vojtáš [14] and Veličković [13] showed that this holds for every  $B$  that has a dense subset of size  $2^{\aleph_0}$ . On the contrary, Jech and Shelah [8] showed that the converse is unprovable in ZFC. Levinski[12] extends the game  $\mathcal{G}(B)$  to the game  $G(B, \kappa)$  of potential length  $\kappa$ . In  $G(B, \kappa)$ , players I and II alternately play a

descending sequence  $\langle a_\alpha \rangle_{\alpha < \kappa}$  of elements of  $B$ . At limit stage II moves first.<sup>2</sup> I wins the game iff at some limit stage  $\alpha$  II can not move; i.e.,  $\bigwedge_{\beta < \alpha} a_\beta = 0$ . Levinski [12] showed

**Theorem B** ([12]). *Let  $\kappa$  be regular. I does not have a winning strategy in  $G(B, \kappa)$  iff  $B$  satisfies the property  $P(\kappa, \infty, \kappa)$ .*

It is observed in [7] that if  $B$  is game-closed, then there exists a partially ordered set  $Q$  such that  $B^+ \times Q$  has a dense  $\sigma$ -closed subset. By using this observation, it is easy to show that for a game-closed complete Boolean algebra  $B$ , II has a winning strategy in  $G(B, \omega_1)$ . Hence we have

**Fact.** (1) *II has a winning strategy in  $\mathcal{G}(B)$  iff II has a winning strategy in  $G(B, \omega_1)$ .*  
 (2) *If I has a winning strategy in  $\mathcal{G}(B)$ , then I has a winning strategy in  $G(B, \omega_1)$ .*

We say that a game  $G$  is *BM-like* if  $G$  satisfies the same fact as above. The converse of (2) does not hold in general. Let  $S \subset \omega_1$  be a stationary and co-stationary set. Let  $\mathcal{P}_S$  be the partially ordered set of closed bounded subsets of  $S$  under end extension. Since  $\mathcal{P}_S$  is  $\sigma$ -Baire, I does not have a winning strategy in  $\mathcal{G}(B(\mathcal{P}_S))$ . On the other hand,  $\mathcal{P}_S$  does not have a branch of length  $\omega_1$ , so that I has a winning strategy in  $G(B(\mathcal{P}_S), \omega_1)$ . So  $G(B, \omega_1)$  is a distinct game from  $\mathcal{G}(B)$ .

**Proposition 1.** *It is consistent to ZFC that  $\neg CH +$  nondeterminacy of  $G(*, \omega_1)$ .*

*Proof.* We use following lemmas due to Levinski[12] and Veličkovič [13].

**Lemma C** ([12]). *Let  $B$  be a complete ccc Boolean algebra and  $P$  be a  $\sigma$ -closed partially ordered set. Then, the completion (in  $V^{(B)}$ )  $C$  of  $P$  satisfies  $P(\omega_1, \infty, \omega_1)$  in  $V^{(B)}$ .*

**Lemma D** ([13]). *Let  $T$  be a tree. If II has a winning strategy in  $G(T)$ , then  $T$  has a dense  $\sigma$ -closed subset.*

We assume that the ground model satisfies GCH. Let  $B$  be the completion of  $Fn(\omega_2 \times \omega, 2)$ . Then  $B$  is ccc and  $2^\omega = \omega_2$  in  $V^{(B)}$ . Let  $T$  be a complete binary tree of height  $\omega_1$ . Obviously  $T$  is  $\sigma$ -closed. By the assumption, the cardinality of  $T$  is  $\omega_1$ . Since  $B$  preserves cardinals,  $|T| = \omega_1 < 2^\omega$  in  $V^{(B)}$ . Hence  $T$  does not have a dense  $\sigma$ -closed subset in  $V^{(B)}$ . By Lemma D, II does not have a winning strategy in  $G(T)$ <sup>3</sup>. Hence II does not have a winning strategy in  $G(B(T))$  in  $V^{(B)}$ , so that II does not have a winning strategy in  $G(B(T), \omega_1)$  in  $V^{(B)}$ . On the other hand, by Lemma C,  $B(T)$  satisfies  $P(\omega_1, \infty, \omega_1)$  in  $V^{(B)}$ . Hence I also does not have a winning strategy in  $G(B(T), \omega_1)$  in  $V^{(B)}$ .

Nondeterminacy of  $G(*, \omega_1)$  follows from PFA (Proper Forcing Axiom) also. There exists a proper  $\sigma$ -Baire Boolean algebra  $B$  which does not have a dense  $\sigma$ -closed subset (see Example 2).  $P(\omega_1, \infty, \omega_1)$  is equivalent to  $\sigma$ -Baire +  $P(\omega_1, \infty, \omega)$ , so that PFA implies that every proper  $\sigma$ -Baire Boolean algebra satisfies  $P(\omega_1, \infty, \omega_1)$ . Hence, in  $B$ , both players do not have a winning strategy. algebras. However, games we consider can be defined on partially ordered sets. Let  $P$  be a partially ordered set with no least element.

<sup>2</sup>The definition of  $G(B, \kappa)$  given here is slightly modified.

<sup>3</sup>This follows from Foreman's Theorem [2] also.

**3.1.** For  $X \subset B^+$ , we say that  $f : X \rightarrow B^+$  is a *d-function* if  $f(a) \leq a$  for every  $a \in X$  and that  $g : X \rightarrow P(B^+)$  is a  $\lambda$ -*ds-function* if  $g(a) \subset (B \setminus a)^+$  and  $0 < |g(a)| \leq \lambda$  for every  $a \in X$ . A  $\lambda$ -ds-function  $g$  is said to be a  $\lambda$ -*pd-function* if  $g(a)$  is a pairwise disjoint family.

The game  $SG(B, \lambda)$  is played as follows: At the beginning, I chooses some  $a \in B^+$ . In the first move, I chooses a  $\lambda$ -ds-function  $g_0 : \{a\} \rightarrow P(B^+)$  and II chooses a d-function  $f_0 : X_0 \rightarrow B^+$  where  $X_0 = \bigcup g_0(\{a\}) = g_0(a)$ . Then I chooses a  $\lambda$ -ds-function  $g_1 : f_0(X_0) \rightarrow P(B^+)$ , and II chooses a d-function  $f_1 : X_1 \rightarrow B^+$  where  $X_1 = \bigcup g_1(f_0(X_0))$ . The game proceeds like this ad infinitum. I chooses, as his  $(n+1)$ th move, a  $\lambda$ -ds-function  $g_n : \text{Im}(f_{n-1}) \rightarrow P(B^+)$  and II chooses, as his  $(n+1)$ th move, a d-function  $f_n : X_n \rightarrow B^+$  where  $X_n = \bigcup \text{Im}(g_n)$ .

Let  $P : g_0, f_0, g_1, f_1, \dots, g_n, f_n, \dots$  be a play of  $SG(B)$ . A descending sequence  $\{a_n\}_{n \in \omega}$  of elements of  $B^+$  is said to be a *path* of  $P$  if for every  $n$ ,  $a_{2n} \in \bigcup \text{Im}(g_n)$  and  $a_{2n+1} \in \text{Im}(f_n)$ . II wins the play  $P : g_0, f_0, g_1, f_1, \dots, g_n, f_n, \dots$  iff  $\exists a > 0 \forall n \in \omega [a_n \geq a]$  for every path  $\{a_n\}_{n \in \omega}$  of  $P$ .

In a play of  $SG(B, \lambda)$ , if we pay attention to a path of the play, then the game proceeds same as  $\mathcal{G}(B)$ . In particular,  $SG(B, 1)$  is  $\mathcal{G}(B)$ . Hence  $SG(B, \lambda)$  is a BM-like game. We denote  $SG(B, |B|)$  by  $SG(B)$ .

**Theorem 2.** *If  $B$  has a dense subset of cardinality  $\lambda$ , then  $SG(B, \lambda)$  is determined. In particular,  $SG(B)$  is determined for every Boolean algebra  $B$ .*

*Proof.* Suppose that II does not have a winning strategy in  $SG(B, \lambda)$  for some  $B$ . Then, II does not have a winning strategy in  $\mathcal{G}(B)$ . Let  $D \subset B^+$  be a dense subset of cardinality  $\lambda$ . Let us consider the following strategy for I in  $SG(B, \lambda)$ . At the beginning, I chooses  $1_B$  and at his  $(n+1)$ th move, I chooses  $g_n$  such that  $g_n(a) = \{b \in D \mid b \leq a\}$  for every  $a \in \text{Im}(f_{n-1})$ . This strategy is actually a winning strategy for I. Suppose not. Then there exists a play

$$P : g_0, f_0, g_1, f_1, \dots, g_n, f_n, \dots$$

such that I plays according to above strategy but II wins. We can construct a winning strategy for II in  $\mathcal{G}(B)$  as follows: Let  $a_0$  be the first move of I in  $\mathcal{G}(B)$ .  $g_0(1_B) = D$ , so that there exists  $b_0 \leq a_0$  such that  $b_0 \in g_0(1_B)$ . II picks  $a_1 = f_0(b_0)$ . Let  $a_2$  be the second move of I such that  $a_0 \geq a_1 \geq a_2$ . Then there exists  $b_2 \leq a_2$  such that  $b_2 \in g_1(a_1)$ . II picks  $a_3 = f_1(b_2)$ . We continue this ad infinitum. If I chooses  $a_{2n}$  at his  $n+1$ th move, then there exists  $b_{2n} \leq a_{2n}$  such that  $b_{2n} \in g_n(a_{2n-1})$ . II picks  $a_{2n+1} = f_n(b_{2n})$  at his  $n+1$ th move. It is easy to see that  $\{a_n\}_{n \in \omega}$  is a path of  $P$ . Since II wins the play, it holds that  $\exists a > 0 \forall n \in \omega [a_n \geq a]$ . So II wins the play  $a_0, a_1, a_2, a_3, \dots$  in  $\mathcal{G}(B)$ . Hence II has a winning strategy in  $\mathcal{G}(B)$ . But this contradicts to the assumption. Hence I has a winning strategy in  $SG(B, \lambda)$ . Therefore  $SG(B, \lambda)$  is determined.

The game  $DG(B, \lambda)$  is played in the same way as  $SG(B, \lambda)$  except that I is restricted to choose  $\lambda$ -pd-functions instead of  $\lambda$ -ds-functions. So  $DG(B, \lambda)$  is also BM-like and easier for I than  $\mathcal{G}(B)$ . We denote  $DG(B, |B|)$  by  $DG(B)$ . If  $P$  is a partially ordered set, we can define the game  $DG(P, \lambda)$  played on  $P$  in the obvious way.

**Lemma 3.** *Let  $P$  be a partially ordered set. I has a winning strategy in  $DG(P, \lambda)$  iff he has one in  $DG(B(P), \lambda)$ .*

*Proof.* The proof is similar to the proof for  $\mathcal{G}(B)$ . (See [6, p136].)

**3.2.** There are many Boolean algebras for which I does not have a winning strategy in  $\mathcal{G}(B)$  but has a winning strategy in  $DG(B)$ . However, it remains open whether  $DG(B)$  is determined for all Boolean algebras .

**Example 1.** Let  $S$  be an uncountable subset of  $\omega_1$ . We consider the following game  $G_S(G_S^*)$  on  $S$ . I and II alternately play an increasing sequence of ordinals in  $S$ .

$$\begin{array}{c|cccc} \text{I} & \alpha_0 & \alpha_2 & \alpha_4 & \cdots \\ \hline \text{II} & & \alpha_1 & \alpha_3 & \cdots \\ \hline & & (\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq \cdots) & & \end{array}$$

I wins the play  $G_S(G_S^*)$  iff  $\bigcup \alpha_i \notin S (\in S)$ . It is easy to show that I has a winning strategy in  $G_S(G_S^*)$  iff II has a winning strategy in  $G_S^*(G_S)$ .

**Proposition 4.** *If  $S$  is stationary and co-stationary, then neither I nor II has a winning strategy in  $G_S$ .*

*Proof.* Since  $S$  is stationary and co-stationary, it is enough to show that II does not have a winning strategy. Suppose that II has a winning strategy  $\sigma$ . Then we can show that II has a winning strategy in  $G(\mathcal{P}_S)$ . Given a partial play  $p_0, p_1, p_2, \dots, p_{2i}$  in  $G(\mathcal{P}_S)$ , II plays  $p_{2i+1} = p_{2i} \cup \{\sigma(\alpha_0, \alpha_2, \dots, \alpha_{2i})\}$  where  $\alpha_j = \max p_j$ . Let  $p_0, p_1, p_2, \dots$  be a play according to the above strategy for II. Since  $\sigma$  is a winning strategy,  $\alpha = \bigcup \alpha_i \in S$ . Hence we have  $p = \bigcup_i p_i \cup \{\alpha\} \in \mathcal{P}_S$ . Therefore II wins the play above. So II has a winning strategy in  $G(\mathcal{P}_S)$ , but this contradicts to the result of Jech[6].

**Theorem 5.** *I has a winning strategy in  $DG(B(\mathcal{P}_S), \omega_1)$ , but does not have one in  $SG(B(\mathcal{P}_S), \omega)$ .*

*Proof.* Let us consider a strategy for I in  $DG(\mathcal{P}_S, \omega_1)$  as follows. for every  $b \in B(\mathcal{P}_S)^+$ . At the beginning I picks  $\emptyset \in \mathcal{P}_S$ . In the first move, I plays a  $\omega_1$ -pd-function  $g_0$  such that  $\text{Im}(g_0) = \{\{\alpha\} | \alpha \in S\} \subset \mathcal{P}_S$  which is a pairwise disjoint family in  $\mathcal{P}_S$ . Let  $f_{n-1}$  be a II's  $n$ th move and  $X_{n-1} = \bigcup \text{Im}(f_{n-1})$ . Then I plays a  $\omega_1$ -pd-function  $g_n$  such that  $\text{Im}(g_n) = \{p \cup \{\alpha\} | p \in X_{n-1}, \max p < \alpha, \alpha \in S\}$ . Suppose that this strategy is not a winning strategy for I. There exists a play

$$P: g_0, f_0, g_1, f_1, \dots, g_n, f_n, \dots$$

such that

$$\exists a > 0 \forall n \in \omega [a_n \geq a] \text{ for every path } \{a_n\}_{n \in \omega} \text{ of } P.$$

We will show that II has a winning strategy in  $G_S$ . This contradicts to Theorem 4.

Let  $\alpha_0 \in S$  be the first move of I in  $G_S$ . Let  $a_0 = \{\alpha_0\}$ . Then  $a_0 \in \text{Im}(g_0) = \{\{\alpha\} | \alpha \in S\}$ . II picks  $\alpha_1 = \max f_0(a_0)$ . Put  $a_1 = f_0(a_0)$ . Let  $\alpha_2 \in S$  be the second move of I such that  $\alpha_2 \geq \alpha_1$ . Let  $a_2 = a_1 \cup \{\alpha_2\}$ . We have  $a_2 \in g_1(a_1)$ . II picks  $\alpha_3 = \max f_1(a_2)$ . Put  $a_3 = f_1(a_2)$ . We continue this ad infinitum. When I chooses  $\alpha_{2n}$  at his  $n + 1$ th move,  $a_{2n} = f_{n-1}(a_{2n-2}) \cup \{\alpha_{2n}\} \in g_n(a_{2n-1})$ . Then II picks  $\alpha_{2n+1} = \max f_n(a_{2n})$ .  $\{a_n\}$  is a path of P and II wins the play P, we have  $\exists a > 0 \forall n \in \omega [a_n \geq a]$ . Hence  $\beta = \bigcup \alpha_n = \bigcup a_n \in S$ . So II can win in  $G_S$  according to this strategy.

Now suppose that I has a winning strategy  $\tau$  in  $SG(B(\mathcal{P}_S), \omega)$ . We will show that I has a winning strategy in  $G_S$ . This contradicts to Theorem 4. Let  $g_0 = \tau(\langle \rangle)$ ,  $\text{Dom}(g_0) = \{q\}$  and  $\alpha_0 = \min\{\alpha \in S | \forall p \in g_0(q) [\max p \leq \alpha]\}$ . Since  $|g_0(q)| \leq \omega$ ,  $\alpha_0$  exists. I chooses  $\alpha_0$  at his first move in  $G_S$ . Let  $\alpha_1 \in S$  be II's first move. Put  $f_0(p) = p \cup \{\alpha_1\}$  for every  $p \in g_0(q)$  and  $g_1 = \tau(f_0)$ . Then there exists  $\alpha \in S$  such that  $\forall p \in g_0(q) [\max g_1(p \cup \{\alpha_1\}) \leq \alpha]$ . Let  $\alpha_2$  be such least  $\alpha$ . I chooses  $\alpha_2$  at his second move in  $G_S$ . We continue this and get the play  $P: g_0, f_0, g_1, f_1, \dots, g_n, f_n, \dots$  where I plays according to  $\tau$ . Since  $\tau$  is a winning strategy

in  $SG(B(\mathcal{P}_S), \omega)$ , for some path  $\{a_n\}$  in  $P$ ,  $\{a_n\}$  does not have a lower bound in  $\mathcal{P}_S$ ; i.e.,  $\sup \bigcup_{n \in \omega} a_n \notin S$ . By the definition of  $f_n$  above, we have  $\sup \bigcup_{n \in \omega} a_n = \bigcup_{n \in \omega} \alpha_n$ , so that I wins in the play  $\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq \dots$ . Hence I has a winning strategy in  $G_S$ .

**Example 2.** Let  $\mathcal{L} = (C_\alpha)_{\alpha \in S}$  be a ladder on  $S \subseteq \text{Lim}(\omega_1)$ ; i.e.  $C_\alpha$  is a cofinal subset of  $\alpha$  of order type  $\omega$  for each limit ordinal  $\alpha \in S$ . Let  $\mathcal{P}_\mathcal{L}$  be a partially ordered set as follows:  $p \in \mathcal{P}_\mathcal{L}$  iff  $p$  is a closed bounded subset of  $\omega_1$  and  $C_\alpha \cap p$  is finite for every  $\alpha \in S$ . We define an ordering on  $\mathcal{P}_\mathcal{L}$  by end extension.  $\mathcal{P}_\mathcal{L}$  is proper and  $\sigma$ -Baire (see [1, p96]) but does not have a dense  $\sigma$ -closed subset.

Now we consider the game  $G_\mathcal{L}$  which is a variant of  $G_S^*$ . I and II alternately play a increasing sequence  $\{\alpha_i\}_{i \in \omega}$  of ordinals smaller than  $\omega_1$ . I wins the play iff  $\alpha = \bigcup \alpha_i \in S$  and  $|C_\alpha \cap \{\alpha_i\}_{i \in \omega}| = \omega$ .

**Proposition 6.** *If  $S$  is stationary, then neither I nor II has a winning strategy in  $G_\mathcal{L}$ .*

*Proof.* By Proposition 4 and the remark of the definition of  $G_S^*$ , I does not have a winning strategy in  $G_S^*$ . In  $G_\mathcal{L}$ , I and II do not necessarily play an element of  $S$ , but it is easy to see that I also does not have a winning strategy in  $G_\mathcal{L}$ . Suppose that II has a winning strategy  $\tau$  in  $G_\mathcal{L}$ . We will show that II has a winning strategy in  $\mathcal{G}(B(\mathcal{P}_\mathcal{L}))$ . Then, by the fact that  $\mathcal{P}_\mathcal{L}$  does not have a  $\sigma$ -closed dense subset and Lemma D, we have a contradiction.

Suppose that  $p_0, p_1, \dots, p_k \in \mathcal{P}_\mathcal{L}$  and every  $p_{i+1}$  is an end extension of  $p_i$ . We define an ordinal  $o(p_0, p_1, \dots, p_k)$  as follows: Let  $q_0 = p_0$  and  $q_{i+1} = p_{i+1} \setminus p_i$  for  $i = 1 \dots k$ , and let  $X_k = q_0 \times q_1 \times \dots \times q_k$ . Note that for every  $\bar{\alpha} = (\alpha_0, \dots, \alpha_k) \in X_k$ ,  $\alpha_0 < \dots < \alpha_k$ . Put  $o(p_0, p_1, \dots, p_k) = \sup\{\tau(\bar{\alpha}) \mid \bar{\alpha} \in X_k\}$ . Then  $o(p_0, p_1, \dots, p_k) < \omega_1$ . Let us consider a strategy  $\sigma$  for II in  $\mathcal{G}(\mathcal{P}_\mathcal{L})$  which is defined by  $\sigma(p_0, p_1, \dots, p_k) = p_k \cup \{o(p_0, p_1, \dots, p_k)\} \in \mathcal{P}_\mathcal{L}$ . Suppose that  $\sigma$  is not a winning strategy. Then, there is a play that II plays according to  $\sigma$  but I wins. Let  $p_0, p_1, \dots$  be a I's move in this play. Let  $\beta = \sup \bigcup_{n < \omega} p_n$ . Since I wins

this play,  $p = \bigcup_{n < \omega} p_n \cup \{\beta\} \notin \mathcal{P}_\mathcal{L}$ . Hence there exists  $\alpha \in S$  such that  $|C_\alpha \cap p| = \omega$ . Since

$C_\alpha$  is a cofinal subset of  $\alpha$  of order type  $\omega$ , if  $\gamma < \alpha$ , then  $|C_\alpha \cap \gamma| < \omega$ . Hence we have  $\beta = \alpha \in S$  and  $|C_\beta \cap \bigcup_{n < \omega} p_n| = \omega$ . Let  $C_\beta \cap \bigcup_{n < \omega} p_n = \{\gamma_0, \gamma_1, \gamma_2, \dots\}$  such that  $\gamma_i < \gamma_{i+1}$

for every  $i < \omega$ . There is  $n_i < \omega$  such that  $\gamma_i \in p_{n_i}$ . Without loss of generality, we can assume that  $n_i$ 's are distinct and moreover  $\gamma_i \in q_i$  for every  $i < \omega$ . Note that, for every  $i < \omega$ , there exists  $\bar{\alpha}_i \in X_i$  such that  $\bar{\alpha}_i$  enumerates elements of  $C_\beta \cap p_i$ . Then, we have

$$\gamma_0 < \tau(\gamma_0) \leq o(p_0) \leq \gamma_1 < \tau(\gamma_0, \gamma_1) \leq o(p_0, p_1) \leq \gamma_2 < \dots$$

Since  $\beta = \sup \bigcup_{n < \omega} p_n = \bigcup_{n < \omega} o(p_0, p_1, \dots, p_n) = \bigcup_{n < \omega} \gamma_n$  and  $\tau$  is a winning strategy for II in  $G_\mathcal{L}$ , we have  $|C_\beta \cap \{\gamma_0, \gamma_1, \dots\}| < \omega$ . This is a contradiction.

**Theorem 7.** *I has a winning strategy in  $DG(B(\mathcal{P}_\mathcal{L}), \omega_1)$ , but does not have in  $SG(B(\mathcal{P}_\mathcal{L}), \omega)$ .*

*Proof.* The proof goes on similar to the proof of Theorem 5 by using above theorem.

**Example 3.** In [3], Gregory constructed a tree  $T_W$  of height  $\omega_1$  which is  $\sigma$ -Baire but if  $2^\omega < 2^{\omega_1}$ , then  $B(T_W)$  does not satisfies  $P(\omega_1, \omega_1, \omega_1)$ . Hence I has a winning strategy in  $G(B(T_W), \omega_1)$  under  $2^\omega < 2^{\omega_1}$ .  $T_W$  is constructed from  $W \in {}^\omega(\omega_2)$  and a function  $F$  such that the following condition are satisfied.

- (1)  $\text{Dom}(F) \subseteq {}^{<\omega_1}(\omega_2) (= \bigcup_{\alpha < \omega_1} \text{func}\alpha(\omega_2))$ ;
- (2)  $\text{Im}(F) \subseteq {}^{<\omega_1}(\omega_2)$ ;
- (3) if  $f \in \text{Dom}(F)$ , then  $\text{Dom}(F(f)) = \text{Dom}(f) + 1$ ;
- (4) if  $f \in \text{Dom}(F)$ , then  $F(f|\alpha) = F(f)|(\alpha + 1)$  for all  $\alpha \leq \text{Dom}(f)$ .
- (5)  $\forall \gamma < \omega_1 \forall f \in \gamma^{+1}(\omega_2) \forall t$ : countable subset of  $\gamma(\omega_2)$  [ $\forall k \in t [F(k) \subset f]$   
and there are  $2^\omega$  many  $h \in \gamma(\omega_2)$  such that  $h$  is a union of element of  $t$   
 $\rightarrow \exists h \in \gamma(\omega_2) [h \text{ is a union of element of } t \text{ and } F(h) = f]$ .];
- (6)  $T_W = \{f | F(f) \subset W\}$ .

**Lemma 8.** *If  $t \subseteq T_W \upharpoonright \gamma$  is countable and there are  $2^\omega$  many  $h \in \gamma(\omega_2)$  such that  $h$  is a union of element of  $t$ , then there exists  $h \in \gamma(\omega_2) \setminus T_W$  such that  $h$  is a union of element of  $t$ .*

*Proof.* Let  $f \in \gamma^{+1}(\omega_2)$  be such that  $f \upharpoonright \gamma = W \upharpoonright \gamma$  but  $F(\gamma) \neq W(\gamma)$ . Since  $t \subset T_W \upharpoonright \gamma$ ,  $\forall k \in t [F(k) \subset W \upharpoonright \gamma]$ . Hence by the condition 5 above, we have  $\exists h \in \gamma(\omega_2) [h \text{ is a union of } t \text{ and } F(h) = f \not\subseteq W]$ . Therefore, there exists  $h \in \gamma(\omega_2) \setminus T_W$  such that  $h$  is a union of element of  $t$ .

Using this lemma, we can easily show the following

**Theorem 9.** *I has a winning strategy in  $DG(T_W, 2)$ .*

**Example 4.** Let  $T$  be an  $\omega_2$ -Suslin tree. We say that  $T$  is essentially  $\sigma$ -closed if there is a club  $C \subseteq \omega_2$  such that  $T \upharpoonright C$  is  $\sigma$ -closed. In [9], Jensen constructed an  $\omega_2$ -Suslin tree  $T_G$  which is not essentially  $\sigma$ -closed under  $V=L$ . Later Gregory[4] showed that there exists an  $\omega_2$ -Suslin tree under  $GCH + \square_{\omega_1}$ . We will show that I has a winning strategy in  $DG(T, 2)$ .

Let  $E \subset \omega_2$  be stationary and  $(C_\alpha | \alpha \in \text{Lim}(\omega_2))$  satisfy  $\square_{\omega_1}(E)$ . Let  $(\gamma_\alpha(\beta) | \beta < \eta_\alpha)$  be the monotone enumeration of  $C_\alpha$ . Then the  $\omega_2$ -Suslin tree  $T$  is defined such that the following conditions are satisfied[9].

- (1)  $T \subseteq \omega_2$  and  $T$  is normal;
- (2)  $T_0 = \{0\}$ ;
- (3)  $\forall x \in T_\alpha [x \text{ has two successors in } T_{\alpha+1}]$ ;
- (4)  $\forall x \in T \forall y \in T [x <_T y \rightarrow x < y \text{ (as an ordinal)}]$ ;
- (5)  $\forall \alpha \in \text{Lim}(\omega_2) \forall x \in T_\alpha \exists y <_T x \forall \beta < \eta_\alpha [ht(y) \leq \gamma_\alpha(\beta) \rightarrow \forall z \in T_{\gamma_\alpha(\beta+1)} [x \upharpoonright_{\gamma_\alpha(\beta)} <_T z \rightarrow x \upharpoonright_{\gamma_\alpha(\beta+1)} \leq z \text{ (as an ordinal)}]]$ .

**Theorem 10.** *I has a winning strategy in  $DG(T, 2)$ .*

*Proof.* Note that  $W = \{\alpha | T \upharpoonright \alpha \text{ is club in } \omega_2\}$  is club in  $\omega_2$ . We describe a strategy for I. At the beginning, I picks 0. In the first move, I plays a 2-pd-function  $g_0$  such that  $\text{Im}(g_0) = \{\alpha_0, \alpha_1\}$  where  $\alpha_0 \neq \alpha_1$  and  $\alpha_0, \alpha_1 \in T_{\min} W$ . Let  $f_{n-1}$  be a II's  $n$ th move and  $X_n = \bigcup \text{Im}(f_{n-1})$ . Then I plays a 2-pd-function  $g_n$  such that  $\text{Im}(g_n) = \bigcup \{\{\alpha_{\rho \wedge 0}, \alpha_{\rho \wedge 1}\} | \rho \in {}^n 2, \alpha_{\rho \wedge 0} \neq \alpha_{\rho \wedge 1} \in T_{\min}(W \setminus \sup X_n), f_{n-1}(\alpha_\rho) <_T \alpha_{\rho \wedge i} \text{ for } i \in \{0, 1\}\}$ . Suppose that this strategy is not a winning strategy for I. Then, there exists a play  $P$ :  $g_1, f_1, g_2, f_2, \dots, g_n, f_n, \dots$  such that  $\forall u \in {}^\omega 2 \exists \alpha_u \in T_\alpha \forall n \in \omega [\alpha_{f_n} <_T \alpha_u]$  where  $\alpha = \sup\{\alpha_\rho | \rho \in {}^{<\omega} 2\}$ . Note that  $\text{cof}(\alpha) = \omega$ . For every  $\alpha_u (u \in {}^\omega 2)$ , let  $y_u \in T$  be an element which satisfies condition 5. By the Pigeonhole Principle, there are  $X \subset {}^\omega 2$  with  $|X| = 2^\omega$ ,  $\beta < \eta_\alpha, \rho \in {}^{<\omega} 2$  such that  $y_u <_T \alpha_\rho <_T \alpha_u$  and  $ht(y_u) \leq \gamma_\alpha(\beta) \leq ht(\alpha_\rho)$  for every  $u \in X$ . Then, for every  $u, v \in X$  and  $\nu > \beta$ , we have  $\alpha_u \upharpoonright_{\gamma_\alpha(\nu)} = \alpha_v \upharpoonright_{\gamma_\alpha(\nu)}$ . Hence, by the normality of  $T$ , we have  $\alpha_u = \alpha_v$  for every  $u, v \in X$ , a contradiction.

## REFERENCES

1. K. Devlin, The Yorkshireman's guide to proper forcing, in: *Surveys in Set Theory*, A.R.D. Mathias(ed.) London Math. Soc. Lec. Note vol 87, Cambridge Univ. Press, 1983
2. M. Foreman, Games played on Boolean algebras, *J. of Symbolic Logic* 41 (1976), 636–671. *J. of Symbolic Logic* 60 (1995), 58–73.
3. J. Gregory, A Countably Distributive Complete Boolean Algebra Not Uncountably Representable, *Proc. Amer. Math. Soc.* 42 (1974), 42–46.
4. J. Gregory, Higher Souslin trees and the generalized continuum hypothesis, *J. of Symbolic Logic* 41 (1976), 663–671.
5. T. Jech, *Set Theory*, Academic Press, 1978
6. T. Jech, A game-theoretic property of Boolean algebras, in: A. Macintyre et al., eds., *Logic Colloquium 77*, North-Holland, Amsterdam, 1978, 135–144.
7. T. Jech, More game-theoretic properties of Boolean algebras, *Annals of Pure and Appl. Logic* 26 (1984), 11–29.
8. T. Jech and S. Shelah, On Countably Closed Complete Boolean Algebras, *J. of Symbolic Logic* 61 (1996), 1380–1386.
9. R.B. Jensen, The Fine Structure of the Constructible Hierarchy, *Ann. Math. Logic* 4 (1972), 229–308.
10. S. Koppelberg, General Theory of Boolean Algebras, in: *Handbook of Boolean Algebras*, J. D. Monk with R. Bonnet(eds.) vol 1, North-Holland, 1989
11. K. Kunen, *Set Theory*, (North-Holland, 1980)
12. J. P. Levinski, Filters and large cardinals, *Annals of Pure and Appl. Logic* 72 (1995), 177–212. *Math. Soc. Japan* 40(1988), 445–456.
13. B. Velicković, Playful Boolean Algebras, *Trans. of AMS* 296 (1986), 727–740.
14. P. Vojtáš, Game properties of Boolean algebras, *Comment. Math. Univ. Carol.*, 24(1983),349–369.

DIVISION OF MATHEMATICS AND INFORMATICS, DEPARTMENT OF HUMAN ENVIRONMENT, FACULTY OF HUMAN DEVELOPMENT, KOBE UNIVERSITY, TSURUKABUTO 3-11, NADA, KOBE 657-8501