ON STRONG IDEALS AND P–IDEALS IN BCI–ALGEBRAS

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Abstract. In this note we discuss the relation between strong ideals and p-ideals. The following results are proved: (1) An ideal I of a BCI-algebra X is strong if and only if I is a closed p-ideals; (2) In a periodic BCI-algebra, an ideal is strong if and only if it is a p-ideal; (3) If I is a strong ideal of a BCI-algebra X then the quotient algebra $(X/I;*,I)$ is a p-semisimple BCI-algebra. Also some of other characterizations of strong ideals are given.

In the developement of BCI-algebras, several classes of ideals have occurred, such as closed ideals, strong ideals, p-ideals, and so on. The aim of the note is to expound the relation between strong ideals and p-ideals, and to discuss further properties of strong ideals.

An algebra $(X;*,0)$ of type (2, 0) is called a *BCI-algebra* if it satisfies the following conditions:

(a_1) $((x*y)*(x*z))*(z*y)=0$,
(a_2) $(x*(x*y))*y=0$,
(a_3) $x*x=0$,
(a_4) $x*y=0$ and $y*x=0$ imply $x=y$,
(a_5) $x\le y$ if and only if $x*y=0$.

Any BCI-algebra X has the following properties:

(b_1) $x*0=x$,
(b_2) $(x*y)*z=(x*z)*y$,
(b_3) $0*(x*y)=(0*x)*(0*y)$.

In what follows X would mean a BCI-algebra unless otherwise specified. In [10], J. Meng and X. L. Xin established fundamental properties of atoms and branchs in BCI-algebras. An element a of X is said to be an *atom* if, for all $x\in X$, $x*a=0$ implies $x=a$. The set of all atoms of X is denoted by $L(X)$. For any $a\in L(X)$, the set $V(a)=\{x\in X : a\le x\}$ is called a *branch* of X. Obviously, $0\in L(X)$ and $V(0)=B(X)$, the BCK-part of X. The following were proved.

(b_4) a is an atom of X if and only if $a=x*(x*a)$ for all $x\in X$,
(b_5) $L(X)=\{0*(0*x) : x\in X\}=\{0*x : x\in X\}$,
(b_6) $x\in V(a)$ and $y\in V(b)$ imply $x*y\in V(a*b)$,
(b_7) $L(X)$ is the greatest p-semisimple subalgebra of X,
(b_8) x and y belong to the same branch if and only if $x*y\in B(X)$,
(b_9) For $a,b\in L(X)$ and $x\in V(b)$, we have $a*x=a*b$.

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Denote $a_x = 0 * (0 * x)$ for all $x \in X$. A nonempty subset I of X is called an *ideal* of X if (a_6) $0 \in I$ and (a_7) $x * y \in I$ and $y \in I$ imply $x \in I$. Any ideal I satisfies (b_{10}) $x \in I$ and $y \leq x$ imply $y \in I$. Note that an ideal I of X may not be a subalgebra of X. If I is a subalgebra of X then we call I a *closed ideal* of X ([2] - [4]). An ideal I is closed if and only if $a \in I$ implies $0 * a \in I$ ([4]). The notion of p-ideals in BCI-algebras was introduced by X. H. Zhang and H. Jiang [11] and recently investigated by J. Meng, Y. B. Jun and E. H. Roh [8].

Definition 1 ([11]). A nonempty subset I of X is called a *p-ideal* of X if it satisfies:
(a_8) $0 \in I$,
(a_9) $(x * z) * (y * z) \in I$ and $y \in I$ imply $x \in I$.

A p-ideal of X must be an ideal ([11; Theorem 1.1]). For a subset A of X, denote $L(A) = \{0 * (0 * x) : x \in A\}$.

Proposition 2 ([9]). *If I is an ideal of X then*
(b_{11}) *$L(I) = I \cap L(X)$,*
(b_{12}) *$L(I)$ is an ideal of $L(X)$.*

Proposition 3 ([8]). *If I is an ideal of X then the following are equivalent:*
(b_{13}) *I is a p-ideal.*
(b_{14}) *$B(X) \subseteq I$.*
(b_{15}) *$x \in I$ implies $V(a_x) \subseteq I$.*
(b_{16}) *$I = \cup\{V(a) : a \in L(I)\}$.*

Now we give a lemma which is useful in the discussion of relation between p-ideals and strong ideals.

Lemma 4. *Let I be a p-ideal of X. Then $x \in I$ implies $0 * x \in I$ if and only if $x \in X - I$ implies $0 * x \in X - I$.*

Proof. Suppose that $x \in I$ implies $0 * x \in I$ for all $x \in X$, and let $a \in X - I$. We claim that $0 * a \in X - I$. If $0 * a \notin X - I$ then $0 * a \in I$. By the hypothesis we obtain $0 * (0 * a) \in I$. Noticing that I is a p-ideal and $a * (0 * (0 * a)) \in B(X) \subseteq I$, we have $a \in I$, which is a contradiction. So $0 * a \in X - I$. This shows that $a \in X - I$ implies $0 * a \in X - I$.

Conversely, suppose that $x \in X - I$ implies $0 * x \in X - I$, and let $a \in I$. We assert $0 * a \in I$. If not then $0 * a \in X - I$. It follows from hypothesis that $0 * (0 * a) \in X - I$. Since I is a p-ideal, it follows from (b_{16}) that $X - I = \cup\{V(b) : b \in L(X) - L(I)\}$, and so $V(0 * (0 * a)) \subseteq X - I$. Observing that $a \in V(0 * (0 * a))$ we obtain $a \in X - I$, which contradicts to $a \in I$. Thus we should have $0 * a \in I$. This proves that $a \in X - I$ implies $0 * a \in X - I$. The proof is complete.

Another equivalent statement of this lemma is as follows.

Corollary 5. *A p-ideal I is closed if and only if $x \in X - I$ implies $0 * x \in X - I$.*

We will adopt the following notations. For a subset A of X, $(A]$ (resp. $(A]_p$) denotes the least ideal (resp. least p-ideal) containing A in X.

Theorem 6. *If I is an ideal of X, then $(I]_p = \cup\{V(a) : a \in L(I)\}$.*

Proof. This is a consequence of Propositions 2 and 3.

S. A. Bhatti [1] introduced the notion of strong ideals and obtained some results about it. In what follows we discuss the relation between strong ideals and p-ideals.

Definition 7 ([1]). An ideal I of X is said to be *strong* if $x \in I$ and $a \in X - I$ imply $x * a \in X - I$.

Lemma 8. *Every strong ideal is a p-ideal.*

Proof. Let I be a strong ideal of X and let $a \in B(X)$. We claim $a \in I$. If $a \notin I$, then $a \in X - I$. Since I is strong and $0 \in I$, it follows from Definition 7 that $0 * a \in X - I$. But $a \in B(X)$ implies $0 * a = 0 \in I$, a contradiction. Hence $a \in I$. This shows that $B(X) \subseteq I$. By Proposition 3 we know that I is a p-ideal. This completes the proof.

The inverse of Lemma 8, in general, does not hold as shown in the following example.

Example 9. Let $X = \{2^n : n = 0, \pm 1, \pm 2, ...\}$, $\div$ the usual division. Then $(X; \div, 1)$ is a p-semisimple BCI-algebra and $I = \{1, 2, 2^2, ...\}$ is a p-ideal of X, but I is not a strong ideal as $2^{-1} \in X - I$ and $1 \div 2^{-1} = 2 \in I$.

Next we will prove that for a closed ideal the inverse of Lemma 8 holds.

Proposition 10 ([1]). *Every strong ideal is closed.*

Note that, in Example 9, I is not closed.

Theorem 11. *An ideal I of X is strong if and only if I is a closed p-ideal.*

Proof. The part "only if " follows from Lemma 8 and Proposition 10.

Conversely suppose that I is a closed p-ideal of X. Let $a \in I$. Then $0 * a \in I$ as I is closed. Taking use of Lemma 4 we know that for any a in X, $a \in X - I$ implies $0 * a \in X - I$. Now we prove that $x \in I$ and $a \in X - I$ imply $x * a \in X - I$. If not, then there are $x \in I$ and $a \in X - I$ such that $x * a \in I$. Since I is closed and $x \in I$, we have $0 * a = (x * a) * x \in I$, which is impossible. Therefore I is a strong ideal. The proof is complete.

Because every ideal of a finite BCI-algebra is closed (see [2; Theorem 2.9]), we have the following corollary.

Corollary 12. *An ideal of a finite BCI-algebra is strong if and only if it is a p-ideal.*

This corollary can be improved. J. Meng and S. M. Wei [9] introduced the notion of periodic BCI-algebras (see [9; Definition 3]) and gave a characterization as follows.

Proposition 13. ([9]). *Every ideal of X is closed if and only if X is periodic.*

Theorem 14. *Suppose X is a periodic BCI-algebra and let I be an ideal of X. Then I is strong if and only if I is a p-ideal.*

Proof. It follows from Theorem 11 and Proposition 13.

This theorem shows that in a periodic BCI-algebra the notion of strong ideals and p-ideals coincide.

Next we will give other characterizations of strong ideals.

Theorem 15. *Let I be a p-ideal of X. Then I is strong if and only if $L(I)$ is a strong ideal of $L(X)$.*

Proof. ($\Rightarrow$) It is easy and omitted.

($\Leftarrow$) Suppose $L(I)$ is a strong ideal of $L(X)$. To prove that I is strong in X, it is sufficient to show that I is closed in X by Theorem 11. Since I is a p-ideal of X, we have $I = \cup\{V(a) : a \in L(I)\}$ by Proposition 3. For any $x \in I$ we know $a_x \in L(I)$. Since $L(I)$ is closed in $L(X)$, it follows that $0*a_x \in L(I)$. On the other hand, $0*a_x = 0*(0*(0*x)) = 0*x$, so $0*x \in L(I)$. Hence $0*x \in I$. Thus I is closed in X and so I is a strong ideal of X. This completes the proof.

Theorem 16. *A closed ideal I of X is strong if and only if $a \in X-I$ implies $0*a \in X-I$.*

Proof. ($\Rightarrow$) It is trivial as $0 \in I$.

($\Leftarrow$) Suppose that $a \in X - I$ implies $0 * a \in X - I$. For any $x \in I$ and $a \in X - I$, $(x * a) * x = 0 * a \in X - I$, which implies that $x * a \in X - I$. Because if $x * a \notin X - I$, then $x * a \in I$. It follows from $x \in I$ and I being closed that $(x * a) * x \in I$, a contradiction. Therefore I is a strong ideal, completing the proof.

We will close this paper with quotient algebras of BCI-algebras via strong ideals.

Suppose I is a closed ideal of X. Define $x \sim y$ if and only if $x * y \in I$ and $y * x \in I$. Then $\sim$ is a congruence relation on X and $(X/I; *, I)$ is a BCI-algebra, which is called the *quotient algebra* of X via I. For the details of quotient algebras we refer to Z. M. Chen and H. X. Wang [3].

Theorem 17. *Suppose I is a strong ideal of X. Then the quotient algebra $(X/I; *, I)$ is a p-semisimple BCI-algebra and $X/I \cong L(X)/L(I)$.*

Proof. Let $C_x \in B(X/I)$, then $I * C_x = I$. We claim $x \in I$. If $x \notin I$, then $x \in X - I$. Since I is strong, therefore $0*x \in X-I$. This contradicts to $I*C_x = I$. Hence $x \in I$, that is, $C_x = I$. This means $B(X/I) = \{I\}$. Therefore X/I is p-semisimple, and $X/I \cong L(X)/L(I)$ is easy and omitted. The proof is complete.

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