AMENABILITY AND REPRESENTATION OF LOCALLY COMPACT GROUPS

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Abstract. Let G be a locally compact topological amenable group and $\{\pi, H\}$ a representation of $M(G)$, then

$$\text{Inf}\ \{\|\pi(\mu)x\| : \mu \in M(G)\} = \text{dis}(x, K_{H,\pi}) \text{ for all } x \in H$$

where $K_{H,\pi}$ is the linear span of $\{y - \pi(\mu)y : y \in H, \mu \in M_0(G)\}$ and $\text{dis}(x, K_{H,\pi})$ is the distance of x from $K_{H,\pi}$. If G is a nontrivial compact group then the regular representation is reducible, also $P : L^2(G) \to \overline{K_{\pi,H}}$ the orthogonal projection of $L^2(G)$ onto $\overline{K_{H,\pi}}$ commutes with convolution, that is $P(\mu * f) = \mu * Pf$ for all $\mu \in M(G)$ and all f in $L^2(G)$.

§**1 Definitions and Notations**. Let G be a locally compact group and $M(G)$ be the Banach algebra of all bounded regular Borel measures on G with variation norm and convolution as multiplication and $M_0(G)$ the set of probability measures in $M(G)$ ($\mu \in M_0(G)$ is and only if $\mu \in M(G), \mu \geq 0$ and $\|\mu\| = 1$). Let $M(G)^*$ be the continuous dual of $M(G)$ and 1 the linear functional in $M(G)^*$ such that $1(\mu) = \mu(G)$, for all μ in $M(G)$. A linear functional M in $M(G)^{**}$, is called a mean on $M(G)^*$ if

$$M(1) = \|M\| = 1.$$

For each μ in $M(G)$ define $\mu \odot F : M(G)^* \to M(G)^*$ by

$$(\mu \odot F)(\nu) = F(\mu * \nu) \text{ for all } \nu \in M(G), F \in M(G)^*.$$

A mean M is called topological left invariant on $M(G)^*$ if $M(\mu \odot F) = M(F)$, for all $F \in M(G)^*$ and $\mu \in M_0(G)$. $F \odot \mu$ and topological right invariant means on $M(G)^*$ are defined in a similar way (see [5]). G is said to be topological left (right) amenable if there is a topological left (right) invariant mean on $M(G)^*$. G is called topological amenable if there is a topological invariant (both topological left and topological right invariant) mean on $M(G)^*$. It is easy to show that G is topological left amenable if and only if G is topological right amenable if and only if G is topological amenable.

Let H be a Hilbert space and $\{\pi, H\}$ be a representation of $M(G)$ (see [3] for definition of representation). It is known that π is continuous, in fact $\|\pi(\mu)\| \leq \|\mu\|$ for all $\mu \in M(G)$ (see [3]). A subspace K of H is said to be invariant under π if $\pi(\mu)K \subseteq K$ for all $\mu \in M(G)$. π is called irreducible if $\{0\}$ and H are the only closed invariant subspaces of H. We say that π is faithful if π is one-to-one. Let $H = L^2(G)$ and consider $\pi : M(G) \to B(L^2(G))$

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defined by $\pi(\mu)f = \mu * f$, where $(\mu * f)(x) = \int f(y^{-1}x)d\mu(y)$, then π is called regular representation of $M(G)$. The regular representation is faithful [3].

In this paper G denotes a locally compact topological group.

§**2 Main results**. Let $\{\pi, H\}$ be a representation of $M(G)$ and $K_{H,\pi}$ be the linear span of $\{y - \pi(\mu)y : y \in H, \mu \in M_0(G)\}$. For $x \in H$, let $\text{dis}(x, K_{H,\pi})$ denote the distance of x from $K_{H,\pi}$.

Theorem 2.1. *Let G be a topological amenable group, then*

$$Inf\{\|\pi(\mu)x\| : \mu \in M_0(G)\} = dis(x, K_{H,\pi}) \quad \text{for all} \quad x \in H.$$

Proof. Let $\{\mu_\alpha\}$ be a net in $M_0(G)$ converging to topological right invariance in norm. That is, $\|\mu_\alpha * \mu - \mu_\alpha\| \to 0$ for any $\mu \in M_0(G)$ (see [4, Theorem 3.1]). For any $y \in H$ and $\mu \in M_0(G)$ we have

$$\begin{aligned} \|\pi(\mu_\alpha)(y - \pi(\mu)y)\| &= \|\pi(\mu_\alpha)y - \pi(\mu_\alpha * \mu)y\| \\ &= \|\pi(\mu_\alpha - \mu_\alpha * \mu)y\| \\ &\leq \|\mu_\alpha - \mu_\alpha * \mu\|\|y\| \to 0 \end{aligned}$$

hence, by linearity of $\pi(\mu_\alpha)$ we conclude that $\|\pi(\mu_\alpha)z\| \to 0$ for all $z \in K_{H,\mu}$.

Now given $\epsilon > 0$ there is $z \in K_{H,\pi}$ such that $\|x + z\| < \text{dis}(x, K_{H,\pi}) + \epsilon$ and since $\|\pi(\mu_\alpha)z\| \to 0$, there is α_0 such that $\|\pi(\mu_{\alpha_0}z\| < \epsilon$. hence

$$\begin{aligned} \|\pi(\mu_{\alpha_0})x\| &\leq \|\pi(\mu_{\alpha_0})(x + z)\| + \|\pi(\mu_{\alpha_0})z\| \\ &\leq \|x + z\| + \epsilon < \text{dis}(x, K_{H,\pi}) + 2\epsilon. \end{aligned}$$

Thus $\text{Inf}\{\|\pi(\mu)x\| : \mu \in M_0(G)\} \leq \text{dis}(x, K_{H,\pi})$. Clearly $\{\pi(\mu)x : \mu \in M_0(G)\} \subseteq x + K_{H,\pi}$ for all $x \in H$, thus $\text{Inf}\{\|\pi(\mu)x\| : \mu \in M_0(G)\} \geq \text{dis}(0, x + K_{H,\pi}) = \text{dis}(x, K_{H,\pi})$ hence the result follows.

Remark. For antirepresentation of discrete semigroups, Theorem 2.1 can be found in Glicksberg [2, p.99 – 104]. The proof here is different but analogue to that in Granirer [1] who considers antirepresentation of the semigroup S instead of the measure algebra.

Definition 2.2. Let $N(H) = \{x \in H : \text{Inf}\{\|\pi(\mu)x\| : \mu \in M_0(G)\} = 0\}$. When $H = L^2(G)$ we denote $N(H)$ by $N(G)$.

Theorem 2.3. *Suppose G is a nontrivial toplogical amenable group if $\pi : M(G) \to B(H)$ is a faithful representation, then $N(H) = H$ or π is reducible.*

Proof. By Theorem 2.1 $N(H) = \bar{K}_{H,\pi}$. Now we show that $N(H)$ is an invariant subspace of H.

For each $\nu \in M(G)$ there are $\alpha_1, \alpha_2 \geq 0$ and $\nu_1, \nu_2 \in M_0(G)$ such that $\nu = \alpha_1\nu_1 - \alpha_2\nu_2$. Let $x \in H$ and $\mu \in M_0(G)$, then

$$\pi(\nu_1)(\pi(\mu)x - x) = \pi(\nu_1 * \mu)x - x + x - \pi(\nu_1)x \in K_{H,\pi},$$

and since π is linear we therefore conclude that $\pi(\nu)K_{H,\pi} \subseteq K_{H,\pi}$ for all $\nu \in M(G)$, that is $K_{H,\pi}$ is an invariant subspace of H. So by continuity of $\pi, N(H) = \bar{K}_{H,\pi}$ is also an invariant subspace of H.

Since G is nontrivial group and π is faithful $N(H) \neq \{0\}$. In fact let $\mu \in M_0(G)$ be such that $\mu \neq \epsilon_e$ (ϵ_e is the Dirac measure at the identity of G). Without loss of generality we may assume that $\pi(\epsilon_e) = I$ the identity operator of $B(H)$, (see [3]) so $\pi(\mu) \neq I$, that is there is $x \in X$ such that $\pi(\mu)x \neq x$ or $\pi(\mu)x - x \neq 0$, so $0 \neq \pi(\mu)x - x \in K_{H,\pi} \subseteq N(H)$. If $N(H) \neq H$ then $N(H)$ is a nontrivial closed invariant subspace of H, so π is reducible.

Corollary 2.4. *Let G be a nontrivial topological amenable group and $\pi : M(G) \to B(H)$ be a faithful irreducible representation of $M(G)$, then $K_{H,\pi}$ is dense in H.*

Corollary 2.5. *If G is a compact group, then the regular representation is reducible.*

Proof. If G is trivial then $M(G)$ is the linear span of ϵ_e and using the fact that $\pi(\epsilon_e) = I$ it is easy to show that any nontrivial closed subspace of H is invariant under π, so π is reducible. So without loss of generality we may assume that G is nontrivial. If we show that $N(G) \neq L^2(G)$ then by Theorem 2.3, π is reducible to end this, we notice that since G is compact $1 \in L^2(G)$, but $1 \notin N(G)$ because $\mathrm{Inf}\{\|\mu * 1\| : \mu \in M_0(G)\} = \sqrt{\lambda(G)} > 0$, where λ is a left Haar measure on G.

Corollary 2.6. *Let G be a compact group and $P : L^2(G) \to N(G)$ be the orthogonal projection of $L^2(G)$ onto $N(G)$. Then $P(\mu * f) = \mu * Pf$ for all $\mu \in M(G)$ and $f \in L^2(G)$.*

Proof. By Corollary 2.5, regular representation is reducible, hence by [3], $P \in (\pi(M(G)))'$ that is $P\pi(\mu) = \pi(\mu)P$ for all $\mu \in M(G)$. Hence

$$P\pi(\mu f) = \pi(\mu)Pf \quad \text{for all} \quad \mu \in M(G) \quad \text{and} \quad f \in L^2(G)$$
$$\text{so} \quad P(\mu * f) = \mu * Pf \quad \text{for all} \quad \mu \in M(G) \quad \text{and} \quad f \in L^2(G).$$

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