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SOME RESULTS ON MULTIPLIERS OF G^p SPACES

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Abstract. In this paper, we study some new properties of the coefficient multipliers of H^p, G^p and B^p to B^q, H^q, A^q and G^q.

1. Introduction. All the functions we consider are supposed to be analytic in D. We use the definitions and the notations in [1] . Moreover, we define Bergman spaces:

$$A^p = \{f : \|f\|_{A^p} = (\pi^{-1} \iint\limits_D |f(z)|^p dxdy)^{1/p} < \infty\}, 0 < p < \infty.$$

$f \in A^p$ if and only if

$$\int_0^1 M_p^p(r, f)dr < \infty.$$

The multipliers of G^p to H^q and B^q,of H^p, G^p and B^p to G^1, and (G^1, G^1) have been studied in [1,2]. In the paper, we give some new properties of the multipliers of H^p, G^p and B^p to B^q, H^q and A^q, of H^p and G^p to G^q.

Let $[\alpha]$ denote the maximum integer not exceeding α, C denote a positive number depending only on the indices p, q, or other parameters of the argument.

2. Multipliers of H^p, G^p and B^p

Theorem 2.1. *Suppose* $0 < p < 1$, $m = [1/p]$, $X = H^p, G^p$ *or* B^p. *Then*

(1) $(X, B^q) = \{g : M_1(r, g^{(m)}) = O(1-r)^{1/p-1/q-m}\}, 0 < q < 1.$
(2) $(X, H^q) = \{g : M_q(r, g^{(m)}) = O(1-r)^{1/p-m-1}\}, 1 \le q \le \infty.$
(3) $(X, G^1) = \{g : M_1(r, g^{(m)}) = O(1-r)^{1/p-m-1}\}.$
(4) $(X, A^q) = \{g : M_q(r, g^{(m)}) = O(1-r)^{1/p-1/q-m-1}\}, 1 \le q < \infty.$

Proof. (1), (2). See the proof of Theorem 3, 4 in [3] respectively; An argument similar to that used in the proof of Theorem 4 in [3] leads to (3), since $H^p \subset G^p \subset B^p$.

The following proves (4). Remember that

$$\{g : M_q(r, g^{(m)}) = O(1-r)^{1/p-1/q-m-1}\} = A.$$

If $f \in B^p, g \in A$, let $h = f * g$. Then

$$\begin{aligned} r^{mq} M_q^q(r^2, h^{(m)}) &\le M_q^q(r, g^{(m)}) M_1^{q-1}(r, f) M_1(r, f) \\ &\le C(1-r)^{1/p-2-mq} M_1(r, f). \end{aligned}$$

Hence

$$\int_0^1 (1-r)^{mq} M_q^q(r, h^{(m)}) dr < \infty.$$

But by successive applications of Lemma ([3, p.75]) and Theorem 5.6 ([4]), it implies $h \in A^q$. This proves that $A \subset (B^p, A^q)$.

Conversely, by the method similar to that used in the proof of Theorem 3 ([3]), we can prove that $(H^p, A^q) \subset A$.

It follows from $(B^p, A^q) \subset (G^p, A^q) \subset (H^p, A^q)$ that (4) holds.

Corollary 2.2. Suppose $0 < p < 1, 2 \leq q < \infty, \beta = 1/p - 2/q$. If $f \in X$, then its fractional integral $f_{[\beta]} \in A^q$.

Proof. Let

$$g(z) = \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+1+\beta)} z^n$$

. Then

$$g(z) = \frac{1}{\Gamma(\beta)} \int_0^1 (1-t)^{\beta-1}(1-tz)^{-1} dt.$$

Differentiation with respect to z gives

$$g^{(m)}(z) = \frac{m!}{\Gamma(\beta)} \int_0^1 t^m (1-t)^{\beta-1}(1-tz)^{-m-1} dt.$$

It follows from Minkowski's inequality that

$$\begin{aligned} M_q(r, g^{(m)}) &\leq C \int_0^1 (1-t)^{\beta-1}(1-tr)^{1/q-m-1} dt \\ &\leq C(1-r)^{1/p-1/q-m-1}. \end{aligned}$$

By Theorem 2.1 (4), this implies that $f_{[\beta]} = f * g \in A^q$.

3. Multipliers of H^p, G^p to G^q.

Theorem 3.1. *If* $0 < p \leq q \leq 1, -1 < b < \infty$,

$$\int_0^1 (1-r)^{b+p} M_q^p(r, f') dr < \infty,$$

then

$$\int_0^1 (1-r)^b M_q^p(r, f) dr < \infty.$$

Proof. Without loss of generality, assume $f(0) = 0$, so that

$$f(re^{i\theta}) = \int_0^r f'(te^{i\theta}) e^{i\theta} dt.$$

Let $t_n = (1 - 2^{-n})r, n = 0, 1, 2, \cdots$. Then

$$\begin{aligned} M_q^p(r, f) &\leq [(2\pi)^{-1} \int_0^{2\pi} (\sum_{n=1}^{\infty} \int_{t_{n-1}}^{t_n} |f'(te^{i\theta})| dt)^q d\theta]^{p/q} \\ &\leq \sum_{n=1}^{\infty} 2^{-np} r^p [(2\pi)^{-1} \int_0^{2\pi} F^q(t_n, \theta) d\theta]^{p/q} \\ &\leq C \sum_{n=1}^{\infty} 2^{-np} r^p M_q^p(t_n, f'), \end{aligned}$$

where $F(t_n,\theta)=\max_{t\le t_n}|f'(te^{i\theta})|$, Theorem 32 (2) in [5] has been used. But

$$\begin{aligned}\int_0^r (r-t)^{p-1}M_q^p(t,f')dt &\ge \sum_{n=1}^{\infty} M_q^p(t_n,f')\int_{t_n}^{t_{n+1}}(r-t)^{p-1}dt\\ &=(1-2^{-p})p^{-1}\sum_{n=1}^{\infty}2^{-np}r^pM_q^p(t_n,f'),\end{aligned}$$

thus

$$M_q^p(r,f)\le C\int_0^r (r-t)^{p-1}M_q^p(t,f')dt.$$

Hence an interchange of the order of integration shows that

$$\int_0^1 (1-r)^bM_q^p(r,f)dr\le C\int_0^1 M_q^p(t,f')\int_t^1 (1-r)^b(r-t)^{p-1}drdt.$$

Let $r=1-(1-t)s$. We have that

$$\int_t^1 (1-r)^b(r-t)^{p-1}dr=B(b+1,p)(1-t)^{b+p}.$$

Consequently,

$$\int_0^1 (1-r)^bM_q^p(r,f)dr\le C\int_0^1 (1-r)^{b+p}M_q^p(r,f')dr.$$

This easily gives the desired conclusion.

Corollary 3.2. For $0<p<1$, if $f\in G^p$, then

$$\int_0^1 (1-r)^{-p}M_1^p(r,f)dr<\infty,$$

and

$$M_1(r,f)=O(1-r)^{1-1/p}.$$

Theorem 3.3. *Suppose* $0<p\le q<1, X=H^p$ *or* $G^p, m=[1/p]$. *Then*

$$(X,G^q)=\{g:M_1(r,g^{(m)})=O(1-r)^{1/p-1/q-m}\}.$$

Proof. Remember

$$\{g:M_1(r,g^{(m)})=O(1-r)^{1/p-1/q-m}\}=B.$$

Let $f\in G^p, g\in B, h=f*g$. Then by Corollary 3.2,

$$\begin{aligned}r^{mq}M_1^q(r^2,h^{(m)}) &\le M_1^{q-p}(r,f)M_1^q(r,g^{(m)})M_1^p(r,f)\\ &\le C(1-r)^{-q(m-1)-p}M_1^p(r,f),\\ \int_0^1 (1-r)^{q(m-1)}M_1^q(r,h^{(m)})dr&<\infty.\end{aligned}$$

But by successive applications of Theorem 3.1, this implies $h \in G^q$. Hence $B \subset (G^p, G^q)$.

Conversely, since $H^p \subset G^p, G^q \subset B^q$, it follows from Theorem 2.1 (1) that

$$(G^p, G^q) \subset (H^p, G^q) \subset (H^p, B^q) = B.$$

Remark. For $0 < p < q \leq 1$, by Theorem 2.1 (3), the conclusion of Theorem 3.3 remains to be true.

Corollary 3.4. For $0 < p < 1$,

$$(G^p, G^p) = (H^p, G^p) = (H^p, B^p) = (G^p, B^p) = (B^p, B^p) = (G^1, G^1).$$

Proof. By Theorems 5.5 ([4]),

$$\{g : M_1(r, g^{(m)}) = O(1-r)^{-m}\} = \{g : M_1(r, g') = O(1-r)^{-1}\}.$$

Theorem 3.3, Theorem 2.1 (1) and Theorem 3.2 ([2]) give the desired result.

Corollary 3.5. Suppose $0 < p < q \leq 1, \beta = 1/p - 1/q$. If $f \in H^p$ or G^p, then $f_{[\beta]} \in G^q$.

Proof. See Remark and the proof of Corollary 2.2.

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