Scientiae Mathematicae Japonicae Online, **Vol. 10**, (2004), 1–11

THREE-PERSON GAMES OF ODD-MAN-WINS

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Received September 9, 2003

Abstract. By introducing a specified definition of the equilibrium value of three-person two-choice games of "odd-man-wins" and "odd-man-out" are formulated and solved and then results are applied to the sequential n-stage-game version. It is shown that, in the equilibrium play of the n-stage Odd-Man-Wins each player chooses R for small offers and randomizes R and A, for other offers, whereas in the n-stage Odd-Man-Out, each player randomizes R and A for every offer of any size, and pure-strategy triple A-A-A doesn't appear (expect at the last stage) even when players face a very large offer.

1 Statement of the Problem. Let $X_i, i = 1, 2, \cdots, n$, be *i.i.d.* random variables each with uniform distribution on $[0, 1]$. As each X_i comes up, each player I, II and III must choose simultaneously and independently of other players' choices, either to accept (A) or to reject (R) it. If all players accept the X_i, then they get $\frac{1}{3}X_i$ each, and the game terminates. If all players reject the X_i this is rejected and the next X_{i+1} is presented and the game continues. If players' choices are different, the odd-man gets the whole and the others get nothing, and the game terminates. If all of the first $n-1$ random variables are rejected, all players must accept the n-th. Each player aims to maximize the expected reward he can get, and the problem is to find a reasonable solution to this three-person n-stage game.

Let (v_n, v_n, v_n) be the eq.values for the game (*c.f.*, the game is symmetric for the players). The Optimality Equation is

$$(1.1) \qquad (v_n, v_n, v_n) = E[\text{eq.val.}\mathbf{M}_n(X)] \qquad (n \geq 1, v_1 = \frac{1}{6})$$

where the payoff matrix $\mathbf{M}_n(x)$ is such that

(1.2)

R by I:

	R by III	A by III
R by II	$v_{n-1}, v_{n-1}, v_{n-1}$	$0, \quad 0, \quad x$
A by II	$0, \quad x, \quad 0$	$x, \quad 0, \quad 0$

A by I:

	R by III	A by III
R by II	$x, \quad 0, \quad 0$	$0, \quad x, \quad 0$
A by II	$0, \quad 0, \quad x$	$x/3, \quad x/3, \quad x/3$

This game may be called odd-man-wins game. Closely-related another game is may-be, odd-man-out. The Optimality Equation is

$$(1.3) \qquad (w_n, w_n, w_n) = E[\text{eq.val.}\mathbf{M}_n(X)] \qquad (n \geq 1, w_1 = \frac{1}{6})$$

where the payoff matrix $\mathbf{M}_n(x)$ is

2000 *Mathematics Subject Classification.* 60G40, 90C39, 90D10.

Key words and phrases. Optimal stopping game, Nash equilibrium, three-person competitive game, Prisoners' Dilemma.

(1.4)

I	II	R by III			A by III		
R by I	R by II	$w_{n-1},$	$w_{n-1},$	w_{n-1}	$x/2,$	$x/2,$	0
	A by II	$x/2,$	$0,$	$x/2$	$0,$	$x/2,$	$x/2$
A by I	R by II	$0,$	$x/2,$	$x/2$	$x/2,$	$0,$	$x/2$
	A by II	$x/2,$	$x/2,$	0	$x/3,$	$x/3,$	$x/3$

The odd-man gets nothing, and the "even-men" divide the X_i (at the i-th stage) equally. Each player wants not to become the odd-man.

Each player must think about : (1) He wants to become the odd-man (an even-man), if the game is Odd-Man-Wins (Odd-Man-Out), especially when he faces a very large X_i, and (2) Since X_i is a random variable, he can expect a larger one may come up in the future.

The game(1.1)-(1.2) and (1.3)-(1.4) are solved in Sections 2 and 3, respectively. We need a specified definition of the eq.val.in these optimality equation, as in Assumption A stated in Section 2, since the equilibrium is often undetermined in Nash theory of competitive games. We have found in Section 3 that the game of odd-man-out yields a seemingly unreasonable solution, which surely comes from our Theorem 3. The result may be called a three-person game version of Prisoners' Dilemma, or may be led from the inadequacy of our Assumption A.

One of the fundamental and elaborate literature in game theory (including cooperative theory of games) is Petrosjan and Zenkevich [2]. Assumption A was first used in Ref.[5] in a multistage two-person two-choice game. There are a few mathematical literature which discuss three-person competitive games, and one of them is Vorobjev [7] and others are Sakaguchi [3, 4, 6] and Mazalov and Banin [1]. The present paper owes much on Vorobjev's work.

2 Odd-Man-Wins.

2 Odd-Man-Wins. Now we consider the game presented by (1.1)-(1.2). In order to compute the r.h.s.of (1.1), let us consider the simplified game with payoff matrix

(2.1)

R by I:

	III's R			III's A		
II's R	c,	c,	c	0,	0,	1
II's A	0,	1,	0	1,	0,	0

A by I:

	R			A		
R	1,	0,	0	0,	1,	0
A	0,	0,	1	1/3,	1/3,	1/3

where c is a given constant.

Let $V(c)$ be the CEV (common eq.value) of the game (2.1). Then since (1.1)-(1.2) results

(2.1′) $$v_n = E[xV(c)|_{c=x^{-1}v_{n-1}}],$$

we want to compute $V(c)$.

Let $\langle\bar{\alpha},\alpha\rangle$, $\langle\bar{\beta},\beta\rangle$ and $\langle\bar{\gamma},\gamma\rangle$ denote the mixed strategies by I, II and III, respectively. Also let $K_1(R,\beta,\gamma)[K_1(A,\beta,\gamma)]$ be the expected payoff to I, when I chooses R [A] and II-III employ the mixed strategies $\langle\bar{\beta},\beta\rangle$ and $\langle\bar{\gamma},\gamma\rangle$.
Then by (2.1) we have

$$\begin{aligned} K_1(R,\beta,\gamma) &= (\bar{\beta},\beta)\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}(\bar{\gamma},\gamma)^{\mathrm{T}}, \\ K_1(A,\beta,\gamma) &= (\bar{\beta},\beta)\begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}(\bar{\gamma},\gamma)^{\mathrm{T}}, \end{aligned}$$

and hence

$$D_1(\beta,\gamma) \equiv K_1(R,\beta,\gamma) - K_1(A,\beta,\gamma) = (\bar\beta,\beta)\begin{bmatrix} c-1 & 0 \\ 0 & 2/3 \end{bmatrix}(\bar\gamma,\gamma)^{\mathrm{T}} \tag{2.2}$$

$$= \begin{cases} (c-\frac{1}{3})\left[\left(\beta - \dfrac{c-1}{c-1/3}\right)\left(\gamma - \dfrac{c-1}{c-1/3}\right) - \dfrac{(2/3)(1-c)}{(c-1/3)^2}\right], & \text{if } c \neq \frac{1}{3}, \\ \dfrac{2}{3}(-\bar\beta\bar\gamma + \beta\gamma) = \dfrac{2}{3}(\beta+\gamma-1), & \text{if } c = \frac{1}{3}. \end{cases}$$

The equation $D_1(\beta,\gamma) = 0$ (if $c \neq 1/3$), is a hyperbola with the asymptotic axies $\beta = a(c)$ and $\gamma = a(c)$, where $a(c) \equiv \frac{c-1}{c-1/3}$, and it passes through the two corner points $(\beta,\gamma) = (1,0)$ and $(0,1)$ in the unit square.

The condition for I that the strategy-triple (α,β,γ) be in equilibrium is $(\alpha,\beta,\gamma) \in S_1$ where

$$S_1 = \{(0,\beta,\gamma)|D_1(\beta,\gamma) > 0\} \cup \{(1,\beta,\gamma)|D_1(\beta,\gamma) < 0\} \cup\{(\alpha,\beta,\gamma)|0 < \alpha < 1 \ \textit{and}\ D_1(\beta,\gamma) = 0\}. \tag{2.3}$$

Conditions for II and III are given by $(\alpha,\beta,\gamma) \in S_2 \cap S_3$ where S_2 and S_3 are defined similarly as for S_1 by beginning from the definitions

$$\begin{aligned} K_2(\alpha,A,\gamma) &= (\bar\alpha,\alpha)\begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}(\bar\gamma,\gamma)^{\mathrm{T}}, \\ K_3(\alpha,\beta,A) &= (\bar\alpha,\alpha)\begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}(\bar\beta,\beta)^{\mathrm{T}}, \\ D_2(\alpha,\gamma) &\equiv K_2(\alpha,R,\gamma) - K_2(\alpha,A,\gamma), \\ D_3(\alpha,\beta) &\equiv K_3(\alpha,\beta,R) - K_3(\alpha,\beta,A), \quad \text{etc.} \end{aligned}$$

By symmetry of the payoff matrix, $S_2[S_3]$ is obtained from S_1 by exchanging $\beta \to \alpha$ and $D_1 \to D_2[\gamma \to \alpha$ and $D_1 \to D_3]$. Thus combining these, the condition that (α,β,γ) be in equilibrium is $(\alpha,\beta,\gamma) \in S \equiv S_1 \cap S_2 \cap S_3$.

As is well-known in the Nash theory of competitive games, the equilibrium is often undetermined, even in three-person two-choice games, which we investigate in the present paper. So we prepare the following assumption.

Assumption A *If the equilibrium consists of some corner and/or edge and a unique inner point, then the latter is adopted for the equilibrium. If equilibrium consists of a single point, either corner or inner point, this is adopted for the equilibrium.*

We prove the following theorem.

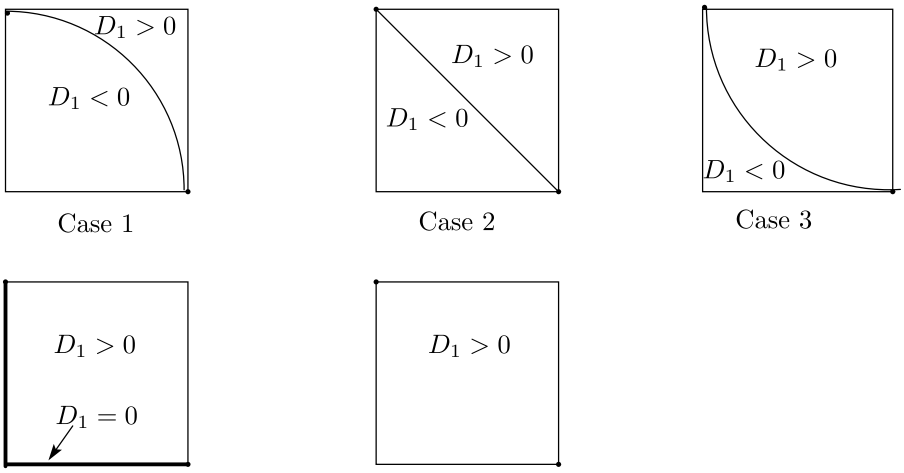
Theorem 1 *The solution to the three-person game* (2.1) *is as follows ; If $c < 1$, the mixed-strategy triple $(\alpha_0,\alpha_0,\alpha_0)$, with $\alpha_0 = \frac{\sqrt{1-c}}{\sqrt{1-c}+\sqrt{2/3}}$ is in eq. If $c \geq 1$, the pure-strategy triple* R-R-R *is in eq. The common eq.value is*

$$V(c) = \begin{cases} (1-c/3) \Big/ \left(\sqrt{1-c}+\sqrt{2/3}\right)^2, & \textit{if } c < 1, \\ c, & \textit{if } c \geq 1. \end{cases} \tag{2.4}$$

$V(c)$ is convex and increasing with values :

$c =$	$-1/3$	0	$1/3$	$1/2$	1
$V(c) =$	$5(1-(2/3)\sqrt{2}) \approx 0.2859$	$3(5-2\sqrt{6}) \approx 0.3031$	$1/3$	$5(7-4\sqrt{3}) \approx 0.3590$	1

Proof. Consider the five cases.
Case 1. $c < 1/3$ and hence $a(c) \equiv \frac{c-1}{c-1/3} > 1$:
Case 2. $c = 1/3$ and hence $D_1(\beta, \gamma) = \frac{2}{3}(\beta + \gamma - 1)$;
Case 3. $1/3 < c < 1$ and hence $a(c) < 0$;
Case 4. $c = 1$ and hence $a(c) = 0$ and $D_1(\beta, \gamma) = \frac{2}{3}\beta\gamma$;
Case 5. $c > 1$ and hence $0 < a(c) < 1$.
We find from (2.2) that the sign of $D_1(\beta, \gamma)$ on $[0, 1]^2$ is :



Case 1 Case 2 Case 3

Case 4 Case 5

¿From these figures and (2.3) we derive S_1 and then we construct $S_1 \cap S_2 \cap S_3$ in each case of $1 \sim 5$.

In Case 1, S_1 consists of the two shaded regions in $\beta - \gamma$ squre and one dotted curved surface (see Figure 1). Therefore S consists of six (not twelve) edges, and one inner point $(\alpha_0, \alpha_0, \alpha_0)$, where $\alpha_0 = \frac{\sqrt{1-c}}{\sqrt{1-c}+\sqrt{2/3}}$ is the *smaller* root of the equation $(\alpha - a(c))^2 = (2/3)(1-c)/(c-\frac{1}{3})^2$. The common eq.value $V(c)$, corresponding to the inner point (of the unit cube) solution is

$$V(c) = K_1(R, \alpha_0, \alpha_0) = K_1(A, \alpha_0, \alpha_0) = \frac{1-(1/3)c}{(\sqrt{1-c}+\sqrt{2/3})^2} \quad \left(= \frac{1}{3}, \text{ if } c = \tfrac{1}{3} - 0\right). \tag{2.5}$$

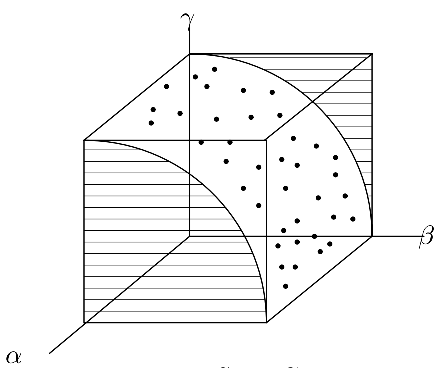


Figure 1. S_1 in Case 1.

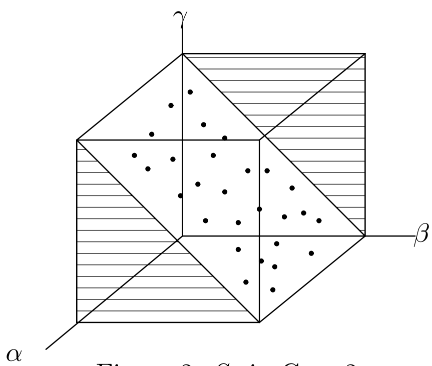


Figure 2. S_1 in Case 2.

In Case 2, S_1 is shown by Figure 2, and so S consists of six edges and one inner point $\alpha_0 = \beta = \gamma = \frac{1}{2}$. The latter gives $V(1/3) = \left(\frac{1}{2}, \frac{1}{2}\right) \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \left(\frac{1}{2}, \frac{1}{2}\right)^{\mathrm{T}} = \frac{1}{3}$.

In case 3, S_1 is shown by Figure 3, and S consists of the same six edges as in Case 1 and one inner point $(\alpha_0, \alpha_0, \alpha_0)$, where $\alpha_0 = \frac{\sqrt{1-c}}{\sqrt{1-c}+\sqrt{2/3}}$ is the *larger* root of the same equation as in Case 1. The inner point solution gives the value (2.5) again. Note that $V(1-0) = 1$.

In Case 4, $S_1 = \{(0, \beta, \gamma) | (\beta, \gamma) \in [0, 1]^2\} \cup \{(\alpha, \beta, \gamma) | 0 < \alpha \le 1, \beta\gamma = 0\}$ consists of three (not six) side sequres and no inner point. So this means $S = \{(\alpha, \beta, \gamma) | \alpha, \beta \ \ or \ \ \gamma = 0\}$.

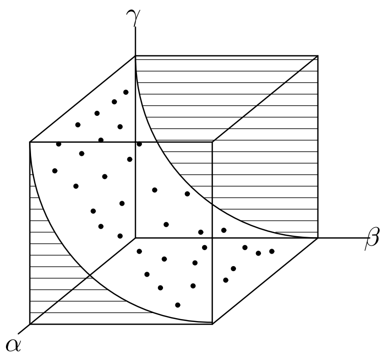


Figure 3. S_1 in Case 3.

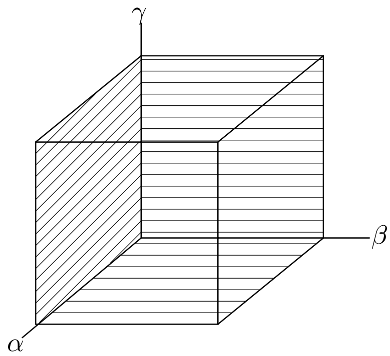


Figure 4. S_1 in Case 4.

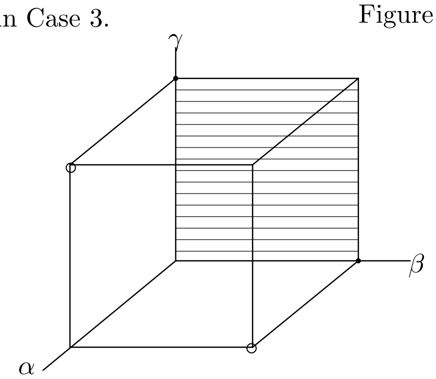


Figure 5. S_1 in Case 5.

In Case 5, $S_1 = \{(0, \beta, \gamma) | (\beta, \gamma) \in [0, 1]^2\} \cup \{(\alpha, \beta, \gamma) | 0 < \alpha < 1, (\beta, \gamma) = (1, 0) \ \ or \ \ (0, 1)\}$ is shown by Figure 5. Hence $S = S_1 \cap S_2 \cap S_3 = \{0, 0, 0\}$ *i.e.* R-R-R is the unique eq.strategy triple, and the common eq.value of the game is c.

Combining all of the above discussions and referring to our Assumption A, we can obtain the result stated in the theorem.

Proof of the rest part is made by elementary calculus. ¿From (2.4),

$$V'(c) = \frac{2 - \sqrt{(2/3)(1-c)}}{3\sqrt{1-c}\left(\sqrt{1-c} + \sqrt{2/3}\right)^2}$$

is positive, and increasing since both of $2 - \sqrt{(2/3)(1-c)}$ and $\frac{1}{\sqrt{1-c}(\sqrt{1-c}+\sqrt{2/3})^2}$ are positive and increasing. □

Note that $V'(0) = 14 - 17\sqrt{2/3} \approx 0.1196$ and $V'(1) = +\infty$.

Corollary 1.1 *$\alpha_0(c)$ is concave and decreasing for $c < 1$.* POA$\equiv$ Pr.{*odd-man appears*} $= 1 - \alpha_0^3 - \bar{\alpha}_0^3$ *is increasing for $c < \frac{1}{3}$; attains maximum at $c = \frac{1}{3}$, and decreasing for $\frac{1}{3} < c < 1$. Computation gives*

$c =$	$-1/3$	0	$1/3$	$1/2$	1
$\alpha_0(c) =$	$2 - \sqrt{2} \approx 0.5858$	$3 - \sqrt{6}) \approx 0.5505$	$1/2$	$2\sqrt{3} - 3$	0
POA=	0.7279	0.7424	$3/4$	0.7461	0

Proof. We see that $\alpha_0'(c) = -\left[\sqrt{6(1-c)}(\sqrt{1-c} + \sqrt{2/3})^2\right]^{-1}$ is negative and decreasing. Also the derivative of POA is

$$-3(\alpha_0^2 - \bar{\alpha}_0^2)\alpha_0'(c) = \frac{1-3c}{\sqrt{6(1-c)}}\left(\sqrt{1-c} + \sqrt{2/3}\right)^{-4},$$

Hence the result follows. □

Define state (n, x) to mean that the first random variable X_1 in the n-stage game turns out to be x.

Theorem 2 *Solution to the three-person n-stage odd-man-wins. The* CES *(common eq.strategy) in state (n, x) is :*

Choose R, *if $x < v_{n-1}$;*

Employ the mixed strategy $(R, A; \bar{\alpha}_0(x), \alpha_0(x))$, with

$$\alpha_0(x) = \frac{\sqrt{x - v_{n-1}}}{\sqrt{x - v_{n-1}} + \sqrt{(2/3)x}}, \quad \textit{if } x > v_{n-1}.$$

The sequence $\{v_n\}$ is determined by the recursion $v_n = T(v_{n-1})$, $(n \geq 2, v_1 = 1/6)$, where

$$T(v) = \begin{cases} v^2 + \displaystyle\int_v^1 \frac{x(x - v/3)}{(\sqrt{x-v} + \sqrt{(2/3)x})^2}dx, & \textit{if } v < 1, \\ v, & \textit{if } v > 1 \end{cases} \tag{2.6}$$

Moreover v_n, as $n \to \infty$ converges to $v_\infty \equiv \sup_v\{v | T(v') > v', \forall v' \in (0, v)\}$.

Proof. We apply Theorem 1 to the r.h.s.of (2.1′). Then

$$v_n = v_{n-1}^2 + \int_{v_{n-1}}^1 xV(v_{n-1}/x)dx, \quad \text{if } v_{n-1} < 1$$

where $V(\cdot)$ is given by (2.4), and therefore (2.6) follows. $T(v)$ is increasing, since

$$\begin{aligned} T'(v) &= v + \int_v^1 x\frac{\partial}{\partial v}\left[\frac{x - v/3}{(\sqrt{x-v} + \sqrt{2x/3})^2}\right]dx \\ &= v + \int_v^1 \frac{x\left\{2x - \sqrt{(2/3)x(x-v)}\right\}}{3\sqrt{x-v}(\sqrt{x-v} + \sqrt{2x/3})^3}dx, \end{aligned} \tag{2.7}$$

(Note that the last integral doesn't diverge, since $\int_v^1 \frac{x^2dx}{\sqrt{x-v}}$ converges to $2\sqrt{1-v} - 4\int_v^1 x\sqrt{x-v}dx$),

$$2x - \sqrt{(2/3)x(x-v)} \geq 2x - \sqrt{2/3}(x - v/2) \geq (2 - \sqrt{2/3})x \geq 0.$$

and hence $T'(v) > v > 0$. Therefore,

$$v_n > v_{n-1} \implies v_{n+1} = T(v_n) > T(v_{n-1}) = v_n$$

and

$$\begin{aligned} v_2 &= T(v_1) = T\left(\frac{1}{6}\right) = \frac{1}{36} + \int_{1/6}^{1} \frac{x(x-\frac{1}{18})}{(\sqrt{x-1/6}+\sqrt{2x/3})^2}dx \\ &\geq \frac{1}{36} + \frac{3}{10}\int_{1/6}^{1} \frac{x(x-\frac{1}{18})}{x-\frac{1}{10}}dx \geq \frac{1}{36} + \frac{3}{10}\cdot\frac{1}{2}\cdot\frac{35}{36} = \frac{25}{144} > \frac{1}{6} \end{aligned}$$

implying $v_n > v_{n-1}, \forall n$. It is clear that $v_n \leq 1$.

Since $T(v)$ is increasing with values $T(0) = \frac{3}{2}(5-2\sqrt{6}) \approx 0.1515, T'(0) = \frac{2-\sqrt{2/3}}{3(1+\sqrt{2/3})^3} \approx 0.1008$ and $T(1) = T'(1-0) = 1, \{v_n\}$ converges to some limit $\alpha \in (0, 1]$, which satisfies $T(\alpha) = \alpha$. □

Computation by computer gives the numerical values

$v =$	1/6	1/5	1/4	1/3	1/2	0.7	0.9
$T(v) =$	0.1896	0.2032	0.2273	0.2765	0.4078	0.6177	0.8717

and also gives $v_\infty \approx 0.206$. The merit for the players of multi-stage play is small, only $0.206 - 1/6 \approx 0.0394$.

3 Odd-Man-Out. We discuss in this section the game of "odd-man-out". Consider the game with payoff matrix

(3.1)

R by I:

	R by III			A by III		
R by II	h,	h,	h	1/2,	1/2,	0
A by II	1/2,	0,	1/2	0,	1/2,	1/2

A by I:

	R			A		
R	0,	1/2,	1/2	1/2,	0,	1/2
A	1/2,	1/2,	0	1/3,	1/3,	1/3

where h is a given constant.

Let $W(h)$ be the CEV of the one-stage game (3.1). Then for the n-stage game, Eq.(1.3)-(1.4) gives

$$w_n = E[xW(h)|_{h=x^{-1}w_{n-1}}]. \tag{3.1'}$$

We want to compute $W(h)$.

Similarly as in Section 2, we have

$$K_1(R,\beta,\gamma) = (\bar{\beta},\beta)\begin{bmatrix} h & 1/2 \\ 1/2 & 0 \end{bmatrix}(\bar{\gamma},\gamma)^{\mathrm{T}},$$

$$K_1(A,\beta,\gamma) = (\bar{\beta},\beta)\begin{bmatrix} 0 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}(\bar{\gamma},\gamma)^{\mathrm{T}},$$

and

$$\begin{aligned} D_1(\beta,\gamma) &= K_1(R,\beta,\gamma) - K_1(A,\beta,\gamma) = (\bar{\beta},\beta)\begin{bmatrix} h & 0 \\ 0 & -1/3 \end{bmatrix}(\bar{\gamma},\gamma)^{\mathrm{T}} \\ &= \begin{cases} (h-\frac{1}{3})\left[(\beta-a(h))(\gamma-a(h)) - \frac{h/3}{(h-1/3)^2}\right], & \text{if } h \neq \frac{1}{3}, \\ \frac{1}{3}(1-\beta-\gamma), & \text{if } h = \frac{1}{3}. \end{cases} \end{aligned} \tag{3.2}$$

where $a(h) \equiv h/(h - \frac{1}{3})$.

The equation $D_1(\beta, \gamma) = 0$ (if $h \neq 1/3$), is a hyperbola with the asymptotic axies $\beta = a(h)$ and $\gamma = a(h)$, and it passes through the two corner points $(\beta, \gamma) = (1, 0)$ and $(0, 1)$.

Under Assumption A stated in the previous section, we obtain the following result. Proof is made by the analogous way as in Theorem 1.

Theorem 3 *The solution to the three-person game* (3.1) *as follows : For any $h \leq 0$, the pure-strategy triple* A-A-A *is in eq. For any $h > 0$, the mixed-strategy triple $\langle \alpha_0, \alpha_0, \alpha_0 \rangle$, with $\alpha_0 = \frac{\sqrt{h}}{\sqrt{h}+\sqrt{1/3}}$ is in eq. The* CEV *is*

$$W(h) = \begin{cases} 1/3, & \text{if } h < 0 \\ \dfrac{\sqrt{h/3} + h/3}{\left(\sqrt{h} + \sqrt{1/3}\right)^2}, & \text{if } h > 0. \end{cases} \tag{3.3}$$

This function is increasing and convex-concave for $0 < h < 3$, attains maximum at $h = 3$, and decreasing and concave-convex for $h > 3$. The two points of inflexion are $h = \frac{1}{3}(9 \pm 4\sqrt{5})(\approx 0.018, 5.981)$. Computation gives ;

$h =$	$0+0$	$1/3$	$1/2$	1	2	3	12	∞
$W(h) =$	0	$1/3$	$3\sqrt{6} - 7 \approx 0.3485$	$\frac{1}{2}(\sqrt{3} - 1) \approx 0.3660$	0.3739	$3/8$	$18/49$	$1/3$

Proof. Consider the four cases.

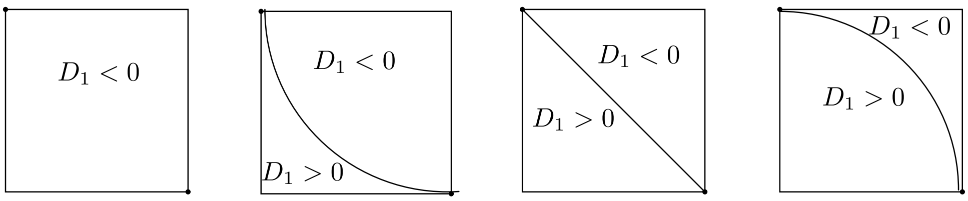
Case 1^+. $h \leq 0$, and hence $0 \leq a(h) < 1$.

Case 2^+. $0 < h < 1/3$, and hence $a(h) < 0$.

Case 3^+. $h = 1/3$, and hence $D_1(\beta, \gamma) = \frac{1}{3}(1 - \beta - \gamma)$.

Case 4^+. $h > 1/3$, and hence $a(h) > 1$.

The sign of $D_1(\beta, \gamma)$ in $(\beta, \gamma) \in [0, 1]^2$ is :



Case 1^+ Case 2^+ Case 3^+ Case 4^+

In Case 1^+, $S_1 = \{(\alpha, \beta, \gamma) | 0 < \alpha < 1, (\beta, \gamma) = (1, 0)$ *or* $(0, 1)\} \cup \{(1, \beta, \gamma) | (\beta, \gamma) \in [0, 1]^2\}$, as in shown by Figure 6. Hence $S = S_1 \cap S_2 \cap S_3 = \{(1, 1, 1)\}$ is the unique pure-strategy eq. The CEV is $\frac{1}{3}$.

In Case 2^+, S_1 consists of the two shaded regions in $\beta - \gamma$ square and one dotted curved surface (see Figure 7). Therefore S consists of the two corners $(0, 0, 0)$ and $(1, 1, 1)$ and one inner point $(\alpha_0, \alpha_0, \alpha_0)$ on the center line $\alpha = \beta = \gamma$, where $\alpha_0 = \frac{\sqrt{h}}{\sqrt{h}+\sqrt{1/3}}$ is the *larger* root of the equation $(\alpha - a(h))^2 = \frac{h/3}{(h-1/3)^2}$. The CEV corresponding to the inner point solution is

$$W(h) = K_1(R, \alpha_0, \alpha_0) = K_1(A, \alpha_0, \alpha_0) = \frac{\sqrt{h/3} + h/3}{(\sqrt{h} + \sqrt{1/3})^2} (= 1/3, \text{if } h = 1/3). \tag{3.4}$$

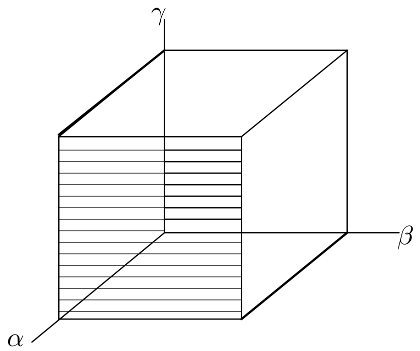


Figure 6. S_1 in Case 1^+.

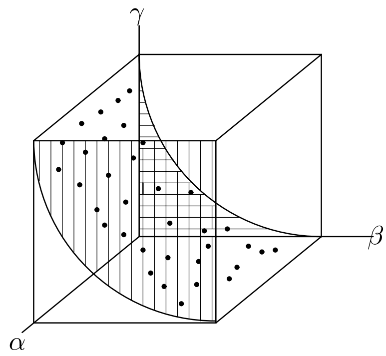


Figure 7. S_1 in Case 2^+.

In Case 3^+, S_1 is shown by Figure 8, and S consists of the two corners $(0,0,0)$ and $(1,1,1)$ and the center $\alpha=\beta=\gamma=1/2$ of the cube. The latter gives

$$W(1/3) = (1/2,1/2)\begin{bmatrix} 1/3 & 1/2 \\ 1/2 & 0 \end{bmatrix}(1/2,1/2)^{\mathrm{T}} = \frac{1}{3}.$$

In Case 4^+, S_1 is as shown by Figure 9, and S consists of the two corners $(0,0,0)$ and $(1,1,1)$ and the inner point $(\alpha_0,\alpha_0,\alpha_0)$, where $\alpha_0 = \frac{\sqrt{h}}{\sqrt{h}+\sqrt{1/3}}$ is the *smaller* root of the same equation as in Case 2^+. The inner point solution give the value (3.4) again.

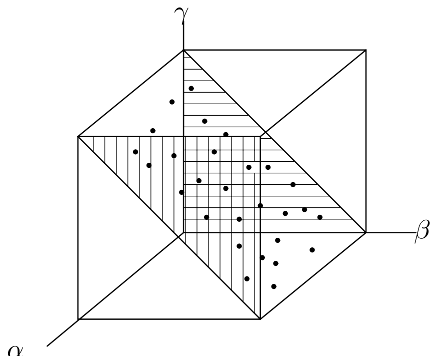


Figure 8. S_1 in Case 3^+.

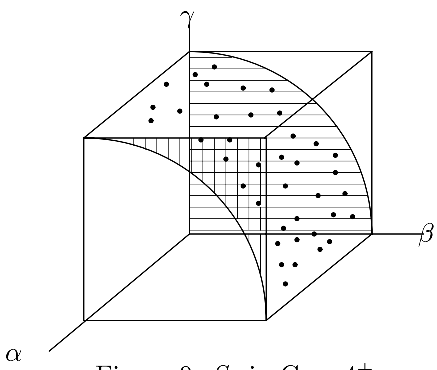


Figure 9. S_1 in Case 4^+.

Combining all of these arguments and referring to our Assumption A we obtain the result stated in the first half of the theorem.

The rest part follows from

$$W'(h) = \frac{\sqrt{1/h}-\sqrt{1/3}}{6(\sqrt{h}+\sqrt{1/3})^3} \quad \textit{and} \quad W''(h) = \frac{3h-4\sqrt{3h}-1}{12\sqrt{3}h^{3/2}(\sqrt{h}+\sqrt{1/3})^4}. \quad \square$$

Corollary 3.1 $\alpha_0(h) \equiv \frac{\sqrt{h}}{\sqrt{h}+\sqrt{1/3}}$ *is concave and increasing for* $h>0$. POA≡ Pr.{*odd-man appears*} $= 1-\alpha_0^3-\bar{\alpha}_0^3$ *is increasing for* $0<h<\frac{1}{3}$*, attains maximum at* $h=\frac{1}{3}$*, and is decreasing for* $h>\frac{1}{3}$*. Computation gives :*

$h=$	0+0	1/3	1	3	4	∞
$\alpha_0(h)=$	0	1/2	0.6340	3/4	0.7760	$1-0$
POA=	0	3/4	0.6962	$9/16\approx 0.5625$	0.5215	0+0

Proof. We have

$$\alpha_0'(h) = \frac{1}{2\sqrt{3}}\left[\sqrt{h}(\sqrt{h}+\sqrt{1/3})^2\right]^{-1},$$

and the derivative of POA is

$$-3(\alpha_0^2 - \bar{\alpha}_0^2)\alpha_0'(h) = \frac{1-3h}{2\sqrt{3h}}(\sqrt{h}+\sqrt{1/3})^{-4}.$$

Hence the result follows □

It is a surprising result that $W(h)$ is decreasing, when the reward h of coincidence of players' choices of R becomes larger beyond some amount (=3, in Theorem 3). This situation may be called a three-person Prisoners' Dilemma. Each player is involved in a loss due to his suspicion that his rivals may forestall him, although they can surely get the highest amount h, if they are admitted to cooperate by choosing R.

Theorem 4 *The solution to the three-person n-stage odd-man-out is : The* CEV *in state (n, x) is to employ the mixed strategy*

$$(R, A; \bar{\alpha}_0(x), \alpha_0(x)), \text{ with } \alpha_0(x) = \sqrt{w_{n-1}}/(\sqrt{w_{n-1}}+\sqrt{x/3}).$$

The sequence $\{w_n\}$ is determined by the recursion $w_n = U(w_{n-1})$ $(n \geq 2, w_1 = 1/6)$, where

$$U(w) = \int_0^1 \frac{\sqrt{wx/3}+w/3}{(\sqrt{w/x}+\sqrt{1/3})^2}dx. \tag{3.5}$$

As $n \to \infty, w_n$ converges to $w_\infty = \inf\left\{w \in (0, \frac{1}{6}) | U(w') < w', \forall w' \in (w, \frac{1}{6})\right\}$.

Proof. We apply Theorem 3 to the r.h.s.of (3.1$'$). Then

$$w_n = \int_0^1 xW(w_{n-1}/x)dx,$$

where $W(t)$ is given by (3.3), and therefore $w_n = U(w_{n-1})$, with

$$U(w) = \int_0^1 x\frac{\sqrt{w/(3x)}+w/(3x)}{(\sqrt{w/x}+\sqrt{1/3})^2}dx.$$

This is (3.5). The derivative is

$$U'(w) = \frac{1}{6}\int_0^1 \frac{\sqrt{x/w}-\sqrt{1/3}}{(\sqrt{w/x}+\sqrt{1/3})^3}dx. \tag{3.6}$$

A comutation by computer shows that $U(w)$ is concave and increasing for $0 \leq w \leq 1$, with values $U(0) = 0, U'(0+0) = +\infty$ and

$w =$	0.03	0.06	0.1	0.14	1/6	1/3	1/2	3/4	1
$U(w) =$	0.113	0.1341	0.1486	0.1571	0.1612	0.1745	0.1799	0.1837	0.1854

Also it gives $w_\infty \approx 0.1603$ as a unique root of the equation $U(w) = w$. Acutually, each player suffers a demerit which glows larger as n becomes larger, such that $\frac{1}{6} - w_\infty \approx 0.0064$ (So, players coordinate, if admitted, to choose A-A-A in the first stage, expecting to get 1/6 each). □

4 Remarks.

Remark 1 Theorems 2 and 4 show the following facts. (1) In the eq.play of the n-stage Odd-Man-Wins, each player chooses R for small offers and randomizes R and A for other offers. (2) In the eq.play of the n-stage Odd-Man-Out each player randomizes R and A for offers of any size, and the pure-strategy triple A-A-A (R-R-R) doesn't appear expect at the last stage, even when players face a very large (small) offer.

Remark 2 Two functions $T(v)$ in Theorem 2, and $U(w)$ in Theorem 4 are roughly shown by Figure 10. An important difference is that $T(\frac{1}{6}) > \frac{1}{6}$, and $U(\frac{1}{6}) < \frac{1}{6}$.

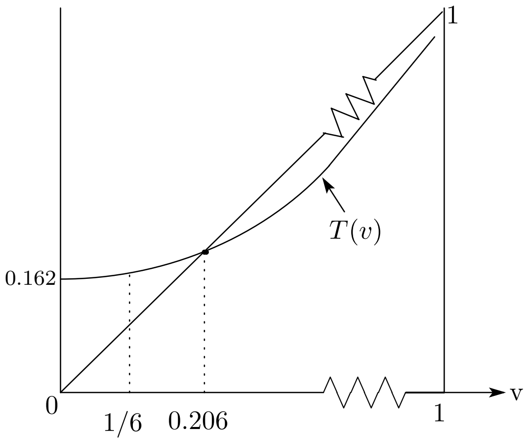


Figure 10 a.

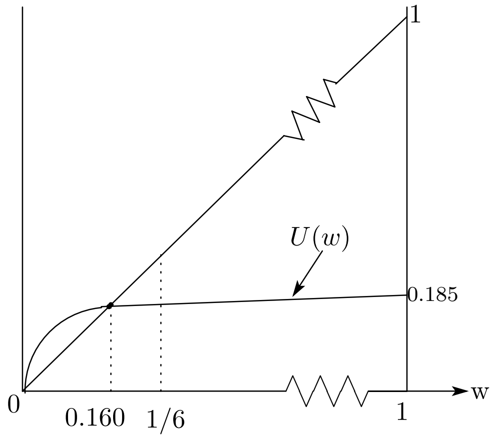


Figure 10 b.

Multistage play yields each player a merit of size $v_\infty - \frac{1}{6} \approx 0.0394$ in Odd-Man-Wins, and a demerit of size $\frac{1}{6} - w_\infty \approx 0.0064$ in Odd-Man-Out.

Acknowledgement The author thanks Atsuko Takano (Dept.of Information Sci., Hyogo University, Hyogo) for computing the numerical values of $T(v)$ in Theorem 2, and $U(w)$ in Theorem 4.

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