Operator Valued Determinant and Hadamard Product

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Abstract. On the lines of Fuglede-Kadison's determinant and ours, we define an operator valued one for positive invertible operators on a Hilbert space: For a unital positive linear map Φ, put $\Delta_\Phi(A) = \exp \Phi(\log A)$. Then we show a parametrized estimation $\Phi(A^t)^{1/t} \geq \Delta_\Phi(A) \equiv \lim_{t\to 0} \Phi(A^t)^{1/t} \geq \Phi(A^{-t})^{-1/t}$. Based on this, we show a reverse Oppenheim inequality and Ando's product formula for Hadamard products.

1 Introduction.

Fuglede-Kadison [4, 5] and Arveson [3] introduced the normalized determinant for invertible operators A in II_1 factors with the canonical trace τ:

$$\Delta_\tau(A) = \exp \tau(\log |A|).$$

Following this, we discussed the (normalized) determinant Δ_φ for positive invertible operators A on a Hilbert space and a fixed (vector) state φ defined by

$$\Delta_\varphi(A) = \exp \varphi(\log A)$$

as a continuous geometric mean in the previous paper [8, 9]. In fact, if A is a positive definite matrix with the eigenvalues $\{t_k\}$ and the corresponding unit eigenvectors $\{e_k\}$, then

$$\Delta_\varphi(A) = \prod_k t_k^{\varphi(e_k)}.$$

¿From now on, all operators are positive and invertible. In this sense, the inequality

$$\varphi(A) \geq \Delta_\varphi(A) \geq \varphi(A^{-1})^{-1}$$

is nothing but the arithmetic-geometric-harmonic mean inequality.

Along this line, we consider an operator valued determinant Δ_Φ defined by

$$\Delta_\Phi(A) = \exp \Phi(\log A)$$

where Φ is a unital positive linear map. One of the important examples is a *conditional expectation* $\mathbf{E}$ from a unital C*-algebra to its subalgebra in the sense of Umegaki [16]. In particular, $\mathbf{E}$ satisfies the *module property* :

$$\mathbf{E}(A\mathbf{E}(B)) = \mathbf{E}(A)\mathbf{E}(B) = \mathbf{E}(\mathbf{E}(A)B).$$

In this note, we show a parametrized estimation and a convergence theorem for this operator valued determinant. Our main interest is to find properties of Δ_Φ when the range of Φ is commutative (It is well-known that Φ is completely positive in this case). Since the determinant for conditional expectations to the diagonal subalgebra is equal to the Hadamard product for the identity, we show inequalities and convergence theorems related to Hadamard products, which is a viewpoint of Ando [1].

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2 Properties of Determinant. First the following properties are shown immediately by definition similarly to the results in [8]:

Lemma 1. *The determinant Δ_Φ for a unital positive linear map Φ has the following properties:*

(i) **continuity:** *The map $A \mapsto \Delta_\Phi(A)$ is norm continuous.*

(ii) **bounds:** $\|A^{-1}\|^{-1} \le \Delta_\Phi(A) \le \|A\|$.

(iii) **power equality** $\Delta_\Phi(A^t) = \Delta_\Phi(A)^t$ *for all real numbers t.*

(iv) **homogeneity:** $\Delta_\Phi(tA) = t\Delta_\Phi(A)$ *and* $\Delta_\Phi(t) = t$ *for all positive numbers t.*

Moreover if the range of Φ is commutative, then:

(vi) **monotonicity:** $A \le B$ *implies* $\Delta_\Phi(A) \le \Delta_\Phi(B)$.

(vii) **multiplicativity:** $\Delta_\Phi(AB) = \Delta_\Phi(A)\Delta_\Phi(B)$ *for commuting A and B.*

In a similar way to [8], we have Ky Fan's inequality:

Theorem 2. *If the range of Φ is commutative, then*

$$\Delta_\Phi((1-\alpha)A + \alpha B) \ge \Delta_\Phi(A)^{1-\alpha}\Delta_\Phi(B)^\alpha$$

for $0 < \alpha < 1$.

Proof. Since log is operator concave, we have

$$\Phi\left(\log\left((1-\alpha)A + \alpha B\right)\right) \ge \Phi\left((1-\alpha)\log A + \alpha \log B\right) = (1-\alpha)\Phi(\log A) + \alpha\Phi(\log B),$$

and hence the required inequality yields by the commutativity of the range of Φ. □

Let $C^*(X)$ be the unital C*-algebra generated by X. Then an arithmetic-geometric-harmonic mean inequality, which is a precise one for (ii), also holds:

Theorem 3. *If the unital algebra $\Phi(C^*(A))$ is commutative, then*

$$\Phi(A) \ge \Delta_\Phi(A) \ge \Phi(A^{-1})^{-1}.$$

Proof. By Jensen's inequality for Φ (e.g. [7]), we have

$$\log\Phi(A) \ge \Phi(\log A) = -\Phi(\log A^{-1}) \ge -\log\Phi(A^{-1}),$$

so that the commutativity implies

$$\Phi(A) = \exp(\log\Phi(A)) \ge \Delta_\Phi(A) \ge \exp(-\log\Phi(A^{-1})) = \Phi(A^{-1})^{-1}.$$

□

The *Specht ratio* $S(h)$ was defined by ([15])

$$S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e\log h}.$$

Though it is considered only for $h > 1$ originally, it is known that $S(1) \equiv \lim_{h\to 1} S(h) = 1$ and $S(1/h) = S(h)$. Note the important fact $\lim_{t\to 0} S(h^t)^{1/t} = 1$ shown by Yamazaki-Yanagida [17]. To obtain a reverse inequality of the above theorem, we need a lemma:

Lemma 4. *If Φ is a unital positive linear map and $h = M/m$ for $0 < m \leq A \leq M$, then*

$$S(h)\Delta_\Phi(A) \equiv S(h)\exp\Phi(\log A) \geq \Phi(A).$$

Proof. Though the proof is quite similar to [9, Lemma](also see [11]), we give a proof for the sake of completeness. It follows that $e^t \leq at + b \leq ae^{\frac{b-a}{a}}e^t$ for $t \in [\log m, \log M]$ where

$$a = \frac{M - m}{\log M - \log m} \quad \text{and} \quad b = \frac{m\log M - M\log m}{\log M - \log m}.$$

Then putting $S = \log A$, we have

$$\Phi(\exp S) \leq \Phi(aS + b) = a\Phi(S) + bI \leq ae^{\frac{b-a}{a}}\exp\Phi(S).$$

Hence we have $\Phi(A) \leq ae^{\frac{b-a}{a}}\exp\Phi(\log A)$ and it is shown in [9] that the number $ae^{(b-a)/a}$ is exactly the Specht ratio $S(h)$. □

So we have a reverse inequality as we showed in [9]. Note that the commutativity of the range of Φ is not needed here:

Theorem 5. *If $0 < m \leq A \leq M$ for positive numbers m and M, then*

$$S(h^t)^{-1}\Phi(A^t) \leq \Delta_\Phi(A^t) \leq S(h^t)\Phi(A^{-t})^{-1}$$

for all real numbers t.

Proof. It suffices to show that

$$S(h)^{-1}\Phi(A) \leq \Delta_\Phi(A) \leq S(h)\Phi(A^{-1})^{-1}.$$

The above lemma shows the former inequality. The latter is obtained by

$$\Delta_\Phi(A) = \Delta_\Phi(A^{-1})^{-1} \leq S(1/h)\Phi(A^{-1})^{-1} = S(h)\Phi(A^{-1})^{-1}.$$

□

Recall that the *chaotic order* $A \gg B$ holds if $\log A \geq \log B$. If the range of Φ is commutative, then $\Delta_\Phi(A) \geq \Delta_\Phi(B)$. Moreover this inequality characterizes the chaotic order (In fact, it holds also even if the maps Φ are restricted to vector states.):

Theorem 6. *If the range of Φ is commutative, then $A \gg B$ holds if and only if $\Delta_\Phi(A) \geq \Delta_\Phi(B)$ for all conditional expectations to commutative subalgebras.*

As an important convergence theorem, we show the following one:

Theorem 7. *If $\Phi(C^*(A))$ is commutative, then $\Phi(A^t)^{1/t}$ converges decreasingly (resp. increasingly) to $\Delta_\Phi(A)$ as $t \downarrow 0$ (resp. $t \uparrow 0$).*

Proof. It suffices to show the case $t > 0$. Suppose $t > s > 0$. Then Jensen's inequality

$$\Phi(A^\alpha) \leq \Phi(A)^\alpha \quad \text{for} \quad 0 < \alpha < 1$$

and the commutativity imply the monotonicity:

$$\Phi(A^t)^{1/t} = \left(\Phi(A^t)^{s/t}\right)^{1/s} \geq \Phi(A^s)^{1/s}.$$

By Theorem 3 and Lemma 1 (iii), we have

$$\Phi(A^t)^{1/t} \geq \Delta_\Phi(A^t)^{1/t} = \Delta_\Phi(A)$$

and hence $\lim_{t\to 0} \Phi(A^t)^{1/t} \geq \Delta_\Phi(A)$. On the other hand, Theorem 5 implies

$$\Delta_\Phi(A) = \Delta_\Phi(A^t)^{1/t} \geq S(h^t)^{-1/t}\Phi(A^t)^{1/t} \longrightarrow 1 \times \lim_{t\to 0} \Phi(A^t)^{1/t},$$

so that the required formula holds. □

3 Determinant for conditional expectation. The noncommutative probability or information theory is initiated in [16] (see also [12]), precisely, the conditional expectation $\mathbf{E}$ for operator algebras was introduced. Usually the operator algebras in discourse are von Neumann ones with the canonical trace τ and the conditional expectation $\mathbf{E}$ is assumed τ-invariant. From Fuglede-Kadison's viewpoint, $\Delta_{\mathbf{E}}$ is a natural extension of their determinant Δ_τ.

But technically there are few difference between Δ_Φ and $\Delta_{\mathbf{E}}$ except the following property for example:

idempotence : $\Delta_{\mathbf{E}}(D) = D$ for all positive invertible operators D in the range of $\mathbf{E}$.

The following theorem suggests us that the range of a conditional expectation should be assumed commutative in our situation:

Theorem 8. *If $\mathbf{E}(\mathcal{A})$ is a maximal abelian $*$-subalgebra of von Neumann algebra $\mathcal{A}$ and W is unitary such that $W^*\mathbf{E}(\mathcal{A})W = \mathbf{E}(\mathcal{A})$, then*

$$W^*\Delta_{\mathbf{E}}(A)W = \Delta_{\mathbf{E}}(W^*AW).$$

Proof. It follows from $W^*\mathbf{E}(A)W = \mathbf{E}(W^*AW)$. □

Remark 1. Note that $W^*\Delta_{\mathbf{E}}(A)W = \Delta_{\mathbf{E}}(W^*AW)$ is generally false for unitary operators $W \in \mathcal{A}$. In fact, define an expectation $\mathbf{E} : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ by

$$\mathbf{E}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}.$$

Put

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad W = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then it follows that $W^*\Delta_{\mathbf{E}}(A)W = \Delta_E(A) = \sqrt{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. On the other hand, we have $W^*AW = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore we have $\Delta_{\mathbf{E}}(W^*AW) \neq W^*\Delta_{\mathbf{E}}(A)W$.

Thus the condition that the range of $\mathbf{E}$ is commutative is important. In particular, the diagonalization, that is, the conditional expectation to the diagonal algebra is closely related to Hadamard product: ¿From now on, we assume that all the operators act on a separable Hilbert space H. Fix a complete orthonormal system (shortly, CONS) $\{e_k\}$. Then the *diagonalization* $\mathbf{D}$ is the conditional expectation to the diagonal subalgebra

$$\left\{ \sum_k t_k e_k \otimes \overline{e_k} \mid \{t_k\} \text{ is bounded} \right\},$$

that is, the pintching map defined by

$$\mathbf{D}(X) = \sum_k (e_k \otimes \overline{e_k}) X (e_k \otimes \overline{e_k}).$$

On the other hand, the Hadamard product $A * B$ with respect to this CONS is defined by

$$(*) \qquad A * B = U(A \otimes B)U^* \qquad \text{for} \quad U = \sum_k (e_k \otimes e_k) \otimes \overline{e_k}$$

(see [13, 6]). Then we have $\mathbf{D}(A) = A * 1_H$ and

$$\Delta_{\mathbf{D}}(A) = \exp\left((\log A) * 1\right).$$

Note that $\mathbf{D}(A * B) = \mathbf{D}(A)\mathbf{D}(B)$ for all operators A and B. So we have the following convergence theorem:

Theorem 9. *If* $\mathbf{D}$ *is a diagonalization, then*

$$\lim_{t \to 0} (A^t * B^t * 1)^{1/t} = \Delta_{\mathbf{D}}(A)\Delta_{\mathbf{D}}(B).$$

Proof. By Theorem 7, we have

$$(A^t * B^t * 1)^{1/t} = (A^t * 1)^{1/t}(B^t * 1)^{1/t} = \mathbf{D}(A^t)^{1/t}\mathbf{D}(B^t)^{1/t} \longrightarrow \Delta_{\mathbf{D}}(A)\Delta_{\mathbf{D}}(B)$$

as $t \longrightarrow 0$. □

Remark 2. The above formula can be shown also by l'Hospital's theorem. In fact,

$$\begin{aligned}\lim_{t \to 0} \frac{\log(A^t * B^t * 1)}{t} &= \lim_{t \to 0} \frac{d(A^t * B^t * 1)}{dt}(A^t * B^t * 1)^{-1} \\ &= \lim_{t \to 0} A^t \log A * B^t * 1 + A^t * B^t \log B * 1 \\ &= \log A * 1 * 1 + 1 * \log B * 1 = (\log A + \log B) * 1,\end{aligned}$$

so that we have the above formula by the continuity of the exponential map.

Recall that the *generalized Kantorovich constant* $K(h,p)$ is defined by

$$K(h,p) = \frac{h^p - h}{(h-1)(p-1)} \left(\frac{(p-1)(h^p-1)}{(h^p-h)p} \right)^p$$

for all real p, which is an important constant for operator inequality, see [10]. Note that Yamazaki-Yanagida [17] showed that $\lim_{p \to 0} K(h,p)^{-1/p} = S(h)$. Then we have

Lemma 10. *If* $0 < m \le A, B \le M$ *for scalars* m *and* M*, then*

$$\mathbf{D}((A * B)^p)^{1/p} \le K(h,p)^{-1/p}\mathbf{D}(A^p)^{1/p}\mathbf{D}(B^p)^{1/p}$$

for all $0 < p < 1$ *and the constant* $h = M/m$.

Proof. By $K(h,p)(A * B)^p \le A^p * B^p$ as in [14], we have

$$K(h,p)\mathbf{D}((A * B)^p) \le \mathbf{D}(A^p * B^p) = \mathbf{D}(A^p)\mathbf{D}(B^p),$$

in the diagonal algebra and hence the required inequality holds. □

In [1, Theorem 18], Ando formulated Oppenheim's inequality

$$\log(A * B) \geq (\log A + \log B) * 1,$$

which implies

$$\Delta_{\mathbf{D}}(A * B) \geq \Delta_{\mathbf{D}}(A)\Delta_{\mathbf{D}}(B).$$

Now taking $p \longrightarrow 0$ in the above lemma, we have the reverse inequality by Theorem 5:

Theorem 11. *If $h = M/m$ for $0 < m \leq A, B \leq M$, then*

$$\Delta_{\mathbf{D}}(A * B) \leq S(h)\Delta_{\mathbf{D}}(A)\Delta_{\mathbf{D}}(B).$$

Theorem 9 is a weaker version of the following Ando's product formula [2] which is rewritten in terms of determinant:

Theorem 12 (Ando). *If $\mathbf{D}$ is a diagonalization, then*

$$\lim_{t\to 0}(A^t * B^t)^{1/t} = \Delta_{\mathbf{D}}(A)\Delta_{\mathbf{D}}(B).$$

Proof. We have only to show under the assumption t is sufficiently small positive number. Let $0 < m \leq A, B \leq M$ and $h = M/m$. Combining the fact that the map $\Phi : X \mapsto U^*XU$ in $(*)$ is a unital positive linear map and the inequality

$$-\log S(h^t) + \log \Phi((A \otimes B)^t) \leq \Phi(\log(A \otimes B)^t)$$

by Theorem 5, we have

$$\begin{aligned}
\log(A^t * B^t) &= \log U^*(A \otimes B)^t U \leq \log S(h^t) + U^*(\log(A \otimes B)^t)U \\
&\leq \log S(h^t) + tU^*\left(\log(A \otimes 1) + \log(1 \otimes B)\right)U \\
&= \log S(h^t) + tU^*\left(\log A \otimes 1 + 1 \otimes \log B\right)U \\
&= \log S(h^t) + t(\log A * 1 + 1 * \log B) = \log S(h^t) + t(\log A + \log B) * 1
\end{aligned}$$

On the other hand, the above Oppenheim's inequality implies

$$\frac{\log(A^t * B^t)}{t} \geq \frac{(\log A^t + \log B^t) * 1}{t} = (\log A + \log B) * 1.$$

It follows that

$$(\log A + \log B) * 1 \leq \frac{\log(A^t * B^t)}{t} \leq (\log A + \log B) * 1 + \log S(h^t)^{1/t}$$

and hence $\log(A^t * B^t)^{1/t} \longrightarrow (\log A + \log B) * 1 = \mathbf{D}(\log A) + \mathbf{D}(\log B)$. Therefore

$$(A^t * B^t)^{1/t} \longrightarrow \exp(\mathbf{D}(\log A) + \mathbf{D}(\log B)) = \Delta_{\mathbf{D}}(A)\Delta_{\mathbf{D}}(B).$$

□

Remark 3. The above theorem can be shown by using the Taylor expansion, which is a natural proof and mentioned in [2]. In fact, since we may assume $\|A^t * B^t - 1\|$ is sufficiently small, then we have

$$\begin{aligned}\lim_{t\to 0}\frac{\log(A^t * B^t)}{t} &= \lim_{t\to 0}\frac{\log(1+A^t * B^t - 1)}{t} = \lim_{t\to 0}\frac{\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}(A^t * B^t - 1)^k}{t} \\ &= \lim_{t\to 0}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}t^{k-1}}{k}\left(\frac{A^t * B^t - 1}{t}\right)^k \\ &= \lim_{t\to 0}\frac{A^t * B^t - 1}{t} + \lim_{t\to 0}\ t\times\sum_{k=2}^{\infty}\frac{(-1)^{k+1}t^{k-2}}{k}\left(\frac{A^t * B^t - 1}{t}\right)^k \\ &= \log A * 1 + 1 * \log B + 0 = (\log A + \log B) * 1.\end{aligned}$$

(Note that the convergence radius of the power series $\displaystyle\sum_{k=2}^{\infty}\frac{(-1)^{k+1}t^{k-2}}{k}z^k$ is equal to $1/|t|$, and hence

$$\sum_{k=2}^{\infty}\frac{(-1)^{k+1}t^{k-2}}{k}\left(\frac{A^t * B^t - 1}{t}\right)^k$$

converges in norm.) It follows that

$$\lim_{t\to 0}(A^t * B^t)^{1/t} = \exp(\log A + \log B) = \Delta_D(A)\Delta_D(B).$$

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