A CHARACTERIZATION OF FULLY BOUNDED DUBROVIN VALUATION RINGS

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Received October 10, 2003

Abstract. Let R be a Dubrovin valuation ring. It is shown that R is fully bounded iff for any prime ideal P of R which is different from the Jacobson radical of R, P is Goldie prime and either it is lower limit or there is a Goldie prime ideal P_1 such that the prime segment $P_1 \supset P$ is Archimedean.

1. Introduction. A ring is called *right bounded* if any essential right ideal contains a non-zero (two-sided) ideal. Similarly, we can define a *left bounded ring.* A ring is just called *bounded* if it is both right bounded and left bounded.

Let S be a ring. We say that S is *fully bounded* if S/P is bounded for any prime ideal P of S. We write $J(S)$ for the Jacobson radical of S and $\mathrm{Spec}(S)$ for the set of all prime ideals of S.

Let R be a Dubrovin valuation ring of a simple Artinian ring Q (see [7, chap. II] for the definition and elementary properties of Dubrovin valuation rings). A prime ideal P of R is called *Goldie prime* if R/P is a prime Goldie ring.

We denote by G-Spec(R) the set of all Goldie prime ideals of R. Now let P_1, $P \in$ G-Spec(R) with $P_1 \supset P$. The pair $P_1 \supset P$ is called a *prime segment* if there are no Goldie primes properly between P_1 and P.

Let $P \in$ G-Spec(R) with $P \neq J(R)$ and set $P_1 = \cap \{P_\lambda \mid P_\lambda \in$ G-Spec(R) with $P_\lambda \supset P\}$. Then, in [2], they have shown that the following four cases only occur:

(1) P is lower limit, i.e., $P = P_1$. Otherwise, $P_1 \supset P$ is a prime segment.

(2) $P_1 \supset P$ is Archimedean ([1, Theorem 6(a)]).

(3) $P_1 \supset P$ is simple ([1, Theorem 6(b)]).

(4) $P_1 \supset P$ is exceptional, i.e., there exists a non-Goldie prime ideal C such that $P_1 \supset C \supset P$ ([1, Theorem 6(c)]).

With this classification, we shall prove that R is fully bounded iff (1) and (2) only hold (Theorem 2.5). (Note that $R/J(R)$ is bounded, because it is a simple Artinian ring). For any regular element c in $J(R)$, we define $P(c) = \cap\{P_\lambda \mid P_\lambda \in$ G-Spec(R) with $c \in P_\lambda\}$, a Goldie prime ideal ([1, Proposition 1]). R is called *locally invariant* if $cP(c) = P(c)c$ for any regular element c in $J(R)$. This concept was defined by Gräter [5] in order to study the approximation theorem in the case where R is a total valuation ring. We shall show that R is fully bounded if and only if it is locally invariant, by using Theorem 2.5 (Proposition 2.6).

If Q is of finite dimensional over its center, then R is always fully bounded. In the end of the paper, we shall give several examples of fully bounded Dubrovin valuation rings of Q with infinite dimension over the center.

2000 *Mathematics Subject Classification.* 16W60.

Key words and phrases. fully bounded, Dubrovin valuation ring, locally invariant.

2. Fully bounded Dubrovin valuation rings.

Throughout this section, R will denote a Dubrovin valuation ring of a simple Artinian ring Q. For any $P \in$ Spec (R), set $C(P) = \{c \in R \mid c$ is regular mod $P\}$. If $P \in$G-Spec(R), then $C(P)$ is localizable and we denote by R_P the localization of R at P. Before starting the lemmas, we note the following: There is a one-to-one correspondence between G-Spec(R) and the set of all overrings of R, which is given by $P \longrightarrow R_P$ with $P = J(R_P)$ and $S \longrightarrow J(S)$ ($P \in$ G-Spec(R) and S is an overring of R). Furthermore, for any P, $P_1 \in$ G-Spec(R), $P \supset P_1$ iff $R_P \subset R_{P_1}$ ([7, §6] and [1, §2]). We will freely use these properties throughout the paper.

Lemma 2.1. *Let S be an order in Q and A be an S-ideal such that $O_r(A) = T = O_l(A)$, where $O_r(A) = \{q \in Q \mid Aq \subseteqq A\}$ and $O_l(A) = \{g \in Q \mid gA \subseteqq A\}$. Suppose that $A = aT$ for some $a \in A$. Then $A = Ta$.*

Proof. $T = O_l(A) = aTa^{-1}$ implies $A = Ta$.

Lemma 2.2. *Let R be a Dubrovin valuation ring of Q and $P \in$ G-Spec(R). Suppose that P is lower limit, i.e., $P = \cap\, \{P_\lambda \mid P_\lambda \in$ G-Spec(R) with $P_\lambda \supset P\}$. Then $R_P = \cup\, R_{P_\lambda}$ and $C(P) = \cup\, C(P_\lambda)$.*

Proof. Since $P_\lambda \supset P$, it follows that $R_P \supset R_{P_\lambda}$ so that $R_P \supseteqq S = \cup\, R_{P_\lambda}$. Suppose that $R_P \supset S$. Then for any P_λ, $P_\lambda = J(R_{P_\lambda}) \supseteqq J(S) \supset J(R_P) = P$ implies $P = \cap\, P_\lambda \supseteqq J(S) \supset P$, a contradiction. Hence $R_P = \cup\, R_{P_\lambda}$ and so $C(P) = \cup\, C(P_\lambda)$ follows.

Lemma 2.3. *Let R be a Dubrovin valuation ring of Q and $P \in$ G-Spec(R). Then*
(1) Spec$(R_P) = \{P_1 \mid P_1 \in$ Spec(R) *with* $P \supseteqq P_1\}$.
(2) *Let P_1 and P_2 be in* Spec(R) *with $P \supseteqq P_1 \supset P_2$. Then $P_1 \supset P_2$ is a prime segment of R if and only if it is a prime segment of R_P.*

Proof. (1) Let $P_1 \in$ Spec(R_P).
Case 1. If P_1 is Goldie prime, then $(R_P)_{P_1}$ is an overring of R_P (and so of R) with $J((R_P)_{P_1}) = P_1$, i.e., $P_1 \in$ Spec(R) and $P = J(R_P) \supseteqq P_1$.
Case 2. If P_1 is non-Goldie prime, then we can construct an exceptional prime segment of R_P, say, $P_2 \supset P_1 \supset P_0$ by [1, Theorem 6]. By case 1, $P \supseteqq P_2$ and P_2, $P_0 \in$ G-Spec(R). It easily follows from note before Lemma 2.1 that there are no Goldie primes properly between P_2 and P_0, which implies $P_2 \supset P_0$ is a prime segment of R. As in [1], let $K(P_2) = \{a \in P_2 \mid P_2aP_2 \subset P_2\}$. Then $K(P_2) = P_1$ by [1, Corollary 7] and so $P_2 \supset P_0$ is an exceptional prime segment of R with $K(P_2) = P_1$, i.e., P_1 is non-Goldie prime of R with $P \supset P_1$. Conversely, let $P_1 \in$ Spec(R) with $P \supseteqq P_1$. Then from note before Lemma 2.1 and the method we have just done, we can easily see that $P_1 \in$ Spec(R_P) and that $P_1 \in$ G-Spec(R) iff $P_1 \in$ G-Spec(R_P).
(2) This is clear from (1).

Lemma 2.4. *Let R be a Dubrovin valuation ring of Q and $P_1 \supset P$ be an Archimedean prime segment. Then for any $c \in P_1 \backslash P$, the following hold:*
(1) $R_{P_1}cR_{P_1} = aR_{P_1} = R_{P_1}a$ *for some* $a \in P_1$.
(2) *If c is a regular element, then $cR_{P_1} = R_{P_1}c$ and $cP_1 = P_1c$.*

Proof. Firstly note that $P_1 \supset P$ is an Archimedean prime segment of R_{P_1} by Lemma 2.3 and [1, Corollary 7].
(1) Let $\tilde{R}_{P_1} = R_{P_1}/P$, a Dubrovin valuation ring of $\overline{R_P} = R_P/P$ (see [7, (6.6)]) such that $J(\tilde{R}_{P_1}) = \tilde{P}_1 = P_1/P$ and $\tilde{P}_1 \supset (\tilde{0})$ is Archimedean. Here for any $a \in R_{P_1}$ we write $\tilde{a}$ for the image of a in $\tilde{R}_{P_1}$. If $\tilde{P}_1 = \tilde{P}_1^2$, then $\tilde{0} \neq \tilde{R}_{P_1}\tilde{c}\tilde{R}_{P_1} = \tilde{a}\tilde{R}_{P_1} = \tilde{R}_{P_1}\tilde{a}$ for some $a \in P_1$ by [2, (2.1)]. If $\tilde{P}_1 \supset \tilde{P}_1^2$, then $\tilde{R}_{P_1}$ is a Noetherian Dubrovin valuation ring and so any ideal of $\tilde{R}_{P_1}$ is power of $\tilde{P}_1$. Thus $\tilde{R}_{P_1}\tilde{c}\tilde{R}_{P_1} = \tilde{a}\tilde{R}_{P_1} = \tilde{R}_{P_1}\tilde{a}$ for some $a \in P_1$, because $\tilde{P}_1$ is principal. Hence, in both cases, $R_{P_1}cR_{P_1} + P = aR_{P_1} + P = R_{P_1}a + P$. However, since $\tilde{a} \in C_{\tilde{R}_{P_1}}(\tilde{0}) = \{\tilde{b} \in \tilde{R}_{P_1} \mid \tilde{b}$ is regular in $\tilde{R}_{P_1}\}$, it follows that $a \in C_{R_{P_1}}(P)$ and so a is a regular element by [7, (22.6)]. Thus we have $aR_{P_1}a^{-1} \subsetneqq aR_Pa^{-1} = R_P$. It follows that aR_{P_1} and P are both left $aR_{P_1}a^{-1}$ and right R_{P_1}-ideals. Hence $aR_{P_1} \supset P$ by [7, (6.4)] and similarly $R_{P_1}a \supset P$. Since $R_{P_1}cR_{P_1}$ and P are both ideals of R_{P_1}, it follows that $R_{P_1}cR_{P_1} \supset P$. Therefore $R_{P_1}cR_{P_1} = aR_{P_1} = R_{P_1}a$ follows.
(2) By (1), $P_1 \supseteqq R_{P_1}cR_{P_1} = R_{P_1}a = aR_{P_1}$ for some $a \in P_1$. Suppose that $cR_{P_1} \subset R_{P_1}cR_{P_1}$. Then, by [7, (6.3)], there is a $b \in R_{P_1}cR_{P_1}$ such that $cR_{P_1} \subsetneqq bP_1 \subsetneqq aP_1$, because $Q_l(cR_{P_1}) = cR_{P_1}c^{-1}$ and $P_1 = J(R_{P_1})$. So $R_{P_1}a^{-1}cR_{P_1} \subsetneqq P_1$. On the other hand, $R_{P_1}cR_{P_1} = aR_{P_1}$ implies that $R_{P_1}a^{-1}cR_{P_1} = R_{P_1}$, a contradiction. Hence, $cR_{P_1} = R_{P_1}cR_{P_1}$ and similarly $R_{P_1}c = R_{P_1}cR_{P_1}$ so that $cR_{P_1} = R_{P_1}c$. Since $cR_{P_1}c^{-1} = R_{P_1}$ and $J(R_{P_1}) = P_1$, we have $cP_1c^{-1} = P_1$ and so $cP_1 = P_1c$.

We are now ready to prove the main result of the paper.

Theorem 2.5. *Let R be a Dubrovin valuation ring of a simple Artinian ring Q. Then R is fully bounded if and only if for any $P \in$ Spec$(R), P \neq J(R)$, the following hold:*
(1) $P \in$ G-Spec(R).
(2) *P is either lower limit or there is a $P_1 \in$ Spec(R) such that $P_1 \supset P$ is an Archimedean prime segment.*

Proof. Suppose that R is fully bounded.
(1) Assume that there is a non-Goldie prime ideal C. Then we have an exceptional prime segment, say, $P_1 \supset C \supset P_2$ by [1, Theorem 6]. R is an n-chain ring by [7, (5.11)] and so is $\overline{R} = R/C$. This implies that $\overline{R}$ has a finite Goldie dimension, say, $m(\leqq n)$. Thus there are non-zero uniform right ideals $\overline{U_i}$ of $\overline{R}$ such that $\overline{U_1} \oplus \ldots \oplus \overline{U_m}$ is an essential right ideal of $\overline{R}$. Since $\overline{R}$ is a prime ring, $\overline{U_i} \cap \overline{P_1} \supseteqq \overline{U_i}\ \overline{P_1} \neq \overline{0}$ and so there are non-zero $\overline{u_i} \in \overline{U_i} \cap \overline{P_1}$, where $u_i \in P_1$. Set $I = u_1R + \ldots + u_mR$. Then $I = aR$ for some $a \in I$, because R is Bezout (cf. [7, (5.11)] and $\overline{I} = \overline{u_1}\overline{R} \oplus \ldots \oplus \overline{u_m}\overline{R} = \overline{a}\overline{R}$ is an essential right ideal of $\overline{R}$. We claim that $\overline{P_1} \supset \overline{I}$. On the contrary, suppose that $\overline{P_1} = \overline{I}$, i.e., $P_1 = aR + C$. Note that $O_l(C) = R_{P_1} = O_r(C)$ by [2, (2.2)] so that C is an ideal of R_{P_1}. If C is a principal right ideal of R_{P_1}, say, $C = cR_{P_1}$ for some $c \in C$, then $P_1 = aR_{P_1} + cR_{P_1} = bR_{P_1}$ for some $b \in P_1$. It follows from Lemma 2.1 that $P_1 = bR_{P_1} = R_{P_1}b$ and so $P_1 \supset P_1^2 \supset C$, which contradicts to the fact that there are no ideals properly between P_1 and C (cf. [1, Theorem 6]). If C is not a principal right ideal of R_{P_1}, then $CP_1 = C$ by [7, (6.9)] and so $P_1 = P_1^2 = aP_1 + CP_1 = aP_1 + C$. Thus we have $a = ap + d$ for some $p \in P_1$ and $d \in C$ and $a(1-p) = d \in C$. It follows that $a \in C$, because $1 - p$ is a unit of R_{P_1}, which shows $\overline{I} = \overline{0}$, a contradiction. We have shown that $\overline{P_1} \supset \overline{I}$ and $\overline{I}$ is an essential right ideal of $\overline{R}$. Hence $\overline{R}$ is not bounded, because there are no ideals properly between P_1 and C. Therefore, any prime ideal of R is Goldie prime.
(2) Let $P \in$ G-Spec(R) and suppose that P is not lower limit. Then there is a $P_1 \in$ G-Spec(R) such that $P_1 \supset P$ is a prime segment, which is not exceptional by (1). Suppose

that this is simple. For any $c \in P_1 \cap C(P)$, it follows that $\overline{c}\overline{P_1}$ is an essential right ideal of $\overline{R} = R/P$, which is a Dubrovin valuation ring of R_P/P (cf. [7, (5.12)]). Suppose that $\overline{c}\overline{P_1} = \overline{P_1}$, i.e., $cP_1 + P = P_1$. Since cP_1 and P are both left $cR_{P_1}c^{-1}$ and right R_{P_1}-ideals (note $cR_{P_1}c^{-1} \subsetneqq cR_Pc^{-1} = R_P$), we have either $cP_1 \supset P$ or $cP_1 \subsetneqq P$ by [7, (6.4)]. The latter case is impossible and so $cP_1 \supset P$. Thus $cP_1 = P_1$ and $c^{-1} \in O_l(P_1) = R_{P_1}$ follows. This is a contradiction, because $c \in P_1$. Hence we have shown that $\overline{P_1} \supset \overline{c}\overline{P_1}$ and $\overline{c}\overline{P_1}$ is an essential right ideal. Therefore, $\overline{R}$ is not bounded, because there are no ideals properly between $\overline{P_1}$ and $(\overline{0})$. Hence either P is lower limit or there is a $P_1 \in$ G-Spec(R) such that $P_1 \supset P$ is an Archimedean prime segment.

Conversely, suppose that the conditions (1) and (2) hold and let $P \in$ Spec(R). Then P is Goldie prime by (1). Firstly, assume that P is lower prime, i.e., $P = \cap\{P_\lambda \mid P_\lambda \in$ G-Spec(R) with $P_\lambda \supset P\}$. Then $C(P) = \cup\, C(P_\lambda)$ by Lemma 2.2. So, for any $c \in C(P)$, we have $c \in C(P_\lambda)$ for some λ. Then $cR \supset P_\lambda$, because cR and P_λ are both left cRc^{-1} and right R-ideals. Hence $\overline{c}\overline{R} \supset \overline{P_\lambda} \neq \overline{0}$ in $\overline{R} = R/P$, showing that $\overline{R}$ is bounded. Secondly, suppose that the prime segment $P_1 \supset P$ is Archimedean and let $c \in C(P)$. Then, as before, $\overline{c}\overline{P_1}$ is an essential right ideal of $\overline{R} = R/P$ and so $cP_1 \cap C(P) \neq \emptyset$. Let $d \in cP_1 \cap C(P)$. Then, by Lemma 2.4 (2) and [7,(22.7)], $cR \supsetneqq cP_1 \supsetneqq dR_{P_1} = R_{P_1}d$ and $dR_{P_1} \supset P$ follows. Therefore, $\overline{R} = R/P$ is bounded and hence R is fully bounded.

As an application of Theorem 2.5, we have the following:

Proposition 2.6. *Let R be a Dubrovin valuation ring of a simple Artinian ring Q. Then R is locally invariant if and only if it is fully bounded .*

Proof. Suppose that R is locally invariant. In order to prove that it is fully bounded, on the contrary, assume that R is not fully bounded. Then there are prime ideals P, P_1 such that either the prime segment $P_1 \supset P$ is simple or $P_1 \in$ G-Spec(R), P is a non-Goldie prime ideal and there are no ideals properly between P_1 and P. In either case, we shall prove that there is a regular element $c \in P_1 \setminus P$. Let c_1 be any element in $P_1 \setminus P$. If c_1R is an essential right ideal. Then $c = c_1$ is regular. If c_1R is not an essential right ideal, then there is a right ideal I such that $cR \oplus I$ is essential. So it follows from Goldie's theorem that $(cR \oplus I)P_1$ is also an essential right ideal which is contained in P_1 but not in P. So there is a regular element $c \in (c_1R \oplus I)P_1$ but not in P by [8, (3.3.7), Corollary]. Now let $c \in P_1 \setminus P$ such that c is regular. Then $cP_1 = P_1c$, because $P_1 = P(c)$. Since $P_1 \supsetneqq cP_1 = P_1c \supset P$, we have $cP_1 = P_1$, which implies $c^{-1} \in O_l(P_1) = R_{P_1}$. Hence $R_{P_1} = cR_{P_1} \subsetneqq P_1$, a contradiction. Therefore, R is fully bounded.

Suppose that R is fully bounded. Let $c \in J(R)$ such that c is regular. By the assumption and Theorem 2.5, $P(c) = \cap\ \{P_\lambda \mid P_\lambda \in$ Spec(R) such that $P_\lambda \ni c\ \}$, which is Goldie prime by [1, Proposition 1]. Suppose that $P(c)$ is upper limit, i.e., $P(c) = \cup\ \{P_\mu \mid P_\mu \in$ G-Spec(R) with $P_\mu \subset P(c)\ \}$. Then there is a P_μ with $P_\mu \ni c$. This contradicts the choice of $P(c)$. Hence $P(c) \supset P = \cup\ \{P_\mu \mid P(c) \supset P_\mu\ \}$ is a prime segment which must be Archimedean by Theorem 2.5. Since $c \in P(c) \setminus P$ and c is regular, we have $cP(c) = P(c)c$ by Lemma 2.4. Hence R is locally invariant.

We say that R is *invariant* if $cRc^{-1} = R$ for any regular element c in R and that it is of *rank n* if there are exactly n Goldie prime ideals. From Lemma 2.4, we have

Proposition 2.7. *Suppose that R is Archimedean and is of rank one. Then it is invariant.*

Proof. Let c be any regular element and let c_1 be any regular element in $J(R)$. Then we have $cRc^{-1} = cc_1R(cc_1)^{-1} = R$ by Lemma 2.4, because $c_1, cc_1 \in J(R)$.

We will give sevaral examples of fully bounded Dubrovin valuation rings.

Example 2.8. Any Dubrovin valuation ring of a simple Artinian ring with finite dimension over its center is fully bounded.

Example 2.9. Any invariant valuation ring of a division ring is fully bounded (see [9, Remarks to examples 2.1 and 2.4] for invariant valuation rings of division rings with infinite dimensions over the centers).

In order to give more general examples, we recall the skew polynomial ring $Q[x, \sigma]$ over Q in an indeterminate x, where $\sigma \in$Aut(Q). Since $Q[x, \sigma]$ is a principal ideal ring, the maximal ideal $P = xQ[x, \sigma]$ is localizable, i.e., $T = Q[x, \sigma]_P = \{f(x)c(x)^{-1} \mid f(x) \in Q[x, \sigma]$ and $c(x) \in C(P)$ $\}$, the localization of $Q[x, \sigma]$ at P, is a Notherian Dubrovin valuation ring with $J(T) = xT$. Since Q is a simple Artinian ring, $C(P) = \{c(x) \in Q[x, \sigma] \mid c(x) = c_0 + c_1x + \ldots + c_nx^n$ such that c_0 is a unit in $Q\}$. For any $t = f(x)c(x)^{-1} \in T$, where $f(x) = f_0 + f_1x + \ldots + f_lx^l$ and $c(x) = c_0 + c_1x + \ldots + c_nx^n$, the map φ: $T \longrightarrow Q$ defined by $\varphi(t) = f_0c_0^{-1}$ is an ring epimorphism. Now let R be a Dubrovin valuation ring of Q. Then, by [9, (1.6)], $\tilde{R} = \varphi^{-1}(R)$, the complete inverse image of R by φ, is a Dubrovin valuation ring of $Q(x, \sigma)$ ($Q(x, \sigma)$ stands for the quotient ring of $Q[x, \sigma]$). Furthermore, let $P = p\tilde{R}$ ($p \in$ Spec(R)). Then $P \in$Spec$(\tilde{R})$ and $\tilde{R}/P \cong R/p$ by [9, (1.6)] and its proof. Thus it follows from [9, (1.6)] that $\tilde{R}$ is fully bounded iff R is fully bounded. Hence we have

Example 2.10. With notation above, suppose that R is a fully bounded Dubrovin valuation ring of Q and that σ is of infinite order ([9, Examples 2.1 $\sim$ 2.6, 2.7 and 2.8]). Then $\tilde{R}$ is a fully bounded Dubrovin valuation ring of $Q(x, \sigma)$ and $Q(x, \sigma)$ is of infinite dimensional over the center.

Finally, we give a few remarks on non-fully bounded total valuation rings: An example of a total valuation ring with a simple segment was first constructed by Mathiak ([6]). See [3] for other examples of total valuation rings with simple segments. Dubrovin constructed an example of a total valuation ring with an exceptional prime segment ([4]).

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