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ASYMPTOTIC IMPROVEMENT OF THE SAMPLE MEAN VECTOR FOR SEQUENTIAL POINT ESTIMATION OF A MULTIVARIATE NORMAL MEAN WITH A LINEX LOSS FUNCTION

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Abstract. This paper considers sequential point estimations of the mean vector of a multivariate normal distribution under a LINEX loss function. It is shown that a sequential procedure with the sample mean as an estimate is asymptotically improved by the procedure with another estimate.

1 Introduction This paper considers a sequential point estimation of the mean vector of a multivariate normal distribution under a LINEX (Linear Exponential) loss function. The LINEX loss function was first proposed by Varian (1975) when it is appropriate to use asymmetric loss functions. Zellner (1986) showed that the sample mean vector is inadmissible for estimating the normal mean vector with known covariance matrix. Furthermore, he showed that the inadmissibility holds even though the covariance matrix is unknown. The purpose of this paper is to show that the same phenomena occur in sequential settings, that is, the sequential procedure with sample mean vector is asymptotically improved by that with another estimate.

Let $X_1, X_2, \cdots$ be independent and identically distributed (i.i.d.) p-dimensional normal random vectors with unknown mean vector μ and unknown covariance matrix Σ ($N_p(\mu, \Sigma)$). We consider two sequential point estimation problems of μ under the LINEX loss function. One is the minimum risk problem and the other is the bounded risk problem. For the univariate case, see Chattopadhyay (1998) and Takada (2000) for these problems. Nagao (2002) considered its extension to a linear regression problem. For another multivariate sequential estimation problems, see Woodroofe (1977) and Nagao and Srivastava (2002).

Zellner (1986) showed that for fixed sample size n, the sample mean vector $\overline{X}_n = (\overline{X}_{n1}, \ldots, \overline{X}_{np})'$ is inadmissible under the LINEX loss function,

$$L(\delta, \mu) = \sum_{i=1}^{n} b_i\{\exp(a_i(\delta_i - \mu_i)) - a_i(\delta_i - \mu_i) - 1\} \tag{1}$$

where $\delta = (\delta_1, \ldots, \delta_p)'$ is an estimate of $\mu = (\mu_1, \ldots, \mu_p)'$ and $a_i \neq 0,\ b_i > 0,\ i = 1, \ldots, p$. If Σ is known, then the sample mean vector is dominated by

$$\delta_n = \overline{X}_n - \frac{\lambda}{2n}, \tag{2}$$

where $\lambda = (a_1\sigma_{11}, \ldots, a_p\sigma_{pp})'$ and $\Sigma = (\sigma_{ij})$. The sample mean vector is also inadmissible even though Σ is unknown, and is dominated by

$$\hat{\delta}_n = \overline{X}_n - \frac{\hat{\lambda}_n}{2n}, \tag{3}$$

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where $\hat{\lambda}_n = (a_1 s_{11,n}, \dots, a_p s_{pp,n})'$ and $S_n = (s_{ij,n}) = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X}_n)(X_i - \overline{X}_n)'$.

The minimum risk problem is as follows. Consider the problem of estimating μ with the LINEX loss function for estimation error and cost $c > 0$ for each observation. If Σ were known, we would estimate μ by δ_n given by (2) for sample size n. Then the risk would be

$$\begin{aligned} R_n &= E_\theta \{L(\delta_n, \mu) + cn\} \\ &= \frac{\tau}{2n} + cn, \end{aligned}$$

where $\theta = (\mu, \Sigma)$ and $\tau = \sum_{i=1}^n a_i^2 b_i \sigma_{ii}$. Then the risk would be minimized at $n_c = \sqrt{\frac{\tau}{2c}}$ with $R_{n_c} = 2cn_c$. Unfortunately, Σ is unknown and hence the best fixed sample size procedure with $n = n_c$ can not be used. However, motivated by the formula for n_c, we propose the following stopping time

$$T_c = \inf\{n \geq m;\ n > \ell_n \sqrt{\frac{\hat{\tau}_n}{2c}}\}, \tag{4}$$

where $m \geq 2$ is the initial sample size, $\hat{\tau}_n = \sum_{i=1}^n a_i^2 b_i s_{ii,n}$ and $\{\ell_n\}$ is a sequence of constants such that

$$\ell_n = 1 + \frac{\ell}{n} + o\left(\frac{1}{n}\right) \qquad \text{as} \to \infty. \tag{5}$$

So we consider a sequential estimation procedure with the stopping time T_c that estimates μ by $\hat{\delta}_{T_c}$ in (3) where n is replaced by T_c. Then the risk of the procedure is

$$R_{T_c} = E_\theta \left\{ L(\hat{\delta}_{T_c}, \mu) + cT_c \right\}. \tag{6}$$

and the regret is $R_{T_c} - R_{n_c}$. We also consider another sequential estimation procedure with the same stopping time T_c that estimates μ by the sample mean vector $\overline{X}_{T_c}$, and compare two procedures.

The bounded risk problem is defined as follows. For a preassigned positive constant W, we want to determine the sample size n and the estimator δ_n such that for all θ

$$E_\theta L(\delta_n, \mu) \leq W. \tag{7}$$

If Σ were known, we would estimate μ by δ_n given by (2) for a sample size n. Since

$$E_\theta L(\delta_n, \mu) = \frac{\tau}{2n},$$

(7) is satisfied if and only if $n \geq n_W = \frac{\tau}{2W}$. Unfortunately, Σ is unknown and hence the best fixed sample size procedure with $n = n_W$ can not be used. However, the formula for n_W suggests us the following stopping time

$$T_W = \inf\{n \geq m;\ n > \ell_n \frac{\hat{\tau}_n}{2W}\}, \tag{8}$$

where $\hat{\tau}_n$ and ℓ_n are the same as in (4). So we consider a sequential procedure which takes T_W observations and estimate μ by $\hat{\delta}_{T_W}$ in (3) where n is replaced by T_W. We also consider another sequential estimation procedure with the same stopping time T_W that estimates μ by the sample mean vector $\overline{X}_{T_W}$, and compare two procedures.

Section 2 provides Woodroofe's (1977) results needed in later sections. Section 3 treats the minimum risk problem. The bounded risk problem is considered in Section 4.

2 Preliminaries Let

$$W_k = \frac{1}{\sqrt{k(k+1)}} \left\{ \sum_{i=1}^{k} X_i - kX_{k+1} \right\}, \quad k \geq 1.$$

Then $W_1, W_2, \ldots$ are i.i.d. $N_p(0, \Sigma)$ and $(n-1)S_n = \sum_{i=1}^{n-1} W_i W_i'$. Hence

$$(n-1)\hat{\tau}_n = \sum_{k=1}^{n-1} U_k,$$

where

$$U_k = \sum_{i=1}^{p} a_i^2 b_i W_{ki}^2$$

and $W_k' = (W_{k1}, \ldots, W_{kp})$. It is easy to see that the expectation of U_k is τ and the variance of U_k is $2\sigma^2$, where $\sigma^2 = \sum_{i=1}^{p} \sum_{j=1}^{p} \sigma_{ij}^2 a_i^2 b_i a_j^2 b_j$.

It is noted that the stopping times considered are of the form

$$N_a = t_a + 1, \tag{9}$$

where

$$t_a = \inf\{n \geq m-1;\ \sum_{i=1}^{n} U_i < an^{\alpha} L(n)\}.$$

For the stopping time T_c in (4), $a = 2c$, $\alpha = 3$ and

$$L(n) = \frac{(n+1)^2}{n^2 \ell_{n+1}^2} = 1 + \frac{2(1-\ell)}{n} + o\left(\frac{1}{n}\right) \quad \text{as } n \to \infty, \tag{10}$$

for the stopping time T_W in (8), $a = 2W$, $\alpha = 2$ and

$$L(n) = \frac{n+1}{n\ell_{n+1}} = 1 + \frac{1-\ell}{n} + o\left(\frac{1}{n}\right) \quad \text{as } n \to \infty. \tag{11}$$

Let $\beta = 1/(\alpha - 1)$ and $n_a = \tau^{\beta} a^{-\beta}$. Then $n_a = n_c$ for T_c and $n_a = n_W$ for T_W. It is well known that $t_a/n_a \to 1$ *a.e.* and that $(t_a - n_a)/\sqrt{n_a}$ converges in distribution to $N(0, 2\beta^2\sigma^2\tau^{-2})$ as $a \to 0$ (see Bhattacharya and Mallik, 1973). Hence we have the following Proposition.

Proposition 1 *As $a \to 0$, $N_a/n_a \to 1$ a.e. and*

$$N_a^* = \frac{N_a - n_a}{\sqrt{n_a}}$$

converges in distribution to $N(0, 2\beta^2\sigma^2\tau^{-2})$.

Let F denote the distribution function of U_k. Then it is easy to see that

$$F(x) \leq Bx^{p/2}, \qquad \text{for all } x > 0$$

for some $B > 0$. Then Lemma 2.3 and Theorem 2.3 of Woodroofe (1977) give the following Propositions.

Proposition 2 *For* $0 < \epsilon < 1$,

$$P_\theta(N_a \le \epsilon n_a) = O\left(n_a^{-\frac{(m-1)p}{2\beta}}\right) \qquad as\ a \to 0.$$

Proposition 3 *If* $(m-1)p/2 > \beta$, *then* N_a^{*2} *are uniformly integrable.*

Propositions 1 to 3 give the following result. See Theorem 3.1 of Woodroofe (1977).

Proposition 4 *If* $(m-1)p/2 > 2\beta$, *then*

$$\lim_{a\to 0} E_\theta\left\{\frac{(N_a - n_a)^2}{N_a}\right\} = 2\beta^2\sigma^2\tau^{-2}.$$

Let

$$R_a = at_a^\alpha L(t_a) - \sum_{i=1}^{t_a} U_i.$$

Then R_a converges in distribution to a distribution H as $a \to 0$. We denote the mean of H by ν. See Theorem 2.2 of Woodroofe (1977) for the explicit form of ν. Let $\nu = \nu_1$ for T_c and $\nu = \nu_2$ for T_W. Theorem 2.4 of Woodroofe (1977) and (9) yield the following asymptotic expansion of the expectation of N_a, in which $L_0 = 2(1-\ell)$ for T_c and $L_0 = 1-\ell$ for T_W.

Proposition 5 *If* $(m-1)p/2 > \beta$, *then*

$$E_\theta(N_a) = n_a + 1 + \beta\tau^{-1}\nu - \beta L_0 - \alpha\beta^2\sigma^2\tau^{-2} + o(1) \qquad as\ a \to 0.$$

Let $N = N_a$. Since the event $\{N = n\}$ is independent of $\overline{X}_n$,

$$\begin{aligned} E_\theta L(\hat{\delta}_N, \mu) &= \sum_{i=1}^{p} b_i E_\theta\left\{\exp\left(-\frac{a_i^2}{2N}(S_{ii,N} - \sigma_{ii})\right) + \frac{a_i^2 S_{ii,N}}{2N} - 1\right\} \\ &= \sum_{i=1}^{p} b_i E_\theta\left\{\exp\left(-\frac{a_i^2}{2N}(S_{ii,N} - \sigma_{ii})\right) + \frac{a_i^2 (S_{ii,N} - \sigma_{ii})}{2N} - 1\right\} \\ &\quad + E_\theta\left(\frac{\tau}{2N}\right) \end{aligned} \tag{12}$$

Lemma 1 *If* $(m-1)p/2 > 2\beta$, *then*

$$E_\theta L(\hat{\delta}_N, \mu) = E_\theta\left(\frac{\tau}{2N}\right) + o(n_a^{-2}) \qquad as\ a \to 0.$$

Proof. From (12) it is enough to show that as $a \to 0$,

$$E_\theta\left\{\exp\left(-\frac{a_i^2}{2N}(S_{ii,N} - \sigma_{ii})\right) + \frac{a_i^2 (S_{ii,N} - \sigma_{ii})}{2N} - 1\right\} = o(n_a^{-2}). \tag{13}$$

Let $C = \{N > \epsilon n_a\} \cap \{|S_{ii,N} - \sigma_{ii}| \le \delta\}$ for some $0 < \epsilon < 1$ and $\delta > 0$. Let C' be the compliment of C. Note that

$$P_\theta(C') \le P_\theta(N \le \epsilon n_a) + P_\theta\left(\sup_{n > \epsilon n_a} |S_{ii,n} - \sigma_{ii}| > \delta\right).$$

From the fact that $\{(S_{ii,n} - \sigma_{ii})^q\}$ is a reversed submartingale for any $q > 1$ and Proposition 2, it follows that

$$P_\theta(C') = O\left(n_a^{-\frac{(m-1)p}{2\beta}}\right) \qquad \text{as } a \to 0. \tag{14}$$

Divide the left side of (13) into two parts such that

$$\begin{aligned}
& \int_C \left\{ \exp\left(-\frac{a_i^2}{2N}(S_{ii,N}-\sigma_{ii}) \right) + \frac{a_i^2(S_{ii,N}-\sigma_{ii})}{2N} - 1 \right\} dP_\theta \\
+ & \int_{C'} \left\{ \exp\left(-\frac{a_i^2}{2N}(S_{ii,N}-\sigma_{ii}) \right) + \frac{a_i^2(S_{ii,N}-\sigma_{ii})}{2N} - 1 \right\} dP_\theta \\
= & \quad I + II \qquad \text{(say)}
\end{aligned}$$

Then

$$\begin{aligned}
I &= \int_C \frac{1}{2} \left(\frac{a_i^2(S_{ii,N}-\sigma_{ii})}{2N} \right)^2 \exp(\Delta_N) dP_\theta \\
&= \left(\frac{a_i^4}{8n_a^3} \right) \int_C \left(\frac{n_a}{N} \right)^3 N(S_{ii,N}-\sigma_{ii})^2 \exp(\Delta_N) dP_\theta,
\end{aligned}$$

where

$$|\Delta_N| \le \frac{a_i^2}{2N}|S_{ii,N}-\sigma_{ii}| \le \frac{a_i^2\delta}{2m}$$

on C. Note that $N/n_a \to 1$ *a.e.*, $\Delta_N \to 0$ *a.e.* and $\sqrt{N}(S_{ii,N}-\sigma_{ii})$ converges in distribution to $N(0, 2\sigma_{ii}^2)$ by Anscombe's theorem (Anscombe, 1952) as $a \to 0$. Then $(n_a/N)^3 N(S_{ii,N} - \sigma_{ii})^2 \exp(\Delta_N)$ converges in distribution to $2\sigma_{ii}^2\chi_1^2$ as $a \to 0$, where χ_1^2 denotes the chi-squared random variable with one degree of freedom. It is easy to see that $(n_a/N)^3 N(S_{ii,N} - \sigma_{ii})^2 \exp(\Delta_N)$ is uniformly integrable on C. Hence it follows that

$$I = O(n_a^{-3}) \qquad \text{as } a \to 0,$$

which means that in order to prove (13), it is sufficient to show that

$$II = o(n_a^{-2}) \qquad \text{as } a \to 0. \tag{15}$$

By Hölder inequality,

$$\begin{aligned}
0 \le II &\le \exp\left(\frac{a_i^2\sigma_{ii}}{2m} \right) P_\theta(C') + \frac{a_i^2}{2m} \int_{C'} |S_{ii,N}-\sigma_{ii}| dP_\theta \\
&\le \exp\left(\frac{a_i^2\sigma_{ii}}{2m} \right) P_\theta(C') + \frac{a_i^2}{2m} K^{1/r} P_\theta(C')^{1/s},
\end{aligned}$$

where $r > 1$ and $s > 1$ with $(1/r) + (1/s) = 1$, and $K = E|S_{ii,N} - \sigma_{ii}|^r$, which is finite for any $r > 1$. Hence it follows from (14) that

$$II = O\left(n_a^{-\frac{p(m-1)}{2\beta s}} \right) \qquad \text{as } a \to 0.$$

The condition of the lemma implies that there exists $s > 1$ such that $p(m-1)/(2\beta s) > 2$, from which (15) follows.

Next we consider the risk of the procedure with $\overline{X}_N$ as an estimate of μ.

Lemma 2 *If* $(m-1)p/2 > 2\beta$, *then*

$$E_\theta L(\overline{X}_N, \mu) = \left(\frac{\sum_{i=1}^p a_i^4 b_i \sigma_{ii}^2}{8} \right) n_a^{-2} + E_\theta \left(\frac{\tau}{2N} \right) + o(n_a^{-2}) \qquad \textit{as } a \to 0.$$

Proof. It follows that

$$\begin{aligned} E_\theta L(\overline{X}_N,\mu) &= \sum_{i=1}^p b_i E_\theta\left\{\exp\left(\frac{a_i^2\sigma_{ii}}{2N}\right)-1\right\} \\ &= \sum_{i=1}^p b_i E_\theta\left\{\exp\left(\frac{a_i^2\sigma_{ii}}{2N}\right)-1-\frac{a_i^2\sigma_{ii}}{2N}\right\}+E_\theta\left(\frac{\tau}{2N}\right). \end{aligned}$$

Hence it is sufficient to show that as $a\to 0$,

$$E_\theta\left\{\exp\left(\frac{a_i^2\sigma_{ii}}{2N}\right)-1-\frac{a_i^2\sigma_{ii}}{2N}\right\}=\frac{a_i^4\sigma_{ii}^2}{8}n_a^{-2}+o(n_a^{-2}). \tag{16}$$

The left hand of (16) is equal to

$$E_\theta\left\{\frac{1}{2}\left(\frac{a_i^2\sigma_{ii}}{2N}\right)^2\exp(\Delta_N')\right\}=\frac{a_i^4\sigma_{ii}^2}{8n_a^2}E_\theta\left\{\left(\frac{n_a}{N}\right)^2\exp(\Delta_N')\right\},$$

where $|\Delta_N'|\le(a_i^2\sigma_{ii})/(2N)\le(a_i^2\sigma_{ii})/(2m)$. So (16) follows if it is shown that

$$E_\theta\left\{\left(\frac{n_a}{N}\right)^2\exp(\Delta_N')\right\}=1+o(1)\qquad\text{as } a\to 0. \tag{17}$$

For any $0<\epsilon<1$, rewrite the left side of (17) as

$$\int_{N<\epsilon n_a}\left(\frac{n_a}{N}\right)^2\exp(\Delta_N')dP_\theta+\int_{N\ge\epsilon n_a}\left(\frac{n_a}{N}\right)^2\exp(\Delta_N')dP_\theta.$$

The first part is bounded by

$$\left(\frac{n_a}{m}\right)^2\exp\left(\frac{a_i^2\sigma_{ii}}{2m}\right)P_\theta(N<\epsilon n_a)=o(1)\qquad\text{as } a\to 0$$

by Proposition 2 and the condition of the lemma. Since $(n_a/N)^2\exp(\Delta_N')$ converges to 1 a.e. as $a\to 0$ and is uniformly integrable on $\{N\ge\epsilon n_a\}$,

$$\int_{N\ge\epsilon n_a}\left(\frac{n_a}{N}\right)^2\exp(\Delta_N')dP_\theta=1+o(1)\qquad\text{as } a\to 0.$$

Hence (17) is proved.

3 Minimum risk problem Now we consider the minimum risk problem. Noting that $a=2c$, $n_a=n_c$, $\alpha=3$, $\beta=1/2$, and $L_0=2(1-\ell)$, Proposition 5 gives the following result.

Theorem 1 *If $m>1+p^{-1}$, then*

$$E_\theta(T_c)=n_c+\ell+(2\tau)^{-1}\nu_1-3\sigma^2(2\tau)^{-2}+o(1)\qquad\text{as } c\to 0.$$

First we consider the sequential estimation procedure with $\hat{\delta}_{T_c}$ as an estimate of μ. It follows from (6) and Lemma 1 that if $m>1+2p^{-1}$, then

$$\begin{aligned} R_{T_c}-R_{n_c} &= E_\theta\left\{\frac{\tau}{2T_c}+cT_c-2cn_c\right\}+o(c) \\ &= cE_\theta\left\{\frac{(T_c-n_c)^2}{T_c}\right\}+o(c)\qquad\text{as } c\to 0. \end{aligned}$$

Hence Proposition 4 yields the following.

Theorem 2 *If $m > 1 + 2p^{-1}$, then*

$$\frac{R_{T_c} - R_{n_c}}{c} = \frac{\sigma^2}{2\tau^2} + o(1) \qquad \text{as } c \to 0.$$

Next we consider the sequential estimation procedure with $\overline{X}_{T_c}$ as an estimate of μ, the risk of which is denoted by R'_{T_c}. Then it follows from Lemma 2 that if $m > 1 + 2p^{-1}$, then

$$\begin{aligned} R'_{T_c} - R_{n_c} &= \frac{c\sum_{i=1}^p a_i^4 b_i \sigma_{ii}^2}{4\tau} + E_\theta\left\{\frac{\tau}{2T_c} + cT_c - 2cn_c\right\} + o(c) \\ &= \frac{c\sum_{i=1}^p a_i^4 b_i \sigma_{ii}^2}{4\tau} + cE_\theta\left\{\frac{(T_c - n_c)^2}{T_c}\right\} + o(c) \qquad \text{as } c \to 0. \end{aligned}$$

Hence we get the following result.

Theorem 3 *If $m > 1 + 2p^{-1}$, then*

$$\frac{R_{T_c} - R_{n_c}}{c} = \frac{\sum_{i=1}^p a_i^4 b_i \sigma_{ii}^2}{4\tau} + \frac{\sigma^2}{2\tau^2} + o(1) \qquad \text{as } c \to 0.$$

Comparing Theorems 2 and 3, it turns out that the sequential estimation procedure with the sample mean as an estimate is asymptotically inadmissible

4 Bounded risk problem In this section we consider the bounded risk problem, for which it is noted that $a = 2W$, $n_a = n_W$, $\alpha = 2$, $\beta = 1$, and $L_0 = 1 - \ell$. Then by Proposition 5 we get the following result.

Theorem 4 *If $m > 1 + 2p^{-1}$, then*

$$E_\theta(T_W) = n_W + \ell + \tau^{-1}\nu_2 - 2\sigma^2\tau^{-2} + o(1) \qquad \text{as } W \to 0.$$

First we consider the sequential estimation procedure with $\hat{\delta}_{T_W}$ as an estimate of μ. It follows from Lemma 1 that if $m > 1 + 4p^{-1}$, then as $W \to 0$,

$$\begin{aligned} E_\theta L(\hat{\delta}_{T_W}, \mu) &= \frac{2W^2}{\tau} E_\theta\left(\frac{n_W^2}{T_W}\right) + o(W^2) \\ &= \frac{2W^2}{\tau}\left\{E_\theta\left[\frac{(T_W - n_W)^2}{T_W}\right] + E_\theta\left(n_W - T_W\right)\right\} + W + o(W^2). \end{aligned}$$

Proposition 4 and Theorem 4 give the following result.

Theorem 5 *If $m > 1 + 4p^{-1}$, then*

$$E_\theta L(\hat{\delta}_{T_W}, \mu) = W + \frac{2W^2}{\tau}\left\{\frac{4\sigma^2}{\tau^2} - \frac{\nu_2}{\tau} - \ell\right\} + o(W^2) \qquad \text{as } W \to 0.$$

It is easy to see that $\sigma^2 \le \tau^2$. Note that $\nu_2 > 0$. Then the following corollary is obtained.

Corollary 1 *If $m > 1 + 4p^{-1}$ and $\ell \ge 4$, then*

$$E_\theta L(\hat{\delta}_{T_W}, \mu) \le W + o(W^2) \qquad \text{as } W \to 0.$$

This corollary shows that the sequential estimation procedure asymptotically satisfies the requirement (7).

Next we consider the sequential estimation procedure with $\overline{X}_{T_W}$ as an estimate of μ. It follows from Lemma 2 that if $m > 1 + 4p^{-1}$, then

$$\begin{aligned} E_\theta L(\overline{X}_{T_W}, \mu) &= \frac{W^2 \sum_{i=1}^p a_i^4 b_i \sigma_{ii}^2}{2\tau^2} + \frac{2W^2}{\tau} E_\theta \left(\frac{n_W^2}{T_W} \right) + o(W^{-2}) \\ &= \frac{2W^2}{\tau} \left\{ \frac{\sum_{i=1}^p a_i^4 b_i \sigma_{ii}^2}{4\tau} + E_\theta \left[\frac{(T_W - n_W)^2}{T_W} \right] + E_\theta (n_W - T_W) \right\} \\ &\quad + W + o(W^2) \qquad \text{as } W \to 0. \end{aligned}$$

Hence we have the following result.

Theorem 6 *If $m > 1 + 4p^{-1}$, then as $W \to 0$,*

$$E_\theta L(\overline{X}_{T_W}, \mu) = W + \frac{2W^2}{\tau} \left\{ \frac{\sum_{i=1}^p a_i^4 b_i \sigma_{ii}^2}{4\tau} + \frac{4\sigma^2}{\tau^2} - \frac{\nu_2}{\tau} - \ell \right\} + o(W^2)$$

Comparing Theorems 5 and 6, it turns out that the sequential estimation procedure with the sample mean needs more observations to achieve (7) asymptotically.

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