PROPERTIES OF TOTAL POSITIVITY AND AN APPLICATION TO JOB SEARCH UNDER UNCERTAINTY

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Abstract. The total positivity of order two is a fundamental property to investigate the sequential decision problem, and it also plays an important role in the Bayesian learning procedure for a partially observable Markov process. For this process, we also deals with a job search, and observe the probability densities on the state space after some additional transitions by employing the optimal policy. This problem is considered as an extension of a job search in a dynamic economy discussed in Lippman and MacCall [8], and we will investigate a problem where the state changes according to a partially observable Markov process. Associated to each state of the process, the wages of a job is a random variable, and information about the unobservable state is obtained through it. All information are summarized by probability distributions on the state space, and we employ the Bayes' theorem as a learning procedure. By using a property called a total positive of order two, some relationships among information, the optimal policy and the probability density on the state space after some additional transitions are obtained.

1 Introduction This paper concerns the total positivity of order two to investigate the learning procedure for a partially observable Markov process, and observes the probability densities on the state space after some additional transitions. We also deals with a job search, and observe these probability densities under the optimal policy. This is one of the optimal stopping problems, and it can be considered as an extension of a job search in a dynamic economy discussed in Lippman and MacCall [8]. For instance, in economics, we consider that the conditions of economy are divided into some classes, and assume them to getting worse. Let's assume the condition of those can not observe directly. That is, it cannot be known which one of these class it is now, but there is some information regarding what a present class is. When each state of this process corresponds the class of the economy, the wages of a job is a random variable depending on these classes. Differ from the problem in [8], the state changes according to a partially observable Markov process, and we will consider the properties of a probability density on the state space after some additional transitions by employing the optimal policy. For job search in which the state is observable, it is known that the maximization is achieved by classifying all possible job offers into two mutually exclusive classes, and the wage of a job offer that separates these two classes is called the reservation wage. It is not, however, always true for this problem since the state of the process is unobservable for the decision maker. In Nakai [15], the author observed these relationships for a partially observable Markov chain.

All information about the unobservable state are summarized by the probability distributions on the state space, and we employ the Bayes' theorem as a learning procedure. By using a property called a total positive of order two, or simply TP_2, which is closely related to the likelihood ratio ordering, we consider some relationships among prior and

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posterior informations, the optimal policy and the probability densities on the state space after some additional transitions. The properties of the total positivity are also investigated by Karlin and McGregor [3], Karlin [2] and Karlin and Rinott [4] and by others regarding the stochastic processes.

In order to observe the case of a partially observable Markov process in which the Bayes' theorem is employed as a learning procedure, we will start to reconsider a job search in a dynamic economy where the state is directly observable. In Section 2, we summarize the properties regarding job search when the state of the process is directly observable. It will be shown that the probability densities on the state space after some additional transitions is TP_2 by employing the optimal policy. In Section 3, we will also investigate the probability densities on the state space after some additional transitions when the state changes according to a partially observable Markov process. We will also observe the similar probabilities by employing the optimal policy of the job search. For the Markov process with the state space $[0, S]$, if the state S designates default, then the probability to become bankrupt will be also observed, and, therefore, this probability for the case with uncertain state information. Finally, we summarize the proofs of several properties concerning TP_2 used in this paper in Section 4.

2 Job Search in a Dynamic Economy

2.1 Optimal Policy and the Expected Reward Consider a Markov process with the state space $[0, S]$ and the transition probability $\boldsymbol{P} = (p_s(t))_{s,t\in[0,S]}$ where $\boldsymbol{p}_s = (p_s(t))_{t\in[0,S]}$ is a probability distribution on $[0, S]$ for any $s \in [0, S]$. Associated to each state s corresponding the class of the economy ($s \in [0, S]$), a wage of a job offer is a random variable X_s. The job search is a problem to find a job in order to maximize the expected reward. We consider an individual who is looking for a job. Each period she pays a fixed amount and receives exactly one job offer. She may continue her search until m jobs appear. In this model, all rejected job offers are immediately withdrawn. The wage of each job offer depends on the state of this process. For a finite state Markov chain with the state space $\{1, 2, \cdots, n\}$, Lippman and MacCall [8] considered this problem in such a dynamic economy under two conditions (1) and (2):

(1) X_i is stochastically decreasing in i, i.e. $F_1(x) \geq F_2(x) \geq \cdots \geq F_n(x)$ for all x,

(2) $\displaystyle\sum_{j=k}^{n} p_{ij}$ is increasing in i for all $k (k = 1, 2, \cdots, K)$.

In this paper, the job search will be considered under an uncertainty condition, i.e. the state of this process can not be observed directly. For this purpose, we introduce two assumptions concerning the transition probability and the random variables X_s ($s \in [0, S]$) as Assumptions 1 and 2, which are the differences to the Lippmann and MacCall's case. For each state s, the random variables X_s are absolutely continuous with density $f_s(x)$ ($s \in [0, S]$). It is possible to generalize it as in Nakai [13], and also apply it to the sequential decision problems (Nakai [10, 11, 12] and so on). In Definition 1, we introduce a stochastic order relation among random variables defined on a complete separable metric space with a total order $\geq$.

Definition 1 *Suppose that two random variables X and Y has the respective probability density functions $f(x)$ and $g(x)$. If $f(y)g(x) \leq f(x)g(y)$ for all x and y where $x \geq y$, then X is said to be greater than Y by means of the likelihood ratio, or simply $X \succeq Y$.*

Definition 2 *Suppose a set function $\boldsymbol{P} = (p_s(t))_{s,t\in[0,S]}$. If $p_s(u)\, p_t(v) \geq p_t(u)\, p_s(v)$, i.e. $\begin{vmatrix} p_s(u) & p_s(v) \\ p_t(u) & p_t(v) \end{vmatrix} \geq 0$ for any s, t, u and v, where $s \leq t$ and $u \leq v$ ($s, t, u, v \in [0, S]$), then this $\boldsymbol{P}$ is said to be total positive of order two, or simply TP_2.*

It is easy to show that this order defined by Definition 1 is a partial order. Next, we introduce two assumptions (Assumptions 1 and 2), since we employ the Bayes' theorem as a learning procedure.

Assumption 1 *For the random variables* $\{X_s\}_{s\in[0,S]}$*, if* $s \le t$*, then* $X_s \succeq X_t$ $(s,t \in [0,S])$*, i.e.* X_s *is decreasing with respect to* s *by means of the likelihood ratio.*

Assumption 2 *The transition probability* $\boldsymbol{P} = (p_s(t))_{s,t\in[0,S]}$ *is* TP_2*.*

In Assumption 1, $X_s \succeq X_t$ implies that if $x > y$, then $f_s(y)f_t(x) \le f_s(x)f_t(y)$ for s and t where $s \le t$ $(s,t \in [0,S])$. From this fact, the random variable X_s takes on smaller values as s becomes larger, and an example of this is where state 0 represents the highest class, $\cdots$, and state S is the lowest class. Assumption 2 is known as TP_2 for this Markov process. This implies that the probability of moving from the current state to 'better' states decreases with improvement in the current state. By this assumption, as the number s associated with each state becomes larger, the probability to make a transition into the lower class increases.

When n jobs remain (i.e. there are n stages to go) and a wage of the currently available job offer is x, if this job is accepted, then a reward $u_n(x)$ will be obtained. We say we are in state (s,x) if the economy is in state s and the currently available job offer is x. The cost c is necessary to search a next job offer, and we induce a discount factor $0 < \beta < 1$. Let $v_n(s,x)$ be a maximal β-discount expected reward attainable when n stages remain and the currently available job offer is x. By the principle of optimality, these $v_n(s,x)$ satisfies the following optimality equation

$$v_n(s,x) = \max\left\{u_n(x), -c + \beta \int_0^S p_s(t)dt \int_0^\infty v_{n-1}(t,y)dF_t(y)\right\}, \tag{1}$$

where $v_1(s,x) = u_1(x)$. We assume $u_n(x)$ to be an increasing function of x and n. For example, $u_n(x) = \dfrac{1-\delta^n}{1-\delta}x$ satisfies these conditions, and this is an amount of the total sum of the capital and interest when we make a deposit of x at an annual interest rate of γ for n years where $\delta = 1+\gamma$. As Lippmann and MacCall [8], it is easy to show that the maximization is achieved by classifying all possible job offers into two mutually exclusive classes. The wage of a job offer that separates these two classes is called the reservation wage, and we denote it as $\alpha_n(s)$ when n jobs remain and the current state of the process is s. Concerning these $\alpha_n(s)$ and $v_n(i,x)$, Assumptions 1 and 2 and the property of $u_n(x)$ imply Lemmas 1 and 2 by the induction principle.

Lemma 1 *For any* $s \in [0,S]$ *and positive integer* n*,* $\alpha_{n+1}(s) \ge \alpha_n(s)$*. If* $s < t$ $(s,t \in [0,S])$*, then* $\alpha_n(s) \ge \alpha_n(t)$ *for any positive integer* n*.*

Lemma 2 *For any positive integer* n*,* $v_{n+1}(s,x) \ge v_n(s,x)$ *and* $v_{n+1}(s,x) \ge v_{n+1}(t,x)$ *where* $x > 0$ *and* $s < t$ $(s,t \in [0,S])$*. If* $x > y$*, then* $v_{n+1}(s,x) \ge v_{n+1}(s,y)$*.*

2.2 Probability Density after Some Additional Transitions In this subsection, we consider a probability density on the state space after n additional transitions when the state is directly observable. First, we only consider a change of the states. For any state s, let $\overline{p}_{s,n}(t)$ be the probability density on the state space after n additional transitions $(s,t \in [0,S], n = 1,2,\cdots)$. It is easy to show that $\overline{p}_{s,n}(t)$ satisfies the recursive equation $\overline{p}_{s,n}(t) = \int_0^S p_s(u)\overline{p}_{u,n-1}(t)du$ with the initial condition $\overline{p}_{s,1}(t) = p_s(t)$.

Let $\overline{\boldsymbol{P}}_n = \left(\overline{p}_{s,n}(t)\right)_{s,t\in[0,S]}$, then $\overline{\boldsymbol{P}}_1 = \boldsymbol{P}$ and $\overline{\boldsymbol{P}}_n = \langle \boldsymbol{P}, \overline{\boldsymbol{P}}_{n-1}\rangle$. Here, we use a notation $\langle \boldsymbol{P}, \boldsymbol{Q}\rangle = \left(\int_0^S p_s(u)q_u(t)du\right)_{s,t\in[0,S]}$ for two functions $\boldsymbol{P} = (p_s(t))_{s,t\in[0,S]}$ and $\boldsymbol{Q} = (q_s(t))_{s,t\in[0,S]}$. Lemma 3 is obtained concerning this notation.

Lemma 3 *If the set function $\boldsymbol{P} = (p_s(t))_{s,t\in[0,S]}$ and $\boldsymbol{Q} = (q_s(t))_{s,t\in[0,S]}$ are TP_2, then $\langle \boldsymbol{P}, \boldsymbol{Q}\rangle$ is also TP_2.*

If $\overline{\boldsymbol{P}}_{n-1} = (p_{s,n-1}(t))_{s,t\in[0,S]}$ is TP_2, then the induction principle on n and Lemma 3 imply that $\overline{\boldsymbol{P}}_n = \langle \boldsymbol{P}, \overline{\boldsymbol{P}}_{n-1}\rangle = \left(\overline{p}_{s,n}(t)\right)_{s,t\in[0,S]}$ is TP_2 since $\boldsymbol{P}$ is TP_2 by Assumption 2.

In this case, $\overline{p}_{s,n}(t)$ is the probability density on the state space after n additional transitions, given that the process starts in state s and the state changes according to the partially observable Markov process. We will next consider a job search in a dynamic economy and observe the similar probabilities by employing the optimal policy. When the state of the process is s and there are n jobs remain, let $\overline{p}_{s,n,m}(t)$ be the probability distribution on the state space after m additional transitions ($s, t \in [0, S]$ and $m \le n, n, m = 1, 2, \cdots$) by employing the optimal policy. For a job search where the state is directly observable, since the optimal policy is determined by the reservation wages $\alpha(s, n)$, $F_s(\alpha(s, n))$ is a probability not to accept the current job offer when the state is s and there are n stages to go. It is, therefore, easy to show that $\overline{\boldsymbol{p}}_{s,n,m} = (\overline{p}_{s,n,m}(t))_{t\in[0,S]}$ satisfies the recursive equation

$$\overline{p}_{s,n,m}(t) = F_s(\alpha(s,n)) \int_0^S p_s(x)\overline{p}_{x,n-1,m-1}(t)dx. \tag{2}$$

The initial condition is $\overline{\boldsymbol{p}}_{s,n,1} = (\overline{p}_{s,n,1}(t))_{t\in[0,S]}$ where $\overline{p}_{s,n,1}(t) = F_s(\alpha(s,n))p_s(t)$. If we put $\overline{\boldsymbol{P}}_{n,m} = \left(\overline{\boldsymbol{p}}_{s,n,m}\right)_{s\in[0,S]}$, then $\overline{\boldsymbol{P}}_{n,1} = (F_s(\alpha(s,n))\boldsymbol{p}_s)_{s\in[0,S]}$ for any $n(>0)$ and Equation (2) implies

$$\overline{\boldsymbol{P}}_{n,m} = (F_s(\alpha(s,n))\langle \boldsymbol{P}, \overline{\boldsymbol{P}}_{n-1,m-1}\rangle_s)_{s\in[0,S]}. \tag{3}$$

Here, we use a notation $\langle \boldsymbol{P}, \boldsymbol{Q}\rangle_s = \left(\int_0^S p_s(u)q_u(t)du\right)_{t\in[0,S]}$ for $\boldsymbol{P} = (p_s(t))_{s,t\in[0,S]}$ and $\boldsymbol{Q} = (q_s(t))_{s,t\in[0,S]}$. The following property is obtained as a corollary of Lemma 3.

Corollary 1 *Suppose the set function $\boldsymbol{P} = (\boldsymbol{P}_s)_{s\in[0,S]}$ is TP_2 and $d(s)$ is a function of s, then $\boldsymbol{Q} = (d(s)\boldsymbol{P}_s)_{s\in[0,S]} = (d(s)p_s(t))_{s,t\in[0,S]}$ is also TP_2.*

These $\overline{\boldsymbol{P}}_{n,m}$ satisfies the next property.

Proposition 1 *$\overline{\boldsymbol{P}}_{n,m} = \left(\overline{\boldsymbol{p}}_{s,m,n}\right)_{s\in[0,S]}$ is TP_2.*

Proof: We employ the induction principle on m. When $m = 1$, Corollary 1 implies that $\overline{\boldsymbol{P}}_{n,1} = (F_s(\alpha(s,n))\boldsymbol{p}_s)_{s\in[0,S]}$ is TP_2. Assume that $\overline{\boldsymbol{P}}_{n,m}$ is TP_2 for any values less than m. Since $\boldsymbol{P} = (p_s(t))_{s,t\in[0,S]}$ and $\overline{\boldsymbol{P}}_{n-1,m-1}$ are TP_2, Corollary 1 induces that $\langle \boldsymbol{P}, \overline{\boldsymbol{P}}_{n-1,m-1}\rangle$ is also TP_2. Lemma 3 implies

$$\overline{\boldsymbol{P}}_{n,m} = (F_s(\alpha(s,n))\langle \boldsymbol{P}, \overline{\boldsymbol{P}}_{n-1,m-1}\rangle_s)_{s\in[0,S]}$$

is also TP_2. □

3 Job Search with Incomplete Information

3.1 Optimal Policy and the Expected Reward A job search on a partially observable Markov process will be considered in thie section, i.e. the state of this process can not be observed. Information about the unobservable state is assumed to be a probability distribution $\boldsymbol{\mu}$ on the state space $[0, S]$. Let $\mathcal{S}$ be a set of all information about the unobservable state, then

$$\mathcal{S} = \left\{ \boldsymbol{\mu} = (\mu(s))_{s\in[0,S]} \,\middle|\, \int_0^S \mu(s) = 1, \mu(s) \geq 0 \, (s \in [0,S]) \right\}.$$

Among informations in $\mathcal{S}$, we introduce an order as was stated in Definition 1, i.e. for two probability distributions $\boldsymbol{\mu}, \boldsymbol{\nu}$ on $[0, S]$, if $\mu(t)\nu(s) \geq \mu(s)\nu(t)$ for any $s, t \, (s \leq t, \; s, t \in [0, S])$ and $\mu(t)\,\nu(s) \geq \mu(s)\,\nu(t)$ at least one pair of s and t, then $\boldsymbol{\mu}$ is said to be greater than $\boldsymbol{\nu}$, or simply $\boldsymbol{\mu} \succ \boldsymbol{\nu}$. This order is a partial order and also said to be total positive of order two, or simply TP_2. By this definition, if $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$ $(\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{S})$, then, as t become large, the ratio $\dfrac{\mu(t)}{\nu(t)}$ of the densities increases whenever $\nu(t) \neq 0$. On the other hand, if we put $\boldsymbol{p}_s = (p_s(u))$ and $\boldsymbol{p}_t = (p_t(u))$, then $\boldsymbol{p}_t \succeq \boldsymbol{p}_s$ for all $s, t \in [0, S]$ since $\boldsymbol{P}$ satisfies Assumption 2. It is possible to generalize this order relation to investigate a partially observable Markov process, and the details are shown in Nakai [13, 14] with the applications to the sequential decision problems. Concerning this order relation, Lemma 4 is also obtained under Assumptions 1 and 2.

Lemma 4 *If* $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$ $(\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{S})$*, then* $\int_0^\infty h(x)dF_{\boldsymbol{\mu}}(x) \leq \int_0^\infty h(x)dF_{\boldsymbol{\nu}}(x)$ *for a non-decreasing non-negative function* $h(x)$ *of* x*.*

In this lemma, $F_{\boldsymbol{\mu}}(x) = \int_0^S \mu(s)F_s(x)$ is a weighted distribution function as in De Vylder [1]. Regarding the unobservable state of the process, there exists an information system or an observation process to obtain information about it. Since the random variables $\{X_s\}_{s\in[0,S]}$ indicate the wage of a job offer depending on the unobservable state, it can be considered as an information system of this process, i.e. we improve information about the unobservable state by using a wage of a current job offer. When prior information is $\boldsymbol{\mu}$, we first observe a wage of a current job offer depending on the state and improve information about it by using the Bayes' theorem. After that, we see time moving forward by one unit and thus this process will make a transition to a new state. It is also possible to formulate and analyze this model by other order. If the wage of a current job offer is x, we improve information as $\boldsymbol{\mu}(x) = (\mu(x, s))_{s\in[0,S]}$ by employing the Byes' theorem, and, after changing to a new state according to $\boldsymbol{P}$, information at the next stage becomes $\overline{\boldsymbol{\mu}(x)} = (\overline{\mu(x, s)})_{s\in[0,S]}$.

For a set function $\boldsymbol{h}(x, s)$, we introduce a monotonicity property as Definition 3.

Definition 3 *For a set valued non-negative function* $\boldsymbol{h}(x) = (h(x, s))_{s\in[0,S]}$ *for all* s *in* $[0, S]$ *and* $x \in \Re_+$*, if* $x < y$ *then* $\boldsymbol{h}(x) \succeq \boldsymbol{h}(y)$ $(\boldsymbol{h}(y) \succeq \boldsymbol{h}(x))$*, i.e.* $h(x, t)\,h(y, s) \geq h(x, s)\,h(y, t)$ $(h(x, t)\,h(y, s) \leq h(x, s)\,h(y, t))$ *for any* t *and* $s \, (s \leq t$ *and* $s, t \in [0, S])$*. This function* $\boldsymbol{h}(x, s)$ *is said to be decreasing* (*increasing*) *function of* x*.*

Since the density functions $\{f_s(x) \mid s \in [0, S]\}$ of $\{X_s\}_{s\in[0,S]}$ satisfy Assumption 1, $\boldsymbol{f}(x) = (f_s(x))_{s\in[0,S]}$ satisfies $\boldsymbol{f}(y) \succeq \boldsymbol{f}(x)$, i.e. if $x > y$, then $f_s(y)f_t(x) \leq f_s(x)f_t(y)$ for any s and $t \, (s \leq t$ and $s, t \in [0, S])$. From this fact, $\boldsymbol{f}(x)$ is an increasing function of x.

Regarding the relationship between prior information $\boldsymbol{\mu}$ and posterior information $\overline{\boldsymbol{\mu}(x)}$, the following essential properties can be obtained under Assumptions 1 and 2, which is known as Lemma 5 of Nakai [13] and so on.

Lemma 5 *If $\boldsymbol{\mu} \succ \boldsymbol{\nu}$, then $\boldsymbol{\mu}(x) \succ \boldsymbol{\nu}(x)$ and $\overline{\boldsymbol{\mu}(x)} \succ \overline{\boldsymbol{\nu}(x)}$ for all x. If $\boldsymbol{\mu} \succ \boldsymbol{\nu}$, then $\boldsymbol{\mu}(x)$ and $\overline{\boldsymbol{\mu}(x)}$ is a decreasing function of x.*

Lemma 5 implies that the order relation among prior information $\boldsymbol{\mu}$ is preserved in $\boldsymbol{\mu}(x)$ and posterior information $\overline{\boldsymbol{\mu}(x)}$. Furthermore, for same prior information $\boldsymbol{\mu}$, as a wage x of a job offer increases, posterior information $\overline{\boldsymbol{\mu}(x)}$ becomes worse.

Suppose a job search with prior information $\boldsymbol{\mu}$. Let $v_n(\boldsymbol{\mu}, x)$ be a maximal β-discount expected reward attainable when there are n stages to go and the currently available job offer is x $(0 < \beta < 1)$. The principle of optimality yields the recursive equation of $v_n(\boldsymbol{\mu}, x)$ as

$$v_n(\boldsymbol{\mu}, x) = \max\left\{u_n(x), c + \beta \int_0^\infty v_{n-1}(\overline{\boldsymbol{\mu}(x)}, y) dF_{\overline{\boldsymbol{\mu}(x)}}(y)\right\} \tag{4}$$

with $v_1(\boldsymbol{\mu}, x) = \mathrm{E}_{\boldsymbol{\mu}}[u_1(X)] = \int_0^\infty u_1(x) dF_{\boldsymbol{\mu}}(x)$. For the state space $[0, S]$, suppose that $p_S(t) \equiv I_S(t)$ as a transition probability and $X \equiv 0$ with probability 1, then it is possible to consider that the state S designates default. In this case, $I_S(t)$ is an indicator function of t. Put $S(\boldsymbol{\mu}, n) = \left\{x \,\middle|\, u_n(x) \geq c + \beta \int_0^\infty v_{n-1}(\overline{\boldsymbol{\mu}(x)}, y) dF_{\overline{\boldsymbol{\mu}(x)}}(y)\right\}$ and $C(\boldsymbol{\mu}, n) = S(\boldsymbol{\mu}, n)^c$, then $S(\boldsymbol{\mu}, n)$ and $C(\boldsymbol{\mu}, n)$ correspond to a stopping region and a continuance region, respectively, for this job search. Here we note that $u_n(x)$ is an increasing function of x, and $\overline{\boldsymbol{\mu}(x)}$ is a decreasing function of x, i.e. $\overline{\boldsymbol{\mu}(y)} \succeq \overline{\boldsymbol{\mu}(x)}$ for $x > y$. If $v_{n-1}(\overline{\boldsymbol{\mu}(x)}, z)$ is an increasing function of z and a decreasing function of $\boldsymbol{\mu}$, then Lemma 4 implies $\int_0^\infty v_{n-1}(\overline{\boldsymbol{\mu}(x)}, z) dF_{\overline{\boldsymbol{\mu}(x)}}(z) \geq \int_0^\infty v_{n-1}(\overline{\boldsymbol{\mu}(y)}, z) dF_{\overline{\boldsymbol{\mu}(y)}}(z)$ for $x > y$. Concerning two regions $S(\boldsymbol{\mu}, n)$ and $C(\boldsymbol{\mu}, n)$, Lemma 6 is obtained.

Lemma 6 *If $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$, then $S(\boldsymbol{\nu}, n) \subset S(\boldsymbol{\mu}, n)$ and $S(\boldsymbol{\mu}, n+1) \subset S(\boldsymbol{\mu}, n)$ for any n.*

Since $S(\boldsymbol{\mu}, n) \cup C(\boldsymbol{\mu}, n) = \Re_+$ and $S(\boldsymbol{\mu}, n) \cap C(\boldsymbol{\mu}, n) = \emptyset$ for any $\boldsymbol{\mu}$ and $n \geq 1$, this lemma implies $C(\boldsymbol{\mu}, n) \subset C(\boldsymbol{\nu}, n)$ and $C(\boldsymbol{\mu}, n) \subset C(\boldsymbol{\mu}, n+1)$. The value $v_n(\boldsymbol{\mu}, x)$ also has a following property.

Lemma 7 *If $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$, then $v_n(\boldsymbol{\mu}, x) \leq v_n(\boldsymbol{\nu}, x)$. If $x > y$, then $v_{n+1}(\boldsymbol{\mu}, x) \geq v_n(\boldsymbol{\mu}, x)$ and $v_n(\boldsymbol{\mu}, x) \geq v_n(\boldsymbol{\mu}, y)$.*

These properties are derived by the induction principle on n as Nakai [13] etc.

3.2 Probability Density after Some Additional Transitions under Uncertainty

Similarly to Section 2.2, we will consider a probability density on the state space after n additional transitions when the state changes according to a partially observable Markov process under Assumptions 1 and 2. Initially, we observe these probabilities leaving the decision and the learning procedure regarding the unobservable state. When prior information is $\boldsymbol{\mu}$, let $\overline{\boldsymbol{P}}_{\boldsymbol{\mu},m}$ be a set function of the probability densities on the state space after m additional transitions. As the initial condition, if $m = 1$, then $\overline{\boldsymbol{P}}_{\boldsymbol{\mu},1} = (\overline{\boldsymbol{P}}_{\boldsymbol{\mu},1}(t))_{t \in [0,S]}$ and $\overline{\boldsymbol{P}}_1(\boldsymbol{\mu})_t = \int_0^S \mu(s) p_s(t) ds = \langle \boldsymbol{\mu}, \boldsymbol{P} \rangle(t)$. Similarly to the previous section, we use a notation $\langle \boldsymbol{\mu}, \boldsymbol{P} \rangle = (\langle \boldsymbol{\mu}, \boldsymbol{P} \rangle(t))_{t \in [0,S]}$ and $\langle \boldsymbol{\mu}, \boldsymbol{P} \rangle(t) = \int_0^S \mu(s) p_s(t) ds$ for $\boldsymbol{\mu} = (\mu(s))_{s \in [0,S]}$ and $\boldsymbol{P} = (p_s(t))_{s,t \in [0,S]}$. It is easy to show that $\langle \langle \boldsymbol{\mu}, \boldsymbol{P} \rangle, \boldsymbol{Q} \rangle = \langle \boldsymbol{\mu}, \langle \boldsymbol{P}, \boldsymbol{Q} \rangle \rangle$. We also define $\boldsymbol{P}^n$ as $\boldsymbol{P}^1 = \boldsymbol{P}$ and $\boldsymbol{P}^n = \langle \boldsymbol{P}, \boldsymbol{P}^{n-1} \rangle$ for $\boldsymbol{P} = (p_s(t))_{s,t \in [0,S]}$. It is also possible to express that

$\overline{\boldsymbol{\mu}} = \langle \boldsymbol{\mu}, \boldsymbol{P} \rangle$ and $\overline{\boldsymbol{\mu}(x)} = \langle \boldsymbol{\mu}(x), \boldsymbol{P} \rangle$. Using this relation yields $\overline{\boldsymbol{P}}_{\boldsymbol{\mu},2} = \overline{\boldsymbol{P}}_{\overline{\boldsymbol{\mu}},1} = \langle \overline{\boldsymbol{\mu}}, \boldsymbol{P} \rangle = \langle \boldsymbol{\mu}, \boldsymbol{P}^2 \rangle$ for $m = 2$, and, therefore, $\overline{\boldsymbol{P}}_{\boldsymbol{\mu},m}$ satisfies the recursive equation as

$$\overline{\boldsymbol{P}}_{\boldsymbol{\mu},m} = \overline{\boldsymbol{P}}_{\overline{\boldsymbol{\mu}},m-1} = \overline{\boldsymbol{P}}_{\langle \boldsymbol{\mu}, \boldsymbol{P} \rangle,m-1} = \langle \langle \boldsymbol{\mu}, \boldsymbol{P} \rangle, \boldsymbol{P}^{m-1} \rangle = \langle \boldsymbol{\mu}, \boldsymbol{P}^m \rangle. \tag{5}$$

Since $\boldsymbol{P}$ is TP_2, the induction principle on m implies $\overline{\boldsymbol{P}}_m(\boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{P}^m \rangle$ is also TP_2. We can obtain the following property.

Lemma 8 *If $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$ and $\boldsymbol{P}$ is TP_2 ($\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{S}$), then $\langle \boldsymbol{\mu}, \boldsymbol{P} \rangle \succeq \langle \boldsymbol{\nu}, \boldsymbol{P} \rangle$.*

By Equation (5), Lemma 8 yields the next proposition.

Proposition 2 *If $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$ ($\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{S}$), then $\overline{\boldsymbol{P}}_{\boldsymbol{\mu},m} \succeq \overline{\boldsymbol{P}}_{\boldsymbol{\nu},m}$.*

Next, we observe similar probabilities leaving the decision out of consideration, i.e. we give consideration to the learning procedure by using a wage of a currently available job offer. When prior information is $\boldsymbol{\mu}$, we first observe a sample depending on the current state and improve information about it by using the Bayes' theorem. After that, we see time moving forward by one unit and thus this process will make a transition to a new state. Hence, whenever we say prior information is $\boldsymbol{\mu}$, the transition to the current state have been finished. When prior information is $\boldsymbol{\mu}$, let $\widehat{P}_{\boldsymbol{\mu},m}(t)$ be a probability density on the state space after m additional transitions ($t \in [0, S]$), and $\widehat{\boldsymbol{P}}_{\boldsymbol{\mu},m} = (\widehat{P}_{\boldsymbol{\mu},m}(t))_{t\in[0,S]}$.

For a set function $\boldsymbol{u}(x) = (u(x,t))_{t\in[0,S]}$, if $\int_a^b u(x,s)dF(x)$ exists for any s ($s \in [0, S]$), then we use a notation $\int_a^b \boldsymbol{u}(x)dF(x) = \left(\int_a^b u(x,t)dF(x)\right)_{t\in[0,S]}$ for the simplicity sake.

When prior information is $\boldsymbol{\mu}$, since $\widehat{\boldsymbol{P}}_{\boldsymbol{\mu},1} = (\widehat{P}_{\boldsymbol{\mu},1}(t))_{t\in[0,S]}$ is a probability distribution on the state space at the next stage, we have, $\widehat{\boldsymbol{P}}_{\boldsymbol{\mu},1} = \int_0^\infty \langle \boldsymbol{\mu}(x), \boldsymbol{P} \rangle dF_{\boldsymbol{\mu}}(x) = \int_0^\infty \overline{\boldsymbol{\mu}(x)} dF_{\boldsymbol{\mu}}(x)$. If prior information at some stage is $\boldsymbol{\mu}$ and a wage x of a current job offer is observed, posterior information will be $\overline{\boldsymbol{\mu}(x)}$ at the next stage. Similarly to the case of $m = 1$, $\widehat{\boldsymbol{P}}_{\boldsymbol{\mu},2} = \int_0^\infty \widehat{\boldsymbol{P}}_{\overline{\boldsymbol{\mu}(x)},1} dF_{\boldsymbol{\mu}}(x)$ where $\widehat{\boldsymbol{P}}_{\boldsymbol{\mu},2}$ is a probability distribution on the state space after 2 stages. Similarly to these cases, when prior information is $\boldsymbol{\mu}$, $\widehat{\boldsymbol{P}}_{\boldsymbol{\mu},m}$ satisfies Equation (6), since $\widehat{\boldsymbol{P}}_{\boldsymbol{\mu},m}$ is a probability distribution on the state space after m stages.

$$\widehat{\boldsymbol{P}}_{\boldsymbol{\mu},m} = \int_0^\infty \widehat{\boldsymbol{P}}_{\overline{\boldsymbol{\mu}(x)},m-1} dF_{\boldsymbol{\mu}}(x), \tag{6}$$

where $\widehat{\boldsymbol{P}}_{\boldsymbol{\mu},1} = \int_0^\infty \overline{\boldsymbol{\mu}(x)} dF_{\boldsymbol{\mu}}(x)$. To obtain the properties about it, we introduce an order.

Definition 4 *For two non-negative set functions $\boldsymbol{g}(x)$ and $\boldsymbol{h}(x)$ of x ($\boldsymbol{g}(x) = (g(x,s))_{s\in[0,S]}$ and $\boldsymbol{h}(x) = (h(x,s))_{s\in[0,S]}$), if $g(x,t)\,h(x,s) \geq g(x,s)\,h(x,t)$ for any s and t ($s \leq t$, $s,t \in [0,S]$), then $\boldsymbol{g}(x)$ is said to be greater than $\boldsymbol{h}(x)$ in the sense of TP_2, or simply $\boldsymbol{g}(x) \succeq \boldsymbol{h}(x)$.*

Concerning this definition, next properties are obtained under Assumptions 1 and 2.

Lemma 9 *If non-negative functions $\boldsymbol{g}(x) = (g(x,s))_{s\in[0,S]}$ and $\boldsymbol{h}(x) = (h(x,s))_{s\in[0,S]}$ are decreasing with respect to x and $\boldsymbol{g}(x) \succeq \boldsymbol{h}(x)$, then $\int_0^\infty \boldsymbol{g}(x)dF(x) \succeq \int_0^\infty \boldsymbol{h}(x)dF(x)$.*

Lemma 10 *If* $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$ *and* $\boldsymbol{h}(x)$ *is a decreasing function of* x*, then* $\int_0^\infty \boldsymbol{h}(x)dF_{\boldsymbol{\mu}}(x) \succeq \int_0^\infty \boldsymbol{h}(x)dF_{\boldsymbol{\nu}}(x)$.

Next two properties are derived from these lemmas.

Corollary 2 *If* $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$*, then* $\int_0^\infty \overline{\boldsymbol{\mu}(x)}dF(x) \succeq \int_0^\infty \overline{\boldsymbol{\nu}(x)}dF(x)$.

Corollary 3 *If* $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$*, then* $\int_0^\infty \boldsymbol{h}(\boldsymbol{\mu}, x)dF_{\boldsymbol{\mu}}(x) \succeq \int_0^\infty \boldsymbol{h}(\boldsymbol{\nu}, x)dF_{\boldsymbol{\nu}}(x)$*, where* $\boldsymbol{h}(\boldsymbol{\mu}, x)$ *is an increasing function of* $\boldsymbol{\mu}$ *and non-increasing function of* x.

If $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$, then $\overline{\boldsymbol{\mu}} \succeq \overline{\boldsymbol{\nu}}$ and $\overline{\boldsymbol{\mu}(x)} \succeq \overline{\boldsymbol{\nu}(x)}$ by Lemma 5, and, therefore, $\widehat{\boldsymbol{P}}_{\boldsymbol{\mu},m}$ has the following property.

Proposition 3 *If* $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$*, then* $\widehat{\boldsymbol{P}}_{\boldsymbol{\mu},m}$ *is an increasing function of* $\boldsymbol{\mu}$*, i.e.* $\widehat{\boldsymbol{P}}_{\boldsymbol{\mu},m} \succeq \widehat{\boldsymbol{P}}_{\boldsymbol{\nu},m}$.

Proof: We employ the induction principle on m. When $m = 1$, if $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$, then $\widehat{\boldsymbol{P}}_{\boldsymbol{\mu},1} = \int_0^\infty \overline{\boldsymbol{\mu}(x)}dF_{\boldsymbol{\mu}}(x)$ and Corollary 2 imply $\widehat{\boldsymbol{P}}_{\boldsymbol{\mu},1} \succeq \widehat{\boldsymbol{P}}_{\boldsymbol{\nu},1}$.

Since $\overline{\boldsymbol{\mu}(x)} \succeq \overline{\boldsymbol{\nu}(x)}$, $\widehat{\boldsymbol{P}}_{\overline{\boldsymbol{\mu}(x)},1} \succeq \widehat{\boldsymbol{P}}_{\overline{\boldsymbol{\nu}(x)},1}$. Corollary 3 implies $\widehat{\boldsymbol{P}}_{\boldsymbol{\mu},2} = \int_0^\infty \widehat{\boldsymbol{P}}_{\overline{\boldsymbol{\mu}(x)},1}dF_{\boldsymbol{\mu}}(x)$ $\succeq \int_0^\infty \widehat{\boldsymbol{P}}_{\overline{\boldsymbol{\nu}(x)},1}dF_{\boldsymbol{\mu}}(x) = \widehat{\boldsymbol{P}}_{\boldsymbol{\nu},2}$, and, therefore, $\widehat{\boldsymbol{P}}_{\boldsymbol{\mu},2} \succeq \widehat{\boldsymbol{P}}_{\boldsymbol{\nu},2}$.

By the induction assumption, if $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$, then $\widehat{\boldsymbol{P}}_{\boldsymbol{\mu},m-1} \succeq \widehat{\boldsymbol{P}}_{\boldsymbol{\nu},m-1}$. Since $\overline{\boldsymbol{\mu}(x)} \succeq \overline{\boldsymbol{\nu}(x)}$, we also have $\widehat{\boldsymbol{P}}_{\overline{\boldsymbol{\mu}(x)},m-1} \succeq \widehat{\boldsymbol{P}}_{\overline{\boldsymbol{\nu}(x)},m-1}$. Corollary 3 implies

$$\begin{aligned}\widehat{\boldsymbol{P}}_{\boldsymbol{\mu},m} &= \int_0^\infty \widehat{\boldsymbol{P}}_{\overline{\boldsymbol{\mu}(x)},m-1}dF_{\boldsymbol{\mu}}(x) \succeq \int_0^\infty \widehat{\boldsymbol{P}}_{\overline{\boldsymbol{\nu}(x)},m-1}dF_{\boldsymbol{\mu}}(x) \\ &\succeq \int_0^\infty \widehat{\boldsymbol{P}}_{\overline{\boldsymbol{\nu}(x)},m-1}dF_{\boldsymbol{\nu}}(x) = \widehat{\boldsymbol{P}}_{\boldsymbol{\nu},m},\end{aligned}$$

and the proof is completed. □

Finally, we will consider the similar probabilities by giving consideration to the learning procedure and the optimal policy for this job search. When prior information is $\boldsymbol{\mu}$, after observing a wage x of the currently available job offer as a sample, information is improved as $\boldsymbol{\mu}(x)$ and the decision maker decides whether to take this job offer or not. If she does not accept this job offer, the time moves forward by one unit and this process will make a transition to a new state according to $\boldsymbol{P}$, and information becomes $\overline{\boldsymbol{\mu}(x)}$.

When there are n jobs remain, suppose that prior information is $\boldsymbol{\mu}$ and a wage of the currently available job offer is not observed at this time. Let $(\widetilde{P}_{\boldsymbol{\mu},n,m}(t))_{t\in[0,S]}$ be a probability density on the state space after m additional transitions by employing the optimal policy $(t \in [0, S], n, m = 1, 2, \cdots, m \le n)$.

For these $\widetilde{\boldsymbol{P}}_{\boldsymbol{\mu},n,m} = (\widetilde{P}_{\boldsymbol{\mu},n,m}(t))_{t\in[0,S]}$, we initially observe the case where $m = 1$. Suppose that a wage of the currently available job offer is x and this job is not accepted. In this time, let improved information about the unobservable state be $\boldsymbol{\mu}^* = (\mu^*(s))_{s\in[0,S]}$. Let $(\widetilde{P}'_{\boldsymbol{\mu}^*,n,1}(t))_{t\in[0,S]}$ be a probability density on the state space after making a transition $(s \in [0, S])$, then $\widetilde{\boldsymbol{P}}'_{\boldsymbol{\mu}^*,n,1} = (\widetilde{P}'_{\boldsymbol{\mu}^*,n,1}(t))_{t\in[0,S]}$ satisfies that $\widetilde{P}'_{\boldsymbol{\mu}^*,n,1}(t) =$

$\int_0^S \mu^*(s)p_s(t)ds$ and $\widetilde{\boldsymbol{P}}'_{\boldsymbol{\mu}^*,n,1} = \langle \boldsymbol{\mu}^*, \boldsymbol{P}\rangle = \overline{\boldsymbol{\mu}^*}$ since the time period moves forward by one unit and this process will make a transition to a new state whenever $x \in C(\boldsymbol{\mu}, n)$. When a wage x is observed as a sample, since improved information can be obtained as $\boldsymbol{\mu}(x)$, we have $\widetilde{P}_{\boldsymbol{\mu},n,1}(t) = \int_{C(\boldsymbol{\mu},n)} \widetilde{P}'_{\boldsymbol{\mu}(x),n,1}(t)dF_{\boldsymbol{\mu}}(x) = \int_{C(\boldsymbol{\mu},n)} dF_{\boldsymbol{\mu}}(x) \int_0^S \mu(x)_s p_s(t)ds$. Thus implies $\widetilde{\boldsymbol{P}}_{\boldsymbol{\mu},n,1} = \int_{C(\boldsymbol{\mu},n)} \overline{\boldsymbol{\mu}(x)} dF_{\boldsymbol{\mu}}(x)$.

Suppose that prior information at the n-th stage is $\boldsymbol{\mu}$, let $(\widetilde{P}_{\boldsymbol{\mu},n,m}(t))_{t\in[0,S]}$ be a probability density on the state space after m additional transitions by employing the optimal policy ($t \in [0, S], n, m = 1, 2, \cdots, m \le n$). In this case, a new job offer will appear with a wage depending on the new state, and then make a decision whether to get it or not. Since this job search continues at least one more stage when $x \in C(\boldsymbol{\mu}, n)$, it is easy to show that $(\widetilde{P}_{\boldsymbol{\mu},n,m}(t))_{t\in[0,S]}$ satisfies the recursive equation as

$$\widetilde{P}_{\boldsymbol{\mu},n,m}(t) = \int_{C(\boldsymbol{\mu},n)} \widetilde{P}_{\overline{\boldsymbol{\mu}(x)},n-1,m-1}(t)dF_{\boldsymbol{\mu}}(x), \tag{7}$$

and $\widetilde{\boldsymbol{P}}_{\boldsymbol{\mu},n,m} = \left(\widetilde{P}_{\boldsymbol{\mu},n,m}(t)\right)_{t\in[0,S]}$. Note that $\int_0^S \widetilde{P}_{\boldsymbol{\mu},n,m}(t)dt \le 1$ because $\int_{S(\boldsymbol{\mu},n)} dF_{\boldsymbol{\mu}}(x)$ is a probability to get a job offer at this stage. Furthermore, $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$ implies $C(\boldsymbol{\mu}, n) \subset C(\boldsymbol{\nu}, n)$, i.e. the probability to continue at least one more stage decreases as $\boldsymbol{\mu}$ increases, and, on the other hand, the probability to make a transition into the lower class increases as $\boldsymbol{\mu}$ increases. When the state is directly observable, Proposition 1 yields that $\overline{\boldsymbol{P}}_{n,m} = (\overline{\boldsymbol{P}}_{s,n,m})_{s\in[0,S]}$ is TP_2. On the contrary to this case, it is difficult to observe such a property for $\widetilde{\boldsymbol{P}}_{\boldsymbol{\mu},n,m}$, since the probabilities $(\widetilde{P}_{\overline{\boldsymbol{\mu}(x)},n-1,m-1}(t))_{t\in[0,S]}$ varies according to the wage x of a new job offer.

4 Total Positivity of Order Two (TP_2) Finally, we summarize the proofs of the lemmas and corollaries concerning TP_2 used in this paper. By using TP_2, we introduced some definitions (Definitions 1, 2, and 3), and these properties are obtained under Assumptions 1 and 2. Lemmas 4, 5, 6 and 7 are obtained in Nakai [13], [14], and, therefore, we omit the proofs of these properties. Concerning the TP_2, Kijima [5] and Kijima and Ohnishi [6, 7] investigate the properties of TP_2 for the financial optimization.

Proof of Lemma 3: Put $\langle \boldsymbol{P}, \boldsymbol{Q}\rangle = (r_s(t))_{s,t\in[0,S]}$. Since $\boldsymbol{P} = (p_s(t))_{s,t\in[0,S]}$ and $\boldsymbol{Q} = (q_s(t))_{s,t\in[0,S]}$ are TP_2, $p_s(x)\,p_t(y) - p_t(x)\,p_s(y) \ge 0$ and $q_x(u)\,q_y(v) - q_y(u)\,q_x(v) \ge 0$ for any s, t, u, v where $u \le v, s \le t$ and $x < y$.

$$\begin{aligned} & r_s(u)r_t(v) - r_s(v)r_t(u) \\ & \qquad = \int_0^S dx \int_x^S (p_s(y)p_t(x) - p_s(x)p_t(y))(q_x(v)q_y(u) - q_x(u)q_y(v))dy \ge 0, \end{aligned}$$

and, therefore, $\langle \boldsymbol{P}, \boldsymbol{Q}\rangle = \left(\int_0^S p_s(u)q_u(t)du\right)_{s,t\in[0,S]}$ is TP_2.□

Proof of Corollary 1: Since $\boldsymbol{P} = (p_s(t))_{s,t\in[0,S]}$ is TP_2, $p_s(u)\,p_t(v) - p_t(u)\,p_s(v) \ge 0$ ($s < t, u < v$). For $\boldsymbol{Q} = (d(s)\boldsymbol{P}_s)_{s\in[0,S]} = (q_s(t))_{s,t\in[0,S]}$, $q_s(u) = d(s)p_s(u)$ and $q_t(v) =$

$d(t)p_t(v)$, and, therefore, $q_s(u)q_t(v) - q_s(v)q_t(u) = d(s)d(t)(p_t(v)p_s(u) - p_s(v)p_t(u)) \geq 0$, for any s, t, u, v where $u \leq v$ and $s \leq t$. $\Box$

Proof of Lemma 8: Since $\langle \boldsymbol{\mu}, \boldsymbol{P} \rangle = (\langle \boldsymbol{\mu}, \boldsymbol{P} \rangle_t)_{t\in[0,S]}$ in which $\langle \boldsymbol{\mu}, \boldsymbol{P} \rangle_t = \int_0^S \mu(s)p_s(t)ds$ for $\boldsymbol{\mu} = (\mu(s))_{s\in[0,S]}$ and $\boldsymbol{P} = (p_s(t))_{s,t\in[0,S]}$, the definition of $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$ and the fact that $\boldsymbol{P}$ is TP_2 imply

$$\begin{aligned} &\langle \boldsymbol{\mu}, \boldsymbol{P} \rangle_t \, \langle \boldsymbol{\nu}, \boldsymbol{P} \rangle_s - \langle \boldsymbol{\mu}, \boldsymbol{P} \rangle_s \, \langle \boldsymbol{\nu}, \boldsymbol{P} \rangle_t \\ &\qquad = \int_0^S \int_x^S (\mu(x)\nu(y) - \mu(y)\nu(x))(p_x(t)p_y(s) - p_x(s)p_y(t)) \geq 0 \end{aligned}$$

for $\langle \boldsymbol{\mu}, \boldsymbol{P} \rangle = (\langle \boldsymbol{\mu}, \boldsymbol{P} \rangle_s)_{s\in[0,S]}$ and $\langle \boldsymbol{\nu}, \boldsymbol{P} \rangle = (\langle \boldsymbol{\nu}, \boldsymbol{P} \rangle_s)_{s\in[0,S]}$. The inequality comes from the fact that $p_x(s)\,p_y(t) - p_y(s)\,p_x(t) \geq 0$ $(s < t, u < v)$ and $\mu(x)\nu(y) - \mu(y)\nu(x) \geq 0$. This completes the proof.$\Box$

Proof of Lemma 9: Note that $\boldsymbol{g}(x) \succeq \boldsymbol{h}(x)$ implies $g(x,t)\,h(x,s) \geq g(x,s)\,h(x,t)$ for any s and $t\,(s \leq t,\ s,t \in [0,S])$. It is also that $\boldsymbol{g}(y) \succeq \boldsymbol{g}(x)$ for $y < x$ implies $g(y,t)\,g(x,s) \geq g(y,s)\,g(x,t)$ for any t and $s\,(s \leq t,\ s,t \in [0,S])$. It is also that $\boldsymbol{h}(y) \succeq \boldsymbol{h}(x)$ for $y < x$ implies $h(y,t)\,h(x,s) \geq h(y,s)\,h(x,t)$ for any t and $s\,(s \leq t,\ s,t \in [0,S])$.

The assumptions of this lemma yield

$$\begin{aligned} &\int_0^\infty g(x,t)dF(x) \int_0^\infty h(y,s)dF(y) - \int_0^\infty g(x,s)dF(x) \int_0^\infty h(y,t)dF(y) \\ &= \int_0^\infty dF(x) \int_0^x (g(x,t)h(y,s) - g(x,s)h(y,t))\, dF(y) \\ &\quad + \int_0^\infty dF(x) \int_0^x (g(y,t)h(x,s) - g(y,s)h(x,t))\, dF(y) \\ &\geq \int_0^\infty g(x,s)h(x,t)dF(x) \int_0^x \left(\frac{h(y,s)}{h(x,s)} - \frac{h(y,t)}{h(x,t)} \right) dF(y) \\ &\quad + \int_0^\infty g(x,s)h(x,t)dF(x) \int_0^x \left(\frac{g(y,t)}{g(x,t)} - \frac{g(y,s)}{g(x,s)} \right) dF(y) \geq 0 \end{aligned}$$

for any $s,t\,(s \leq t,\ s,t \in [0,S])$ in which $\int_0^\infty \boldsymbol{g}(x)dF(x) = \left(\int_0^\infty g_s(x)dF(x) \right)_{s\in[0,S]}$ and $\int_0^\infty \boldsymbol{h}(x)dF(x) = \left(\int_0^\infty h_s(x)dF(x) \right)_{s\in[0,S]}$. These inequalities come from the facts that $g(x,t)h(x,s) \geq g(x,s)h(x,t)$, $h(x,s)g(x,t) \geq g(x,s)h(x,t)$, $g(y,t)\,g(x,s) \geq g(y,s)\,g(x,t)$ and $h(y,t)\,h(x,s) \geq h(y,s)\,h(x,t)$.$\Box$

Proof of Corollary 2: The inequality $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$ implies $\overline{\boldsymbol{\mu}(x)} \succeq \overline{\boldsymbol{\nu}(x)}$, and $\overline{\boldsymbol{\mu}(x)}$ is a decreasing function of x. Lemma 9 yields this corollary.$\Box$

Proof of Lemma 10: By definition of the weighted distribution functions, $dF_{\boldsymbol{\mu}}(x) =$

$\int_0^S \mu(s)dF_s(x)ds$ and $dF_{\boldsymbol{\nu}}(x) = \int_0^S \nu(s)dF_s(x)ds$. Note that

$$\begin{aligned}
&\int_0^\infty h(x,t)dF_{\boldsymbol{\mu}}(x)\int_0^\infty h(y,s)dF_{\boldsymbol{\nu}}(y) - \int_0^\infty h(x,s)dF_{\boldsymbol{\mu}}(x)\int_0^\infty h(y,t)dF_{\boldsymbol{\nu}}(y)\\
&= \int_0^\infty dF_{\boldsymbol{\mu}}(x)\int_0^x (h(x,t)h(y,s)-h(x,s)h(y,t))dF_{\boldsymbol{\nu}}(y)\\
&\quad + \int_0^\infty dF_{\boldsymbol{\nu}}(x)\int_0^x (h(y,t)h(x,s)-h(y,s)h(x,t))dF_{\boldsymbol{\mu}}(y),
\end{aligned}$$

and a simple calculation yields

$$\begin{aligned}
&(h(x,t)h(y,s)-h(x,s)h(y,t))dF_{\boldsymbol{\mu}}(x)dF_{\boldsymbol{\nu}}(y)\\
&\qquad +(h(y,t)h(x,s)-h(y,s)h(x,t))dF_{\boldsymbol{\nu}}(x)dF_{\boldsymbol{\mu}}(y)\\
&= (h(x,t)h(y,s)-h(x,s)h(y,t))\\
&\qquad \times \int_0^S\int_u^S (\mu(u)\nu(v)-\mu(v)\nu(u))(dF_u(x)dF_v(y)-dF_v(x)dF_u(y)) \geq 0,
\end{aligned}$$

for $x > y$. The last inequality is derived from the following three fact that

1. $X_u \succeq X_v$ yields
$$dF_v(x)dF_u(y) - dF_u(x)dF_v(y) = (f_v(x)f_u(y) - f_u(x)f_v(y))dxdy \leq 0$$
for any $u, v\,(u \leq v, u, v \in [0,S])$, and
2. when $x > y$, $\boldsymbol{h}(y) \succeq \boldsymbol{h}(x)$ yields $h(y,t)\,h(x,s) \geq h(y,s)\,h(x,t)$ for any $s,t\,(s \leq t, s,t \in [0,S])$, and
3. $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$ yields $\mu(v)\,\nu(u) \geq \mu(u)\,\nu(v)$ for any $u, v\,(u \leq v, u, v \in [0,S])$.

For $t > s$, these facts imply the inequality

$$\int_0^\infty h(x,t)dF_{\boldsymbol{\mu}}(x)\int_0^\infty h(y,s)dF_{\boldsymbol{\nu}}(y) - \int_0^\infty h(x,s)dF_{\boldsymbol{\mu}}(x)\int_0^\infty h(y,t)dF_{\boldsymbol{\nu}}(y) \geq 0,$$

and this completes the proof. □

If set valued functions $\boldsymbol{h}(x)$ and $\boldsymbol{g}(x)$ are decreasing with respect to x, then Lemmas 9 and 10 imply next corollary since

$$\int_0^\infty \boldsymbol{g}(x)dF_{\boldsymbol{\mu}}(x) \succeq \int_0^\infty \boldsymbol{g}(x)dF_{\boldsymbol{\nu}}(x) \succeq \int_0^\infty \boldsymbol{h}(x)dF_{\boldsymbol{\nu}}(x).$$

Corollary 4 *If $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$ and $\boldsymbol{g}(x) \succeq \boldsymbol{h}(x)$, then $\int_0^\infty \boldsymbol{g}(x)dF_{\boldsymbol{\mu}}(x) \succeq \int_0^\infty \boldsymbol{h}(x)dF_{\boldsymbol{\nu}}(x)$ under Assumptions* 1 *and* 2 *($\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{S}$).*

Proof of Corollary 3: If $\boldsymbol{\mu} \succeq \boldsymbol{\nu}$, then $\boldsymbol{\mu}(x) \succeq \boldsymbol{\nu}(x)$. Lemma 4 and Corollary 4 imply

$$\int_0^\infty \boldsymbol{h}(\boldsymbol{\mu},x)dF_{\boldsymbol{\mu}}(x) \succeq \int_0^\infty \boldsymbol{h}(\boldsymbol{\mu},x)dF_{\boldsymbol{\nu}}(x) \succeq \int_0^\infty \boldsymbol{h}(\boldsymbol{\nu},x)dF_{\boldsymbol{\nu}}(x).\Box$$

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