A MULTIDIMENSIONAL INTEGRATION

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Abstract. We proposed in [6] a multiple integration on a multidimensional interval, named the (LA) integral in the strong sense, which reduces to the special Denjoy integral in the one-dimensional case (cf. [5]). In this paper, we show that in the two-dimensional case, Fubini's theorem holds for the (LA) integral in the strong sense in addition to the following three statements which have been proved in [6]: The indefinite integral of an (LA) integrable function in the strong sense is continuous; the derivative of a finitely additive interval function which is derivable in the strong sense at every point is (LA) integrable in the strong sense; and the indefinite integral of an (LA) integrable function f in the strong sense is, at almost all points p, derivable in the ordinary sense and its derivative coincides with $f(p)$.

In [6], we defined a multiple integration for a real valued function on a multidimensional interval, named the (LA) integral in the strong sense or the strong (LA) integral. In this paper, we discuss the statements shown in [6] which are true in all multidimensional cases, in more detail (Theorems 1, 2 and 3), and show that, in the two-dimensional case, Fubini's theorem holds for the strong (LA) integral (Theorem 6). Theorem 2 is already proved in [6], but in this paper, we show a direct proof of the theorem (Proposition 9).

In general, when a function f is strongly (LA) integrable on an n-dimensional interval R $(n \geq 2)$, for a variable taken arbitrarily if, fixing a point p in the projection of the interval R into the $(n-1)$-dimensional space consisting of the other variables, we consider the function f as a function of the variable taken first, then the function is strongly (LA) integrable for almost all p in the projection of R (Theorem 4).

This paper is a correction of the study for the multiple integral proposed in [4], named the (D) integral (we found recently an error in the study (precisely, in the proof of [4, Théorèm 6])).

We remark that we have defined in the paper [9] a multidimensional multiple integration, named the (D_0) integral, whose integral reduces to the special Denjoy integral in the one-dimensional case and is expressible as the iterated integral of the one-dimensional (D_0) integral.

Throughout this paper, we refer to the terminology and notations indicated in the paper [9]. In this paper, parts of the proof are omitted. The proof of the parts omitted is leaved to the reader to see the corresponding parts in [6] or [9].

We denote the n-dimensional Euclidean space by E_n. A finite system of intervals is called *non-overlapping* if they have mutually no common inner points. An *interval function* in an interval $R_0 \subset E_n$ means a function defined on the family of all sub-intervals of R_0. A *finitely additive interval function,* or in short, an *additive interval function,* in R_0 means an interval function F such that $F(I_1 \cup I_2) = F(I_1) + F(I_2)$ for any pair of non-overlapping intervals

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I_1 and I_2 whose union is an interval. A finite system of intervals I_i $(i = 1, 2, \dots, i_0)$ in E_n is called an *elementary system* if $I_i \cap I_{i'} = \emptyset$ for $i \neq i'$, sometimes the elementary system is denoted by $S : \{I_i\ (i = 1, 2, \dots, i_0)\}$. Throughout this paper, μ_n denotes the Lebesgue measure on E_n, sometimes, the Lebesgue measure of an interval I in E_n is denoted by $|I|$. Sometimes, for an elementary system $S : \{I_i\ (i = 1, 2, \dots, i_0)\}$, S denotes the set $\cup_{i=1}^{i_0} I_i$ and $|S|$ denotes the Lebesgue measure $\sum_{i=1}^{i_0} |I_i|$, and when F is a finitely additive interval function in an interval containing S, $F(S)$ denotes $\sum_{i=1}^{i_0} F(I_i)$. Measure means Lebesgue measure. For a set A in E_n, $\overline{A}$ denotes the closure of A in E_n and A° denotes the interior of A in E_n. For an interval $I = [a_1, b_1; a_2, b_2; \dots ; a_n, b_n]$, *norm*$(I)$ denotes $\max\{b_i - a_i : i = 1, 2, \dots, n\}$, and $d(I)$ denotes $\sup\{\mathrm{dist}(x, y) : x, y \in I\}$. For a closed set F in the one-dimensional Euclidean space E_1, an interval I in E_1 is said to be *contiguous to* F if the both end-points of I belong to F and $I^\circ \cap F = \emptyset$. N denotes the set $\{1, 2, \dots\}$. Sometimes, the empty set is treated as a measurable set or a closed set.

Let the Euclidean space E_n be the product space $E_n = E_{n_1} \times E_{n_2}$ of Euclidean spaces E_{n_1} and E_{n_2} and A a subset of E_n. Then $\underset{E_{n_1}}{\mathrm{proj}_x}(A)$ denotes the projection of A into E_{n_1} and $\underset{E_{n_2}}{\mathrm{proj}_y}(A)$ the projection of A into E_{n_2}, in particular, when $n = 2$ and $n_1 = n_2 = 1$, $\mathrm{proj}_x(A)$ denotes $\underset{E_{n_1}}{\mathrm{proj}_x}(A)$, and $\mathrm{proj}_y(A)$ denotes $\underset{E_{n_2}}{\mathrm{proj}_y}(A)$. For a point $p \in E_{n_1}$, A^p denotes the set $\{(p, q) : (p, q) \in A,\ q \in E_{n_2}\}$ and for a point $q \in E_{n_2}$, A^q denotes the set $\{(p, q) : (p, q) \in A,\ p \in E_{n_1}\}$.

§1 Multidimensional integration

Definition 1 ([6, Definition 5]). Let R_0 be an interval in the n_0-dimensional Euclidean space E_{n_0} and f a real valued measurable function on R_0. The function f is said to be (LA) *integrable in the strong sense* or *strongly* (LA) *integrable* on R_0 if there exist a finitely additive interval function F in R_0, a nondecreasing sequence of measurable sets M_n $(n = 1, 2, \dots)$ such that $M_n \subset R_0$ and $\cup_{n=1}^{\infty} M_n = R_0$, and a nondecreasing sequence of closed sets F_n $(n = 1, 2, \dots)$ such that $F_n \subset M_n$ and $\mu_{n_0}(R_0 - \cup_{n=1}^{\infty} F_n) = 0$, satisfying the following two conditions (1) and (2):

(1) The function f is Lebesgue integrable on F_n for each $n \in N$;

(2) Given $n \in N$ and $\varepsilon > 0$, there exists a $\delta(n, \varepsilon) > 0$ for which the following holds; if I_i $(i = 1, 2, \dots, i_0)$ is a finite system of non-overlapping intervals in R_0 such that

(2.1) $I_i \cap M_n \neq \emptyset$ for $i = 1, 2, \dots, i_0$;

(2.2) $\mu_{n_0}(\cup_{i=1}^{i_0} I_i - M_n) < \delta(n, \varepsilon)$;

(2.3) *norm*$(I_i) < 1/n$ for $i = 1, 2, \dots, i_0$,

then the following inequality holds:

$$\left| \sum_{i=1}^{i_0} F(I_i) - \sum_{i=1}^{i_0} (L) \int_{I_i \cap F_n} f(p) dp \right| < \varepsilon.$$

In this case, $F(R_0)$ is called the (LA) *integral in the strong sense* or the *strong* (LA) *integral,* of $f(p)$ on R_0, and is denoted by

$$(SLA)\int_{R_0} f(p)dp \text{ or } (SLA)\int_{R_0} f(x_1, x_2, \dots, x_{n_0})d(x_1, x_2, \dots, x_{n_0}).$$

In this definition, it does not arise any confusion by Proposition 2 below. The sequence M_n $(n = 1, 2, \dots)$ is called a *characteristic sequence* of the strong (LA) integral and the sequence F_n $(n = 1, 2, \dots)$ a *fundamental sequence* of the strong (LA) integral. In the case when we can choose $\{M_n\}_{n=1}^{\infty}$ and $\{F_n\}_{n=1}^{\infty}$ so that $M_n = F_n$ for every n, the function f is said to be *strongly* (LA^*) *integrable* on R_0.

We remark that in Definition 1 we can suppose that

$$\delta(n, \varepsilon) \geq \delta(m, \varepsilon) \text{ for } m > n, \text{and } \delta(n, \varepsilon) \geq \delta(n, \varepsilon') \text{ for } \varepsilon > \varepsilon'. \tag{1°}$$

When no confusion is possible, we also use the symbols $(SLA)\int_{R_0} f dp$, $(SLA)\int_{R_0} f$, $(L)\int_A f$, etc.

Remark 1. In the definition of the strong (LA) integral we can replace the condition (2.3) with the condition:

(2.3′) $d(I_i) < 1/n$ for $i = 1, 2, \dots, i_0$.

Because, we have $norm(I) \leq d(I) \leq (n_0)^{1/2}\ norm(I)$ for any interval I in E_{n_0}. First, suppose that, if, for $n \in N$ and $\varepsilon > 0$, a system of non-overlapping intervals I_i $(i = 1, 2, \dots, i_0)$ satisfies (2.1), (2.2) and (2.3′), then $\left|\sum_{i=1}^{i_0} F(I_i) - \sum_{i=1}^{i_0}(L)\int_{I_i \cap F_n} f\right| < \varepsilon$ holds, where $\{M_n\}, \{F_n\}$ and $\delta(n, \varepsilon)$ are those indicated in the definition of the (LA) integral. In this case, take a sequence of positive integers m_n $(n = 1, 2, \dots)$ so that $m_n \geq (n_0)^{1/2}n$ and $1 < m_1 < m_2 < \dots$, and put

$$M_1^* = \dots = M_{m_1-1}^* = \emptyset,\ M_{m_1}^* = M_{m_1+1}^* = \dots = M_{m_2-1}^* = M_1, \dots,$$

$$M_{m_n}^* = M_{m_n+1}^* = \dots = M_{m_{n+1}-1}^* = M_n, \dots;$$

$$F_1^* = \dots = F_{m_1-1}^* = \emptyset,\ F_{m_1}^* = F_{m_1+1}^* = \dots = F_{m_2-1}^* = F_1, \dots,$$

$$F_{m_n}^* = F_{m_n+1}^* = \dots = F_{m_{n+1}-1}^* = F_n, \dots.$$

Then, if, for $m_n + k \in N$ and $\varepsilon > 0$, where $0 \leq k \leq (m_{n+1} - 1) - m_n$, a finite system of non-overlapping intervals I_i $(i = 1, 2, \dots, i_0)$ satisfies (2.1), (2.2) and (2.3) for $m_n + k$, $M_{m_n+k}^*$ and $\delta(m_n + k, \varepsilon)$, then

(2.1) $I_i \cap M_n \neq \emptyset$ for $i = 1, 2, \dots, i_0$;

(2.2) $\mu_{n_0}(\cup_{i=1}^{i_0} I_i - M_n) < \delta(m_n + k, \varepsilon) \leq \delta(n, \varepsilon)$ by $m_n + k > n$;

(2.3′) $d(I_i) \leq (n_0)^{1/2} norm(I_i) < (n_0)^{1/2}(1/(m_n + k)) \leq (n_0)^{1/2}(1/m_n)$

$\leq (n_0)^{1/2}/((n_0)^{1/2}n) = 1/n$ for $i = 1, 2, \dots, i_0$.

Hence, $\left|\sum_{i=1}^{i_0} F(I_i) - \sum_{i=1}^{i_0}(L)\int_{I_i \cap F_n} f\right| < \varepsilon$, so $\left|\sum_{i=1}^{i_0} F(I_i) - \sum_{i=1}^{i_0}(L)\int_{I_i \cap F_{m_n+k}^*} f\right| < \varepsilon$. The converse is clear by the inequality indicated first.

By Remark 1, we have:

Proposition 1. A function f is strongly (LA) integrable on R_0 in E_{n_0} in the sense of Definition 1 if and only if it is (LA) integrable in the strong sense on R_0 in the sense of [6, Definition 5], and both integrals coincide.

By Proposition 1 above and [6, Corollary 1, p. 421], we have:

Proposition 2. Tha finitely additive interval function F indicated in the definition of the strong (LA) integral is uniquely determined.

The following Propositions 3-5 follow immediately from the definition of the strong (LA) integral.

Proposition 3. If a function f is strongly (LA) integrable on an interval R_0 in E_{n_0}, then so is it on any sub-interval R of R_0. If F is an interval function indicated in the definition of the strong (LA) integral for f, then $F(R)$ is the strong (LA) integral of f on R for any sub-interval $R \subset R_0$.

Proposition 4. Let $f = g$ almost everywhere in $R_0 \subset E_{n_0}$. Then, if one of them is strongly (LA) integrable on R_0, then so is the other, and the strong (LA) integrals of f and g on R_0 coincide.

Proposition 5. If f and g are strongly (LA) integrable on an interval R_0 in E_{n_0}, then so is $\alpha f + \beta g$, where α and β are real numbers, and $(SLA)\int_{R_0}(\alpha f + \beta g) = \alpha(SLA)\int_{R_0} f + \beta(SLA)\int_{R_0} g$.

Proposition 6. Let f be a function on an interval I_0 in the one-dimensional Euclidean space E_1. Then it is strongly (LA) integrable on I_0 if and only if it is (D_0) integrable ([9], Definition 1) (so special Denjoy integrable by [9, Proposition 4]) on I_0, and both integrals coincide.

Proof. It is clear that if f is (D_0) integrable, then f is strongly (LA) integrable and both integrals coincide, because a finite system of non-overlapping intervals is classified into two parts so that each part is an elementary system. Next, we prove that if f is strongly (LA) integrable on I_0, then f is (D_0) integrable on I_0. Let F, $\{M_n\}_{n=1}^{\infty}$, $\{F_n\}_{n=1}^{\infty}$ and $\delta(n,\varepsilon)$ be those indicated in the definition of the strong (LA) integral for f. Now, given $n \in N$ and $\varepsilon > 0$, put

$$\delta^*(n,\varepsilon) = (1/2)\min\{1/n, \delta(n,\varepsilon/2)\}.$$

Next, we shall prove that:

If I_i $(i = 1, 2, \ldots, i_0)$ is an elementary system of intervals in I_0 such that

(2.1) $I_i \cap M_n \neq \emptyset$ for $i = 1, 2, \ldots, i_0$;

(2.2) $\mu_1(\cup_{i=1}^{i_0} I_i - M_n) < \delta^*(n,\varepsilon)$,

then

$$\left|\sum_{i=1}^{i_0} F(I_i) - \sum_{i=1}^{i_0}(L)\int_{I_i \cap F_n} f\right| < \varepsilon.$$

In this case, without loss of generality, we can suppose that there exists an integer i_1 with $0 \le i_1 \le i_0$ for which

$$\mu_1(I_i \cap M_n) = 0 \text{ for } i = 1, 2, \dots, i_1 \text{ and } \mu_1(I_i \cap M_n) > 0 \text{ for } i = 1_1 + 1, \dots, i_0.$$

First:

(i) For I_i $(i = 1, 2, \dots, i_1)$: We have $\mu_1(\cup_{i=1}^{i_1} I_i - M_n) < \delta^*(n, \varepsilon) < \delta(n, \varepsilon/2)$, and for $i = 1, 2, \dots, i_1,\ \mu_1(I_i) = \mu_1(I_i - M_n) < \delta^*(n, \varepsilon)$ and so $norm(I_i) < 1/n$. Hence, with (2.1) above by the definition of the strong (LA) integral we have

$$\left| \sum_{i=1}^{i_1} F(I_i) - \sum_{i=1}^{i_1} (L) \int_{I_i \cap F_n} f \right| < \varepsilon/2.$$

Next:

(ii) For I_i $(i = i_1 + 1, \dots, i_0)$: For each $i \in \{i_1 + 1, \dots, i_0\}$, by the Vitali's covering theorem we can find finite intervals $J_1^i, \dots, J_{k_0(i)}^i$ such that: $|J_k^i| < 1/n,\ J_k^i \subset I_i$ and both end-points of J_k^i belong to M_n for $k = 1, 2, \dots, k_0(i)$; $\mu_1((I_i \cap M_n) - \cup_{k=1}^{k_0(i)} J_k^i) < 1/2n$; and $J_1^i, \dots, J_{k_0(i)}^i$ are mutually disjoint. Denote the family of intervals contiguous to the closed set consisting of $\cup_{k=1}^{k_0(i)} J_k^i$ and the both end-points of I_i by H_h^i $(h = 1, 2, \dots, h_0(i))$. Then

$(\cup_{k=1}^{k_0(i)} J_k^i) \cup (\cup_{h=1}^{h_0(i)} H_h^i) = I_i$ for each i.

Further

$J_k^i \cap M_n \ne \emptyset$ for each pair i, k;

$H_h^i \cap M_n \ne \emptyset$ for each pair i, h;

$\mu_1(\cup_{i=i_1+1}^{i_0}((\cup_{k=1}^{k_0(i)} J_k^i) \cup (\cup_{h=1}^{h_0(i)} H_h^i)) - M_n) = \mu_1(\cup_{i=i_1+1}^{i_0} I_i - M_n) < \delta^*(n, \varepsilon) < \delta(n, \varepsilon/2)$;

$norm(J_k^i) = |J_k^i| < 1/n$ for each pair i, k;

$$norm(H_h^i) = |H_h^i| \le \mu_1(I_i - \cup_{k=1}^{k_0(i)} J_k^i) \le \mu_1(I_i - (I_i \cap M_n)) + \mu_1((I_i \cap M_n) - \cup_{k=1}^{k_0(i)} J_k^i) < \delta^*(n, \varepsilon) + 1/2n \le 1/2n + 1/2n = 1/n \text{ for each pair } i, h.$$

The system of intervals $\{J_k^i, H_h^i$, where $i = i_1 + 1, \dots, i_0,\ k = 1, 2, \dots, k_0(i)$ and $h = 1, 2, \dots, h_0(i)\}$ is a finite system of non-overlapping intervals in I_0. Hence by the definition of the strong (LA) integral

$$\left| \sum_{i=i_1+1}^{i_0} F(I_i) - \sum_{i=i_1+1}^{i_0} (L) \int_{I_i \cap F_n} f \right| = \left| \sum_{i=i_1+1}^{i_0} \left(\sum_{k=1}^{k_0(i)} F(J_k^i) + \sum_{h=1}^{h_0(i)} F(H_h^i) \right) - \sum_{i=i_1+1}^{i_0} \left(\sum_{k=1}^{k_0(i)} (L) \int_{J_k^i \cap F_n} f + \sum_{h=1}^{h_0(i)} (L) \int_{H_h^i \cap F_n} f \right) \right| < \varepsilon/2.$$

Thus, by (i) and (ii), $|\sum_{i=1}^{i_0} F(I_i) - \sum_{i=1}^{i_0} (L) \int_{I_i \cap F_n} f| < \varepsilon$.

The *indefinite integral* of a strongly (LA) integrable function f on R_0 is the interval function F in R_0 defined by $F(I) = (SLA) \int_I f$ for every interval $I \subset R_0$.

An interval function F in R_0 is said to be *continuous on* R_0 if, given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $|F(R)| < \varepsilon$ for every interval $R \subset R_0$ with $|R| < \delta(\varepsilon)$, to be *continuous from the inside for an interval* $R \subset R_0$ if, given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $|F(R) - F(J)| < \varepsilon$ for every interval $J \subset R$ with $\mu_{n_0}(R - J) < \delta(\varepsilon)$, and to be *continuous from the inside on* R_0 if F is continuous from the inside for every intereval $R \subset R_0$.

Theorem 1. The indefine integral of a strongly (LA) integrable function on an interval R_0 in E_{n_0} is continuous on R_0.

This theorem is true by Propositions 7 and 8 below. Proposition 7 is proved in [10, Théorèm, p. 282].

Proposition 7. An additive function F in an interval $R_0 \subset E_{n_0}$ is continuous on R_0 if and only if it is continuous from the inside on R_0.

Proposition 8. The indefinite integral of a strongly (LA) integrable function in an interval R_0 is continuous from the inside on R_0.

In order to prove Proposition 8, it is sufficient to prove the following lemma (the proof is a correction of [6, Proposition 12]).

Lemma 1. Let F be the indefinite integral of a strongly (LA) integrable function f on an interval R_0 in E_{n_0} and $A \subset R_0$ an (n_0-1)-dimensional interval contained in a hyperplane of E_{n_0} written for some $i \in \{1, 2, \dots, n_0\}$ in the from

$$A = \{(\xi_1, \dots, \xi_{n_0}) : \xi_i = c \text{ and } a_j \le \xi_i \le b_j \text{ for } j \ne i\}.$$

Then given $\varepsilon > 0$, there exists $\rho(\varepsilon) > 0$ such that for any intervals $A_{+\varepsilon}$ and $A_{-\varepsilon}$ in E_{n_0} written in the form

$$A_{+e} = \{(\xi_1, \dots, \xi_{n_0}) : c \le \xi_i \le c + e \text{ and } a_j \le \xi_j \le b_j \text{ for } j \ne i\};$$

$$A_{-e} = \{(\xi_1, \dots, \xi_{n_0}) : c - e \le \xi_i \le c \text{ and } a_j \le \xi_j \le b_j \text{ for } j \ne i\}.$$

where $0 < e < \rho(\varepsilon)$, and contained in R_0, we have $|F(A_{+e})| < \varepsilon$ and $|F(A_{-e})| < \varepsilon$.

Proof. We shall prove only for the case of $i = 1$. Let $\{M_n\}_{n=1}^{\infty}$ and $\{F_n\}_{n=1}^{\infty}$ be the sequences of measurable sets and closed sets indicated in the definition of the strong (LA) integral of f. In this case, by the definition given $n \in N$ and $\varepsilon > 0$, there exists a $\delta^*(n, \varepsilon) > 0$ such that for any interval $I \subset R_0$ such that $I \cap M_n \ne \emptyset$, $|I| < \delta^*(n, \varepsilon)$, and $norm(I) < 1/n$, we have $|F(I)| < \varepsilon$. For each $p \in A$, take the $n(p) \in N$ with $p \in M_{n(p)} - M_{n(p)-1}$, where $M_0 = \emptyset$. Define a function g on A by $g(p) = 1/3n(p)$ for $p \in A$. Then, by [3, Compatibility theorem, p. 168] for example, there exists a g-fine division of the $(n_0 - 1)$-dimensional interval A, written (D_s, p_s) $(s = 1, 2, \dots, s_0)$ with $p_s \in D_s$. Given $\varepsilon > 0$, put

$$\rho(\varepsilon) = \left[\min\left\{\min_{1\le s\le s_0} \delta^*(n(p_s), \varepsilon/s_0),\ \min_{1\le s\le s_0} 1/n(p_s)\right\}\right] /3\max\{1, (n_0 - 1)\text{-dimensional measure of } A\}.$$

Let $0 < e < \rho(\varepsilon)$ and put $D_s^* = [c-\rho(\varepsilon),\ c+\rho(\varepsilon)] \times \mathrm{proj}_y(D_s)$. Then, the set A_{+e} is the union of non-overlapping intervals $D_s^* \cap A_{+e}$ $(s = 1, 2, \dots, s_0)$. In this case, $p_s \in (D_s^* \cap A_{+e}) \cap M_{n(p_s)}$, $norm(D_s^* \cap A_{+e}) \le d(D_s^*) < 1/n(p_s)$ and $|D_s^* \cap A_{+\varepsilon}| < \rho(\varepsilon)$ $((n_0 - 1)$-dimensional measure of $A) < \delta^*(n(p_s), \varepsilon/s_0)$ for each $s \in \{1, 2, \ldots, s_0\}$. Hence, $|F(D_s^* \cap A_{+e})| < \varepsilon/s_0$ for each $s \in \{1, 2, \ldots, s_0\}$. Therefore, $|F(A_{+\varepsilon})| < \varepsilon$ for every e with $0 < e < \rho(\varepsilon)$. Similarly, $|F(A_{-\varepsilon})| < \varepsilon$ for every e with $0 < e < \rho(\varepsilon)$.

For an interval function F in an interval $R \subset E_{n_0}$ and a $p \in R$, consider a variable interval $I \subset R$ with $p \in I$. Then, if there exists the limit value $\lim_{d(I)\to 0} F(I)/|I|$ as a finite value, the interval function F is said to be *derivable in the strong sense* at p, and the limit value is called the *strong derivative* of F at p and is denoted by $F_s'(p)$.

Theorem 2 ([6, Proposition 3]). Let F be a finitely additive interval function in an interval R_0 in the n_0-dimensional Euclidean space E_{n_0} $(n_0 \geq 1)$ which is derivable in the strong sense at every point of R_0. Then the strong derivative F_s' of F is strongly (LA) integrable, more precisely, strongly (LA^*) integrable, on R_0, and $F(R_0) = (SLA)\int_{R_0} F_s'$ holds.

This theorem is proved in [6, Proposition 3], but in what follows, we shall show a direct proof of the theorem, as an immediate consequence of Proposition 9 below.

For a finitely additive interval function F in an interval R_0 which is derivable in the strong sense at every point of R_0, we denote, for each $n \in N$, by A_n the set of all $p \in R_0$ at which

$[A, n] : |F(I)|/|I| < n$ for any interval $I \subset R_0$ such that $p \in I$ and $d(I) < 4/n$.

Lemma 2. Let F be a finitely additive interval function in an interval R_0 in E_{n_0} $(n_0 \geq 1)$. Suppose that F is derivable in the strong sense at every point of R_0. Then:

(1) $\overline{A}_n \subset A_{2n}$ for every $n \in N$;

(2) $|F_s'| \leq n$ on A_n for every $n \in N$;

(3) F_s' is measurable and bounded on $\overline{A}_n$ for every $n \in N$;

(4) $\overline{A}_n \uparrow R_0$ as $n \to \infty$.

Proof. It is proved in [11, p. 112, (4.2), Theorem] that F_s' is measurable on R_0. For (1), see [6, p. 415]. The other parts are clear.

Let $R = [a_1, b_1; \ldots; a_{n_0}, b_{n_0}]$ be an interval in E_{n_0}. Corresponding to each $s \in \{0, 1, \ldots\}$, consider a *grating*(s) of R obtained by the family of hyperplanes:

$$x_i = a_i + k(b_i - a_i)2^{-s} \ (1 \leq i \leq n_0,\ 0 \leq k \leq 2^s,\ i, k \text{ are integers}),$$

where x_i is ith coordinate of point of E_{n_0}. We denote the family of hyperplanes indicated above by $\mathfrak{H}_s(R)$. An interval in R written

$$[a_1 + k_1(b_1 - a_1)2^{-s}, a_1 + (k_1+1)(b_1 - a_1)2^{-s}; \ldots; a_i + k_i(b_i - a_i)2^{-s},\ a_i + (k_i + 1)$$
$$(b_i - a_i)2^{-s}; \ldots; a_{n_0} + k_{n_0}(b_{n_0} - a_{n_0})2^{-s}, a_{n_0} + (k_{n_0} + 1)(b_{n_0} - a_{n_0})2^{-s}],$$

where $k_1, \ldots, k_{n_0}$ are integers with $0 \leq k_i \leq 2^s - 1$ $(i = 1, 2, \ldots, n_0)$, is called a *mesh of grating*(s) of R. An interval in R is called a *mesh* of R if it is a mesh of grating(s) of R for some $s \in \{0, 1, \ldots\}$. For an interval $I = [c_1, d_1; \ldots; c_i, d_i; \ldots; c_{n_0}, d_{n_0}]$, the intersection of the interval I and any one of the $2n_0$ hyperplanes $x_i = c_i$ and $x_i = d_i$ is called a *face* of the interval I.

We denote the family of all intervals I in R for which there exists an s such that each face of I is contained in some hyperplane belonging to $\mathfrak{H}_s(R)$, by $\mathfrak{J}(R)$.

Lemma 3 Under the same assumption as in Lemma 2, for each $n \in N$ and any interval $R \subset R_0$, the following (1) and (2) hold:

(1) For each interval $I \in \mathfrak{J}(R)$ such that $I \cap \overline{A}_n \neq \emptyset$, let us choose a sequence of meshes of $R; R_j^n$ $(j = 1, 2, \dots)$ (possibly empty or finite) as in [A], (a) and (b) indicated in [6, p. 409]. In this case, we have:

(i) $\cup_{j=1}^{\infty} R_j^n = I - \overline{A}_n$, and R_j^n $(j = 1, 2, \dots)$ is non-overlapping;

(ii) $\sum_{j=1}^{\infty} F(R_j^n)$ is convergent;

(iii) If I is a mesh of R with $d(I) < 1/n$, then

$$\sum_{j=1}^{\infty} |F(R_j^n)| \leq (n\kappa_{n_0}) \sum_{j=1}^{\infty} |R_j^n|,$$

where κ_{n_0} is a number depending only on the dimension of E_{n_0} such that $\kappa_{n_0} > 1$.

(2) For each interval $I \in \mathfrak{J}(R)$ such that $I \cap \overline{A}_n = \emptyset$, let us choose a finite sequence R_j^n $(j = 1, 2, \dots, j_0)$ as in [A], (c) indicated in [6, p. 410]. In this case, we have

(iv) $F(I) = \sum_{j=1}^{j_0} F(R_j^n)$.

Proof. The case of (1) is proved in a quite similar way as in [6, p. 410], replacing F with $\overline{A}_n$ and using Lemma 2, (1) above instead of (2.3) in [6, p. 408]. The case of (2) is clear.

Under the same assumption as in Lemma 2, for each $n \in N$ and any interval $R \subset R_0$, we define a function $G_n(R; I)$ on $\mathfrak{J}(R)$ as follows:

(a) $G_n(R; I) = (L)\int_{I\cap\overline{A}_n} f + \sum_{j=1}^{\infty} F(R_j^n)$ if $I - \overline{A}_n \neq \emptyset$ and $I \cap \overline{A}_n \neq \emptyset$;

(b) $G_n(R; I) = (L)\int_{I\cap\overline{A}_n} f$ if $I - \overline{A}_n = \emptyset$ and $I \cap \overline{A}_n \neq \emptyset$;

(c) $G_n(R; I) = F(I)$ if $I - \overline{A}_n = \emptyset$,

where R_j^n $(j = 1, 2, \dots)$ is the sequence of meshes of R chosen in Lemma 3.

Let $R \subset R_0$ be an interval, K a function of mesh of interval $R \subset R_0$ and $p \in R$. Let I be a variable mesh of R with $p \in I$. Then, if there exists a unique limit: $\lim_{d(I)\to 0} K(I)/|I|$ as a finite limit, we say that K is *derivable with respect to meshes of* R at p, and denote the limit by $K'_{\mathfrak{R}}(P)$.

Lemma 4. Under the same assumption as in Lemma 2, for each $n \in N$ and any interval $R \subset R_0$, the function $G_n(R; I)$ on $\mathfrak{J}(R)$ has the following properties:

(1) $G_n(R; I)$ is additive;

(2) $(G_n(R))'_{\mathfrak{R}} = F'_s$ almost everywhere in $R \cap \overline{A}_n$.

This is proved in a quite similar way as in the proof of [6. Lemma 2], replacing F with $\overline{A}_n$ and F'_η with F'_s.

Lemma 5. Under the same assumption as in Lemma 2, for each $n \in N$ $F(R) = G_n(R;R)$ holds for any interval $R \subset R_0$, where $G_n(R;R)$ is the value defined above as $I = R$. More precisely

$$F(R) = (L)\int_{R\cap\overline{A}_n} f + \sum_{j=1}^{\infty} F(R_j^n),$$

where$\{R_j^n\}$ (possibly empty or finite) is the sequence of meshes of R indicated in Lemma 3.

Proof. This is proved in a quite similar way as in [6, Lemma 3], replacing F with $\overline{A}_n$, F'_η with F'_s, and $A_{2n}(\eta)$ with A_{2n}, and putting

$$\delta(\varepsilon) = \min\{\delta_1(\varepsilon), \varepsilon/4n\kappa_{n_0},\ (|R|/(norm(R))^{n_0})(1/(n\sqrt{n_0}))^{n_0}\}.$$

Proposition 9. Let F be a finitely additive interval function in an interval $R_0 \subset E_{n_0}$ $(n_0 \geq 1)$. Suppose that F is derivable in the strong sense at every point of R_0. Then, for $\overline{A}_n$ $(n = 1, 2, \dots)$ the following statements hold:

(1) $\overline{A}_n \uparrow R_0$ as $n \to \infty$;

(2) F'_s is measurable and bounded on $\overline{A}_n$ for every $n \in N$;

(3) Given $n \in N$ and $\varepsilon > 0$, there exists $\delta(n,\varepsilon) > 0$ such that, if a finite system of non-overlapping intervals R_i $(i = 1, 2, \dots, i_0)$ in R_0 satisfies:

(3.1) $R_i \cap \overline{A}_n \neq \emptyset$ for $i = 1, 2, \dots, i_0$;

(3.2) $\mu_{n_0}(\cup_{i=1}^{i_0} R_i - \overline{A}_n) < \delta(n,\varepsilon)$; and

(3.3) $d(R_i) < 1/n$ for $i = 1, 2, \dots, i_0$,

then

$$\left|\sum_{i=1}^{i_0} F(R_i) - \sum_{i=1}^{i_0}(L)\int_{R_i\cap\overline{A}_n} F'_s\right| < \varepsilon.$$

Proof. (1) and (2) hold by Lemma 2. Next, we prove (3). For $n \in N$ and $\varepsilon > 0$ given, put $\delta(n,\varepsilon) = \varepsilon/n\kappa_{n_0}$, where κ_{n_0} is the number indicated in (iii) of Lemma 3. For each interval R_i, let $\{R_j^{in}\}$ $(j = 1, 2, \dots)$ be the sequence of meshes of R_i chosen as in Lemma 3 to define $G_n(R_i;R_i)$. Then, by Lemmas 5 and 3 we have

$$\begin{aligned}
&\left|\sum_{i=1}^{i_0} F(R_i) - \sum_{i=1}^{i_0}(L)\int_{R_i\cap\overline{A}_n} F'_s\right| \\
&= \left|\sum_{i=1}^{i_0}(L)\int_{R_i\cap\overline{A}_n} F'_s - \sum_{i=1}^{i_0}\sum_{j=1}^{\infty} F(R_j^{in}) - \sum_{i=1}^{i_0}(L)\int_{R_i\cap\overline{A}_n} F'_s\right| \\
&\leq \sum_{i=1}^{i_0}(n\kappa_{n_0}\sum_{j=1}^{\infty}|R_j^{in}|) = n\kappa_{n_0}\left(\sum_{i=1}^{i_0}\mu_{n_0}(R_i - \overline{A}_n)\right) < n\kappa_{n_0}\delta(n,\varepsilon) = \varepsilon.
\end{aligned}$$

For an interval I in E_n, the *parameter of reguralit*y of I is the number $|I|/|R|$, where R is the minimum cube containing I, and is denoted by $r(I)$. If $r(I) \geq \eta$, the interval I is called

η-regular. A sequence of intervals $\{I_i\}$ is said to be *η-regular* if I_i is η-regular for every i, and we say that the sequence $\{I_i\}$ *tends* to p if $d(I_i) \to 0$ as $i \to \infty$ and $p \in I_i$ for every i. Let F be an interval function in an interval R_0 in E_n and $p \in R_0$. For an η with $1 \geq \eta > 0$, we call the least upper bound [resp. the greatest lower bound] (extended real number) of the numbers l for which there is an η-regular sequence $\{I_i\}$ of intervals tending to p such that $\lim_{i\to\infty} F(I_i)/|I_i| = l$ the *upper derivate* [resp. the *lower derivate*] *in the η-regular sense* of F at p. The upper [resp. lower] derivate of F in the η-regular sense at p is denoted by $\overline{D}_\eta F(p)$ [resp. $\underline{D}_\eta F(p)$]. When $\overline{D}_\eta F(p) = \underline{D}_\eta F(p)$, the common value is called the *derivative in the η-regular sense* of F at p, and is denoted by $F'_\eta(p)$. If further $F'_\eta(p)$ is finite, F is said to be *derivable in the η-regular sanse* at p. As easily seen, F is derivable in the η-regular sense at p if and only if there exists a unique limit $\lim_{d(I)\to 0} F(I)/|I|$ as a finite limit, where I is a variable η-regular interval with $p \in I$. If $\lim_{\eta\to 0} \overline{D}_\eta F(p) = \lim_{\eta\to 0} \underline{D}_\eta F(p)$, then the common value is called the *ordinary derivative* of F at p, and is denoted by $F'(p)$. When $F'(p)$ is finite, F is said to be *derivable in the ordinary sense* at p.

Theorem 3. Let f be strongly (LA) integrable on an interval R_0 in E_{n_0} and F an indefinite integral of f. Then, at almost all $p \in R_0$, F is derivable in the ordinary sense and $F'(p) = f(p)$ holds.

The theorem holds by virtue of [6, Theorem 5, (2), p. 424] and Proposition 1 above. Because, a strongly (LA) integrable function on R_0 is (LA) integrable in the ordinary sense on R_0 (see, for the definition of "(LA) integrable in the ordinary sense", [6, Definition 4, p. 423]). To prove the theorem we extend the idea of the semi-regular integral in the Burkill sense introduced by S.Kempisty in [2] to the η-regular integral in the Burkill sense with any η, where $0 < \eta < 1$([6, p. 416])(cf. [1]).

We improve on the definition of (D) integrability proposed in [4] as in Definition 2 below, by replacing the condition that "M_n is closed" with the condition that "M_n is measurable".

Definition 2. Let R_0 be an interval in E_{n_0} and f a measurable function on R_0. The function f is said to be (D) *integrable* on R_0 if there exist a finitely additive interval function F in R_0, a nondecreasing sequence of measurable sets M_n $(n = 1, 2, \dots)$ such that $M_n \subset R_0$ and $\cup_{n=1}^\infty M_n = R_0$, and a nondecreasing sequence of closed sets F_n $(n = 1, 2, \dots)$ such that $F_n \subset M_n$ and $\mu_{n_0}(R_0 - \cup_{n=1}^\infty F_n) = 0$, satisfying the following two conditions (1) and (2):

(1) The function f is Lebesgue integrable on F_n for each $n \in N$;

(2) Given $n \in N$ and $\varepsilon > 0$, there exists a $\delta(n, \varepsilon) > 0$ for which the following holds: if I_i $(i = 1, 2, \dots, i_0)$ is an elementary system in R_0 such that

(2.1) $I_i \cap M_n \neq \emptyset$ for $i = 1, 2, \dots, i_0$;

(2.2) $\mu_{n_0}(\cup_{i=1}^{i_0} I_i - M_n) < \delta(n, \varepsilon)$;

(2.3) $norm(I_i) < 1/n$ for $i = 1, 2, \dots, i_0$,

then the following inequality holds:

$$\left| \sum_{i=1}^{i_0} F(I_i) - \sum_{i=1}^{i_0} (L) \int_{I_i \cap F_n} f \right| < \varepsilon.$$

In this case, the sequence M_n $(n = 1, 2, \dots)$ is called a *characteristic sequence* of the (D) integral and the sequence F_n $(n = 1, 2, \dots)$ is called a *fundamental sequence* of the (D) integral.

We remark that in Definition 2 we can suppose that

$$\delta(n,\varepsilon) \geq \delta(m,\varepsilon) \text{ for } m > n \text{ and } \delta(n,\varepsilon) \geq \delta(n,\varepsilon') \text{ for } \varepsilon > \varepsilon'. \tag{2°}$$

Proposition 10. Let $f = g$ almost everywhere in an interval $R_0 \subset E_{n_0}$. Then, if one of them is (D) integrable on R_0, then so is the other, and both integrals coincide.

The condition imposed on the definition of (D) integrability is weaker than it imposed on the definition of the (D_0) integral and is weaker than it imposed on the definition of the strong (LA) integral. Hence

Proposition 11. (1) If f is (D_0) integrable on an interval R_0 in E_{n_0}, then it is (D) integrable on R_0.

(2) If f is strongly (LA) integrable on an interval R_0 in E_{n_0}, then it is (D) integrable on R_0.

Since the difference in the definition of strong (LA) integrability and (D) integrability is only the difference of condition given for the system of intervals I_i $(i = 1, 2, \dots, i_0)$: "non-overlapping intervals" and "mutually disjoint intervals". Hence, when $n_0 = 1$, the finitely additive interval function F indicated in the definition of (D) integrability is uniquely determined by Proposition 2, so we may call $F(R_0)$ the (D) integral of f on R_0, and denote $F(R_0)$ by $(D)\int_{R_0} f(p)dp$ etc.

By Proposition 6 and the statement above, we have

Proposition 12. When $n_0 = 1$, for the (LA) in the strong sense, (D), and (D_0) their integrabilities are equivalent with the integrability of Denjoy in the special sense, and their integrals coincide.

In what follows, R_0 denotes an interval in E_{n_0}. When f is (D) integrable on R_0, let F, $\{M_n\}_{n=1}^{\infty}$, $\{F_n\}_{n=1}^{\infty}$ and $\delta(n,\varepsilon)$ be those indicated in the definition of (D) integrability for f.

Lemma 6. Let f be (D) integrable on an interval R_0 in E_{n_0}. Then, if, for $n \in N$ and $\varepsilon > 0$, I_i $(i = 1, 2, \dots, i_0)$ is an elementary system in R_0 such that

(2.1*) $I_i \cap \overline{M}_n \neq \emptyset$ for $i = 1, 2, \dots, i_0$;

(2.2*) $\mu_{n_0}(\cup_{i=1}^{i_0} I_i) < \delta(n,\varepsilon)$;

(2.3*) $norm(I_i) < 1/n$ for $i = 1, 2, \dots, i_0$,

then

$$\left|\sum_{i=1}^{i_0} F(I_i) - \sum_{i=1}^{i_0}(L)\int_{I_i \cap F_n} f\right| < \lambda_{n_0}\varepsilon,$$

where λ_{n_0} is a positive number depending only on the dimension of the space E_{n_0}. In particular $\lambda_2 = 4$.

The lemma follows easily from the definition of (D) integrability.

Let f be (D) integrable on R_0 in E_{n_0}. For $n \in N$ and $\varepsilon > 0$, let $\eta(n, \varepsilon)$ be a positive number such that

$$\text{if } \mu_{n_0}(E) < \eta(n, \varepsilon), \text{ then } (L) \int_{E \cap F_n} |f| < \varepsilon. \tag{3°}$$

Without loss of generality, we can suppose that

$$\eta(n, \varepsilon) \geq \eta(m, \varepsilon) \text{ for } n < m \text{ and } \eta(n, \varepsilon) \geq \eta(n, \varepsilon') \text{ for } \varepsilon > \varepsilon'. \tag{4°}$$

Throughout this paper, let ε_n $(n = 1, 2, \dots)$ be a sequence of positive numbers such that

$$\varepsilon_n \downarrow 0 \text{ and } \sum_{m=n+1}^{\infty} \varepsilon_m \leq \varepsilon_n \text{ for each } n \in N, \tag{5°}$$

and let ε_n^{**} $(n = 1, 2, \dots)$ be the nonincreasing sequence defined by

$$\varepsilon_n^{**} = \min\{1/n, \delta(n, \varepsilon_n/2^{n+5}), \eta(n, \varepsilon_n/2^{n+5})\} \text{ for each } n \in N. \tag{6°}$$

We have $\varepsilon_n^{**} \downarrow 0$

Let J be an interval in the one-dimensional Euclidean space E_1 and A_n $(n = 1, 2, \dots)$ a nondecreasing sequence of closed sets in E_1 such that $\cup_{n=1}^{\infty} A_n = J$. Then we say that a non-empty closed set F_{nm} in E_1, where $n < m$, has the *property* $(\mathbf{B}_1)$ *for* $n < m$ *in* J *associated with* $\{A_n\}_{n=1}^{\infty}$ *and* $\{\varepsilon_n^{**}\}_{n=1}^{\infty}$ if it has the following property $(\mathbf{B}_1)$.

$(\mathbf{B}_1)$: (1) $F_{nm} \subset J$ and $F_{nm} \subset A_m$;

(2) Denote the sequence of intervals contiguous to the set consisting of the set F_{nm} and the both end-points of J by J_j $(j = 1, 2, \dots)$. Then, J_j $(j = 1, 2, \dots)$ are classified into $m - n + 1$ parts written J_{kj} $(j = 1, 2, \dots)$ (possibly empty or finite), where $k = n, n+1, n+2, \dots, m$, so that

1) $\sum_{j=1}^{\infty} |J_{kj}| < \varepsilon_n^{**}$;

2) $(J_{kj})^\circ \cap A_k = \emptyset$ for every $j \in N$;

3) one at least of the end-points of the interval J_{kj} belongs to A_k for each $j \in N$.

In this case, the point taken as one at least of the end-points of J_{kj} in 3) is called the *characteristic point* of J_{kj} and the number k is called the *characteristic number* of J_{kj}.

First, let us spply Lemma 2 in [4, p. 72; 8, p. 2] for the interval R_0, the sequence of closed sets $\{\overline{M}_n\}_{n=1}^{\infty}$ and the sequence of positive numbers $\{\varepsilon_n^{**}\}_{n=1}^{\infty}$. Then, the following statement (I) holds.

(I) There exist two increasing sequences of positive integers

$$n_i \text{ and } m_i \ (i = 1, 2, \dots) \text{ such that } i < n_i \text{ and } n_i < m_i < n_{i+1} \tag{7°}$$

and a nondecreasing sequence of non-empty closed sets

$$F_{n_i m_i} \ (i = 1, 2, \dots)$$

having the following properties (1) and (2):

(1) $F_{n_i m_i} \subset R_0$ and $F_{n_i m_i} \subset \overline{M}_{m_i}$ for every $i \in N$;

(2) Put

$$Y = \cup_{i=1}^{\infty} \operatorname*{proj}_{E_{n_0-1}} {}_y(F_{n_i m_i}) \text{ and } Z = \operatorname*{proj}_{E_{n_0-1}} {}_y(R_0) - Y, \tag{8°}$$

then

a) $\mu_{n_0-1}(Z) = 0$;

b) for each $q \in Y$ and $i \in N$, if $(F_{n_i m_i})^q \neq \emptyset$, the closed set $(F_{n_i m_i})^q$ has the property $(\mathbf{B}_1)$ for $n_i < m_i$ in $(R_0)^q$ associated with $\{(\overline{M}_n)^q\}_{n=1}^{\infty}$ and $\{\varepsilon_n^{**}\}_{n=1}^{\infty}$; and

c) $\cup_{i=1}^{\infty}(F_{n_i m_i})^q = (R_0)^q$ holds for each $q \in Y$.

Next, to each point

$$q \in Z \left(= \operatorname*{proj}_{E_{n_0-1}} {}_y(R_0) - \cup_{i=1}^{\infty} \operatorname*{proj}_{E_{n_0-1}} {}_y(F_{n_i m_i}) \right),$$

let us apply Lemma 1 in [4, p. 72; 8, p. 2] for the one-dimensional interval $(R_0)^q$, the sequence of one-dimensional closed set $\{(\overline{M}_n)^q\}_{n=1}^{\infty}$ and the sequence of positive numbers $\{\varepsilon_n^{**}\}_{n=1}^{\infty}$. Then:

(II) There exist two increasing sequences of positive integers

$$n_i(q) \text{ and } m_i(q) \ (i = 1, 2, \dots) \text{ such that } i < n_i(q) \text{ and } n_i(q) < m_i(q) < n_{i+1}(q)$$

and a nondecreasing sequence of non-empty closed sets

$$F_{n_i(q) m_i(q)} \quad (i = 1, 2, \dots)$$

such that:

a) each $F_{n_i(q) m_i(q)}$ has the property $(\mathbf{B}_1)$ for $n_i(q) < m_i(q)$ in $(R_0)^q$ associated with $\{(\overline{M}_n)^q\}_{n=1}^{\infty}$ and $\{\varepsilon_n^{**}\}_{n=1}^{\infty}$, in particular, $F_{n_i(q) m_i(q)} \subset (R_0)^q$ and $F_{n_i(q) m_i(q)} \subset (\overline{M}_{m_i(q)})^q$; and

b) $\cup_{i=1}^{\infty} F_{n_i(q) m_i(q)} = (R_0)^q$.

An elementary system $S : I_i \ (i = 1, 2, \dots, i_0)$ in E_n is called a $(*)$-*elemetary system* if

$$\operatorname*{proj}_{E_{n-1}} {}_y(I_1) = \operatorname*{proj}_{E_{n-1}} {}_y(I_2) = \dots = \operatorname*{proj}_{E_{n-1}} {}_y(I_{i_0}).$$

An elementary system S is called a $(**)$-*elementary system* if it is composed of finite $(*)$-elemetary systems $S_l \ (l = 1, 2, \dots, l_0)$ such that

$$\operatorname*{proj}_{E_{n-1}} {}_y(S_l) \cap \operatorname*{proj}_{E_{n-1}} {}_y(S_{l'}) = \emptyset \text{ for } l, \ l' \in \{1, 2, \dots, l_0\} \text{ with } l \neq l'.$$

Lemma 7. If f is (D) integrable on an interval R_0 in $E_{n_0}(n_0 \geq 2)$, then there exists a nondecreasing sequence of measurable sets B_h $(h = 1, 2, \dots)$ (the first finite sets may be empty) such that:

(1) $B_h \uparrow R_0$ as $h \to \infty$;

(2) for every $h \in N$, the set $(B_h)^q$ is a closed set for every $q \in \underset{E_{n-1}}{\text{proj}}\,{}_y(B_h)$,

in such a way that the following statement holds:

Corresponding to h, ε with $h \in N$ and $\varepsilon > 0$, there exists a $\rho^*(h, \varepsilon) > 0$ such that:

Given $\varepsilon > 0$, suppose that, for some $h \in N$, a $(**)$-elementary system S consisting of $(*)$-elementary systems S_l $(l = 1, 2, \dots, l_0)$, where for each l

$$S_l \text{ is a } (*)\text{-elementary system consisting of intervals written}$$

$$I_{lj} \ (j = 1, 2, \dots, j_0(l)),$$

satisfies the following conditions:

(a) For each $l \in \{1, 2, \dots, l_0\}$, there exists a $q_l \in \underset{E_{n_0-1}}{\text{proj}}\,{}_y(S_l) \cap \underset{E_{n_0-1}}{\text{proj}}\,{}_y(B_h)$ such that $(I_{lj})^{q_l} \cap (B_h)^{q_l} \neq \emptyset$ for every $j \in \{1, 2, \dots, j_0(l)\}$;

(b) $|\underset{E_{n_0-1}}{\text{proj}}\,{}_y(S)| < \rho^*(h, \varepsilon)$;

(c) $\mathit{norm}(\underset{E_{n_0-1}}{\text{proj}}\,{}_y(S_l)) < 1/h$ for every $l \in \{1, 2, \dots, l_0\}$.

Then the following inequality holds:

$$|F(S)| < \varepsilon.$$

Proof. For simplicity, we prove only for the case when $n_0 = 2$ and $R_0 = [0, 1; 0, 1]$. Denote q_l taken in the assumption (a) of the lemma by y_l.

Let

$$n_i, m_i \text{ and } F_{n_i m_i} \ (i = 1, 2, \dots)$$

be the two sequeces of positive integers and the sequence of non-empty closed sets indicated in (I) above.

Corresponding to each $h \in N$, if there exists an m_i with $m_i \leq h$, then we put

$$i(h) = \max\{i : m_i \leq h, \quad i \in N\}. \tag{9°}$$

In this case, the following holds:

$$F_{n_k m_k} = F_{n_{i(m_k)} m_{i(m_k)}} \quad \text{for every } k \in N.$$

Put, as in (8°)

$$Z = \text{proj}\,{}_y(R_0) - \cup_{i=1}^{\infty} \text{proj}\,{}_y(F_{n_i m_i}).$$

Then

$$\mu_1(Z) = 0.$$

Corresponding to each $y \in Z$, let $n_i(y), m_i(y)$ and $F_{n_i(y)m_i(y)}$ $(i = 1, 2, \dots)$ be the two sequences of positive integers and the sequence of closed sets indicated in (II) above.

Corresponding to $h \in N$, if there exists an $m_i(y)$ with $m_i(y) \le h$, then we put

$$i(y,h) = \max\{i : m_i(y) \le h,\ i \in N\}. \qquad (10°)$$

Given an $h \in N$, put

$$B_h = F_{n_{i(h)}m_{i(h)}} \cup \left(\cup^*_{y\in Z} F_{n_{i(y,h)}(y)m_{i(y,h)}(y)}\right) \text{ when } i(h) \text{ is definable;}$$

$$B_h = \cup^*_{y\in Z} F_{n_{i(y,h)}(y)m_{i(y,h)}(y)} \text{ for the other case,} \qquad (11°)$$

where the union $\cup^*_{y\in Z}$ is over all $y \in Z$ for which $i(y,h)$ is definable. Then, B_h $(h = 1, 2, \dots)$ is a nondecreasing sequence of measurable sets (the first finite sets may be empty) satisfying (1) and (2) of the lemma.

Now, for $h \in N$ and $\varepsilon > 0$, put

$$\rho^*(h,\varepsilon) = \min\{\delta(h, \varepsilon/2^7 h),\ \eta(h, \varepsilon/2^4 h)\}. \qquad (12°)$$

Then

$$\rho^*(h,\varepsilon) \ge \rho^*(k,\varepsilon) \text{ for } k > h \text{ and } \rho^*(h,\varepsilon) \ge \rho^*(h,\varepsilon') \text{ for } \varepsilon > \varepsilon'.$$

Given $\varepsilon > 0$, let, for some $h \in N$, S be a $(**)$-elementary system S_l $(l = 1, 2, \dots, l_0)$ satisfying the conditions (a), (b) and (c) of the lemma for B_h, $\rho^*(h,\varepsilon)$ defined above and h.

First of all, for the $(**)$-elementary system S we suppose the following condition:

(d) Let $l \in \{1, 2, \dots, l_0\}$ and y_l the point of $\mathrm{proj}_y(S_l) \cap \mathrm{proj}_y(B_h)$ taken in the condition (a). For each pair l, j with $l \in \{1, 2, \dots, l_0\}$ and $j \in \{1, 2, \dots, j_0(l) - 1\}$, if we denote by a^l_j the right hand end-point of one-dimensional interval $(I_{lj})^{y_l}$ and by b^l_j the left hand end-point of one-dimensional interval $(I_{l,j+1})^{y_l}$, then $([a^l_j, b^l_j] \times \{y_l\}) \cap B_h \ne \emptyset$ for $j = 1, 2, \dots j_0(l) - 1$.

The proof requires four steps.

(1) Consider the family of all intervals I_{lj}, possibly empty, for which

$$|\mathrm{proj}_x(I_{lj})| < 1/h, \text{ where } l \in \{1, 2, \dots, l_0\} \text{ and } j \in \{1, 2, \dots, j_0(l)\},$$

and denote the family by

$$R^1_m \ (m = 1, 2, \dots, m_1).$$

Then, by the condition (a), we have $R^1_m \cap B_h \ne \emptyset$ for $m = 1, 2, \dots, m_1$. Further since $\overline{M}_{m_{i(h)}} \supset F_{n_{i(h)}m_{i(h)}}$, $\overline{M}_{m_{i(y,h)}(y)} \supset F_{n_{i(y,h)}(y)m_{i(y,h)}(y)}$ by (1) of (I) and a) of (II), $m_{i(h)} \le h$ and $m_{i(y,h)}(y) \le h$, we have $\overline{M}_h \supset B_h$. Therefore

$$R^1_m \cap \overline{M}_h \neq \emptyset \text{ for } m = 1, 2, \dots, m_1.$$

(2) Consider the family of all intervals I_{lj}, possibly empty, for which

$$|\text{proj}_x(I_{lj})| \geq 1/h, \text{ where } l \in \{1, 2, \dots, l_0\} \text{ and } j \in \{1, 2, \dots, j_0(l)\}. \quad (13^\circ)$$

For each interval I_{lj} of the family and y_l taken in (a), denote the sequence of one-dimensional intervals contiguous to the one-dimensional closed set consisting of the non-empty closed set $(I_{lj})^{y_l} \cap (B_h)^{y_l}$ and the both end-points of the interval $(I_{lj})^{y_l}$ by

$$J_{ljr} \quad (r = 1, 2, \dots),$$

where we can suppose that $|J_{ljr}| \geq |J_{ljr+1}|$ for $r = 1, 2, \dots$. Take an index r such that

$$|J_{ljr}| \geq 1/2h \text{ and } |J_{ljr+1}| < 1/2h, \text{ and written } r_0(l, j).$$

Next consider the sequence of one-dimensional intervals contiguous to the one-dimensional closed set consisting of the set $\cup_{r=1}^{r_0(l,j)} J_{ljr}$ and the both end-points of the interval (I_{lj}). Denote the sequence by

$$K_{ljt} \ (t = 1, 2, \dots, t_0(l, j)).$$

Then, for $l \in \{1, 2, \dots, l_0\}$ and $j \in \{1, 2, \dots, j_0(l)\}$, we have

$$\cup_{r=1}^{r_0(l,j)} J_{ljr} \cup \cup_{t=1}^{t_0(l,j)} K_{ljt} = (I_{lj})^{y_l};$$

$$\cup_{t=1}^{t_0(l,j)} K_{ljt} = \cup_{r=r_0(l,j)+1}^{\infty} J_{ljr} \cup ((I_{lj})^{y_l} \cap (B_h)^{y_l}); \quad (14^\circ)$$

$$\{J_{ljr} \ (r = 1, 2, \dots, r_0(l, j)); K_{ljt} \ (t = 1, 2, \dots, t_0(l, j))\} \text{ are non-overlapping.}$$

For K_{ljt} $(l = 1, 2, \dots, l_0, \ j = 1, 2, \dots, j_0(l), \ t = 1, 2, \dots, t_0(l, j))$, first consider

(2.1): the family (possibly empty)

$$\{K_{ljt} : |K_{ljt}| < 1/h, \text{ where } l, \ j \text{ is any pair belonging to the set of all indices } (l, j)$$
$$\text{for which } I_{lj} \text{ is chosen to be } (13^\circ), \text{ and } t \in \{1, 2, \dots, t_0(l, j)\}\},$$

and associate, with each K_{ljt} of the family, the two-dimensional interval

$$\text{proj}_x(K_{ljt}) \times \text{proj}_y(S_l).$$

We denote the family of such two-dimensional intervals by

$$R^2_m \ (m = 1, 2, \dots, m_2).$$

Since then $K_{ljt} \cap B_h \neq \emptyset$ and $\overline{M}_h \supset B_h$, we have

$$R_m^2 \cap \overline{M}_h \neq \emptyset \text{ for } m = 1, 2, \ldots, m_2.$$

Next, consider

(2,2): the family (possibly empty)

$$\{K_{ljt} : |K_{ljt}| \geq 1/h, \text{ where } l,\ j \text{ is any pair belonging to the set of all indices } (l, j)$$
$$\text{for which } I_{lj} \text{ is chosen to be } (13^\circ), \text{ and } t \in \{1, 2, \ldots, t_0(l, j)\}\}.$$

Corresponding to each K_{ljt} of the family, take a finite sequence of non-overlapping one-dimensional intervals K'_{ljts} ($s = 1, 2, \ldots, s_0(l, j, t)$) whose union is K_{ljt} and such that $1/2h \leq |K'_{ljts}| < 1/h$. With each of such intervals K'_{ljts} we associate a two-dimensional interval;

$$\operatorname{proj}_x(K'_{ljts}) \times \operatorname{proj}_y(S_l),$$

and denote the family of all such two-dimensional intervals by

$$R_m^3 \ (m = 1, 2, \ldots, m_3).$$

Then, we have

$$R_m^3 \cap \overline{M}_h \neq \emptyset \text{ for } m = 1, 2, \ldots, m_3.$$

Because, $K'_{ljts} \cap B_h \neq \emptyset$ holds for every K'_{ljts}. Indeed, suppose that $K'_{ljts} \cap (B_h)^{y_l} = \emptyset$. for some K'_{ljts}. Since then $K'_{ljts} \subset \cup_{r=r_0(l,j)+1}^{\infty} J_{ljr}$ by (14°) and J_{ljr} ($r = 1, 2, \ldots$) are the intervals contiguous to the closed set consisting of the non-empty closed set $(B_h)^{y_l} \cap (I_{lj})^{y_l}$ and the both end-points of $(I_{lj})^{y_l}$, there exists an $r^* \geq r_0(l, j) + 1$ for which $K'_{ljts} \subset J_{ljr^*}$. Hence, $|K'_{ljts}| \leq |J_{ljr^*}| < 1/2h$, which contradicts $|K'_{ljts}| \geq 1/2h$. Thus, $K'_{ljts} \cap \overline{M}_h \neq \emptyset$ for every K'_{ljts} by $\overline{M}_h \supset B_h$.

For simplicity, we put

$$\begin{aligned} &\{R_m \ (m = 1, 2, \ldots, m_0)\} \\ &= \{R_m^1 \ (m = 1, 2, \ldots, m_1); R_m^2 \ (m = 1, 2, \ldots, m_2); R_m^3 \ (m = 1, 2, \ldots, m_3)\}. \end{aligned}$$

As easily seen, R_m ($m = 1, 2, \ldots, m_0$) are classified into two parts so that each part is an elementary system. In addition, we have $R_m \cap \overline{M}_h \neq \emptyset$ for $m = 1, 2, \ldots, m_0$; $\sum_{m=1}^{m_0} |R_m| < \delta(h, \varepsilon/2^7 h)$, because $|\operatorname{proj}_y(S)| < \rho^*(h, \varepsilon) \leq \delta(h, \varepsilon/2^7 h)$ by (b) and (12°) and $|\operatorname{proj}_x(R_0)| = 1$; and $norm(R_m) < 1/h$ for $m = 1, 2, \ldots, m_0$ by (c). Hence by Lemma 1, we have

$$\left| \sum_{m=1}^{m_0} F(R_m) - \sum_{m=1}^{m_0} (L) \int_{R_m \cap F_h} f \right| < 4(\varepsilon/2^7 h) \times 2 = \varepsilon/16h.$$

Further, since $\sum_{m=1}^{m_0} |R_m| < \rho^*(h, \varepsilon) \leq \eta(h, \varepsilon/2^4 h)$, $\left| \sum_{m=1}^{m_0} (L) \int_{R_m \cap F_h} f \right| < \varepsilon/16h$. Therefore

$$\left|\sum_{m=1}^{m_0} F(R_m)\right| < \varepsilon/16h + \varepsilon/16h = \varepsilon/8h. \tag{15°}$$

Next, consider

(2.3) the family of one-dimensional intervals J_{ljr} $(l = 1, 2, \dots, l_0,\ j = 1, 2, \dots j_0(l),\ r = 1, 2, \dots, r_0(l, j))$. Then we have $|J_{ljr}| \geq 1/2h$ for each such J_{ljr}. Corresponding to each such interval J_{ljr}, there exists uniquely a one-dimensional interval H_{ljr} having thte following properties:

(a°) H_{ljr} is contained in one of the intervals contiguous to the closed set consisting of $(B_h)^{y_l}$ and the both end-points of $(R_0)^{y_l}$, say H'_{ljr};

(b°) $H_{ljr} \supset J_{ljr}$;

(c°) One end-point of the interval H_{ljr} is an end-point of J_{ljr};

(d°) The other end-point of the interval H_{ljr} is the characteristic point of the interval H'_{ljr} say p_{ljr}.

We denote the characteristic number of the characteristic point p_{ljr} by h_{ljr}. Since then $n_{i(h)} \leq h_{ljr} \leq m_{i(h)}$ or $n_{i(y,h)}(y) \leq h_{ljr} \leq m_{i(y,h)}(y)$ for some $y \in Z$ by the definition of characteristic number, we have $1 \leq h_{ljr} \leq h$ by (9°) and (10°). In this case, by the assumption (d) and the definition of $(*)$-elementary system, H_{ljr} $(j = 1, 2, \dots, j_0(l),\ r = 1, 2, \dots, r_0(l, j))$ are non-overlapping for each $l \in \{1, 2, \dots, l_0\}$.

Next, for each triple l, j, r with $l \in \{1, 2, \dots, l_0\}$, $j \in \{1, 2, \dots, j_0(l)\}$, and $r \in \{1, 2, \dots, r_0(l, j)\}$, put

$$Q_{ljr} = \operatorname{proj}_x(J_{ljr}) \times \operatorname{proj}_y(S_l);$$

$$Q^*_{ljr} = \operatorname{proj}_x(H_{ljr}) \times \operatorname{proj}_y(S_l).$$

First, corresponding to each k with $1 \leq k \leq h$, consider the family of two-dimensional intervals Q^*_{ljr} for which $h_{ljr} = k$, where $l = 1, 2, \dots, l_0$, $j = 1, 2, \dots, j_0(l)$ and $r = 1, 2, \dots, r_0(l, j)$, and denote the family by R^*_{km} $(m = 1, 2, \dots, m_0(k))$. When Q^*_{ljr} is denoted by R^*_{km}, we denote Q_{ljr} by R_{km}. Then for each k with $1 \leq k \leq h$:

R^*_{km} $(m = 1, 2, \dots, m_0(k))$ are non-overlapping;

$R^*_{km} \cap \overline{M}_k \neq \emptyset$ for $m = 1, 2, \dots, m_0(k)$;

$\mu_2(\cup_{m=1}^{m_0(k)} R^*_{km}) \leq \mu_2(S) < \rho^*(h, \varepsilon) \leq \delta(h, \varepsilon/2^7 h) \leq \delta(k, \varepsilon/2^7 h)$ by (b), (12°) and (2°); and

$norm(R^*_{km}) < \max\{\varepsilon^{**}_k, 1/h\} \leq \max\{1/k, 1/h\} = 1/k$ for $m = 1, 2, \dots, m_0(k)$ by virtue of 1) of (2) of the property $(\mathbf{B}_1)$, (c) and (6°).

In addition, R^*_{km} $(m = 1, 2, \dots, m_0(k))$ are classified into two elementary systems. Hence, by Lemma 5 we have

$$\left|\sum_{m=1}^{m_0(k)} F(R^*_{km}) - \sum_{m=1}^{m_0(k)} (L)\int_{R^*_{km} \cap F_k} f\right| < 4(\varepsilon/2^7 h) \times 2 = \varepsilon/16h.$$

On the other hand, since $(1 \leq)k \leq h$, by (b), (12°) and (4°)

$$\sum_{m=1}^{m_0(k)} |R_{km}^*| \leq \mu_2(S) < \rho^*(h, \varepsilon) \leq \eta(h, \varepsilon/2^4 h) \leq \eta(k, \varepsilon/2^4 h).$$

Hence, by (3°)

$$\left| \sum_{m=1}^{m_0(k)} (L) \int_{R_{km}^* \cap F_k} f \right| < \varepsilon/16h.$$

Therefore

$$\left| \sum_{m=1}^{m_0(k)} F(R_{km}^*) \right| < \varepsilon/16h + \varepsilon/16h = \varepsilon/8h.$$

Thus, we obtain

$$\left| \sum_{k=1}^{h} \sum_{m=1}^{m_0(k)} F(R_{km}^*) \right| < \varepsilon/8.$$

Similarly, we obtain

$$\left| \sum_{k=1}^{h} \sum_{m=1}^{m_0(k)} F(R_{km}^* - R_{km}) \right| < \varepsilon/8.$$

Therefore

$$\left| \sum_{k=1}^{h} \sum_{m=1}^{m_0(k)} F(R_{km}) \right| < \varepsilon/4. \tag{16°}$$

By (15°) and (16°)

$$|F(S)| \leq \left| \sum_{m=1}^{m_0} F(R_m) \right| + \left| \sum_{k=1}^{h} \sum_{m=1}^{m_0(k)} F(R_{km}) \right| < \varepsilon/8h + \varepsilon/4 < \varepsilon/2,$$

because $S = \cup_{m=1}^{m_0} R_m \cup \cup_{k=1}^{h} \cup_{m=1}^{m_0(k)} R_{km}$.

In general, the intervals constructing S classified into two parts so that each part satisfies the condition (d). Hence, $|F(S)| < \varepsilon$ holds.

As an application of Lemma 7, we obtain:

Proposition 13. Let f be (D) integrable on an interval R_0 in the two-dimensional Euclidean space, then if I_i $(i = 1, 2, \dots)$ is a sequence of intervals in R_0 such that:

$$I_1 \supset I_2 \supset \dots \text{ and } |\operatorname{proj}_y(I_i)| \to 0,$$

we have $\lim_{i\to\infty} F(I_i) = 0$.

Proof. There exists a point $q \in \cap_{i=1}^{\infty} I_i$. Since $B_h \uparrow R_0$ by (1) of Lemma 7, there exists an h_0 such that $q \in B_{h_0}$. Hence, $I_i \cap B_{h_0} \neq \emptyset$ for every $i \in N$. For $\varepsilon > 0$, take an $i_0 = i_0(\varepsilon)$ so that $|\operatorname{proj}_y(I_i)| < \min\{\rho^*(h_0, \varepsilon), 1/h_0\}$ for every $i \geq i_0$ ($\rho^*(h_0, \varepsilon)$ is the number indicated in Lemma 7). In this case, $norm(\operatorname{proj}_y(I_i)) < 1/h_0$ for every $i \geq i_0$. Hence, by Lemma 7, $|F(I_i)| < \varepsilon$ holds for every $i \geq i_0$.

For an interval $I = [a_1, b_1; a_2, b_2; \dots; a_n, b_n]$ in E_n, we denote by $\boldsymbol{R}_m(I)$ the family of all intervals which are written; $[a_1 + (k_1(b_1 - a_1))/m, a_1 + ((k_1 + 1)(b_1 - a_1))/m; a_2 + (k_2(b_2 - a_2))/m, a_2 + ((k_2 + 1)(b_2 - a_2))/m; \dots; a_n + (k_n(b_n - a_n))/m, a_n + ((k_n + 1)(b_n - a_n))/m]$, where k_i is an integer with $0 \leq k_i \leq m - 1$ for $i = 1, 2, \dots, n$.

Lemma 8. Let f be (D) integrable on an interval R_0 in E_{n_0} ($n_0 \geq 2$). Given a sequence of positive numbers ε_n ($n = 1, 2, \dots$) such that $\varepsilon_n \downarrow 0$ and $\sum_{m=n+1}^{\infty} \varepsilon_m < \varepsilon_n$ for every $n \in N$, there exist:

nondecreasing sequences of closed sets A_i ($i = 1, 2, \dots$) and D_i ($i = 1, 2, \dots$) such that

(1) $\mu_{n_0}(R_0 - \cup_{i=1}^{\infty} A_i) = 0$ and $\mu_{n_0}(R_0 - \cup_{i=1}^{\infty} D_i) = 0$;

(2) $A_i \supset D_i$ for every $i \in N$;

(3) f is Lebesgue integrable on D_i for every $i \in N$,

and a nonincreasing sequence of positive numbers τ_i^* ($i = 1, 2, \dots$) with $\tau_i^* \downarrow 0$,

in such a way that the following statement (4) holds:

(4) For each $i \in N$ the following holds: If S is a $(**)$-elementary system consisting of $(*)$-elementary systems S_l ($l = 1, 2, \dots, l_0$), where for each l

$$S_l \text{ is a } (*)\text{-elementary system consisting of intervals written } I_{lj}\ (j = 1, 2, \dots, j_0(l)),$$

such that

(4.0) $norm\left(\underset{E_{n_0-1}}{\operatorname{proj}}{}_y(S_l)\right) < \tau_i^*$ for $l = 1, 2, \dots, l_0$,

and for which there exists a non-empty measurable set Y in $\underset{E_{n_0-1}}{\operatorname{proj}}{}_y(R_0)$ such that:

(4.1) $Y \subset \underset{E_{n_0-1}}{\operatorname{proj}}{}_y(S^\circ) \cap \underset{E_{n_0-1}}{\operatorname{proj}}{}_y(A_i)$;

(4.2) $\mu_{n_0-1}\left(\underset{E_{n_0-1}}{\operatorname{proj}}{}_y(S) - Y\right) < \tau_i^*$;

(4.3) $Y \cap \underset{E_{n_0-1}}{\operatorname{proj}}{}_y((S_l)^\circ) \neq \emptyset$ for every $l \in \{1, 2, \dots, l_0\}$;

(4.4) for each $l \in \{1, 2, \dots, l_0\}$ if $q \in Y \cap \underset{E_{n_0-1}}{\operatorname{proj}}{}_y((S_l)^\circ)$, then

$(I_{lj})^q \cap (A_i)^q \neq \emptyset$ for every $j \in \{1, 2, \ldots, j_0(l)\}$,

then the following inequality holds:

$$\left| F(S) - (L) \int_{S \cap D_i} f \right| < \varepsilon_i.$$

We emphasize that, in the two-dimensional case, we can remove the assumption (4.0) in the statement (4) of Lemma 8 above. In detail:

Lemma 9. When $n_0 = 2$, for the $\{A_i\}_{i=1}^{\infty}$, $\{D_i\}_{i=1}^{\infty}$ and $\{\tau_i^*\}_{i=1}^{\infty}$ indicated in Lemma 8 the following statement (4*) holds:

(4*) For every $i \in N$, the following holds: Let S be a $(**)$-elementary system with the form indicated in (4) of Lemma 8 (without the assumption of (4.0)). If, for such an S, there exists a non-empty measurable set Y in $\operatorname*{proj}_{E_{n_0-1}} {}_y(R_0)$ satisfying the conditions (4.1), (4.2), (4.3) and (4.4) in Lemma 8, then $\left| F(S) - (L) \int_{S \cap D_i} f \right| < \varepsilon_i$ holds.

Proof of Lemma 8. For simplicity, we prove only for the case when $n_0 = 2$ and $R_0 = [0, 1; 0, 1]$. For the $\{\varepsilon_n\}_{n=1}^{\infty}$ given in the lemma, we define $\{\varepsilon_n^{**}\}_{n=1}^{\infty}$ as in (6°). Let n_i, m_i and $F_{n_i m_i}$ $(i = 1, 2, \ldots)$ be the sequences of integers and the sequence of closed sets obtained as in (I), associating with R_0, $\{\overline{M}_n\}_{n=1}^{\infty}$ and $\{\varepsilon_n^{**}\}_{n=1}^{\infty}$. Put

$$\tau_i = (1/2) \min\{1/m_i,\ \rho^*(m_i,\ \varepsilon_i/2^4)\} \text{ for each } i \in N, \tag{17°}$$

where $\rho^*(h, \varepsilon)$ is the number defined in (12°). Then, τ_i $(i = 1, 2, \ldots)$ is a nonincreasing sequence with $\tau_i \downarrow 0$.

For each $i \in N$, take an $h(i) \in N$ so that

$$h(i) > i,\ h(j) > h(i) \text{ for } j > i \text{ and } \mu_2(F_{n_i m_i} - F_{m_{h(i)}}) < \tau_i. \tag{18°}$$

Put

$$A_i = F_{n_i m_i} \text{ and } D_i = F_{n_i m_i} \cap F_{m_{h(i)}} \text{ for each } i \in N. \tag{19°}$$

Then

$$A_i \supset D_i \text{ and } \mu_2(A_i - D_i) < \tau_i \text{ for each } i \in N.$$

Put

$$\tau_i^* = (1/2) \min\left\{\tau_i,\ \eta(m_{h(i)},\ \varepsilon_{h(i)}/2^5)\right\} \text{ for each } i \in N. \tag{20°}$$

Then, $\{A_i\}_{i=1}^{\infty}$ and $\{D_i\}_{i=1}^{\infty}$ are nondecreasing sequences of closed sets satisfying (1), (2) and (3) of the lemma, and $\{\tau_i^*\}_{i=1}^{\infty}$ is a nonincreasing sequence with $\tau_i^* \downarrow 0$ by (4°). Next, we shall prove that the statement (4) holds for them. The proof requires three steps.

Take an $i \in N$ and fix. Under the assumption of (4) of the lemma:

(i) The case when $\mu_1(Y \cap \operatorname{proj}_y((S_l)^\circ)) > 0$ for $l = 1, 2, \ldots, l_0$: There exists an $m_0(i)$ with

$$m_0(i) > m_i \tag{21°}$$

such that: for each pair l, j with $l \in \{1, 2, \ldots, l_0\}$ and $j \in \{1, 2, \ldots, j_0(l)\}$, there exists a non-empty family of cells belonging to $\boldsymbol{R}_{m_0(i)}(I_{lj})$, denoted by

$$R_{ljs} \ (s = 1, 2, \ldots, s_0(l, j)),$$

such that:

1) $R_{ljs} \cap A_i \neq \emptyset$ for $s = 1, 2, \ldots, s_0(l, j)$;

2) $R \cap A_i = \emptyset$ for the other cells R belonging to $\boldsymbol{R}_{m_0(i)}(I_{lj})$;

3) $\mu_2 \left(\cup_{s=1}^{s_0(l,j)} R_{ljs} - A_i \right) < \tau_{h(i)} / \sum_{l=1}^{l_0} j_0(l)$;

and, further, when we denote the family of R_{ljs} for which

$$R_{ljs} \cap D_i \neq \emptyset, \text{ where } s \in \{1, 2, \ldots, s_0(l, j)\}$$

by R_{ljs} $(s = 1, 2, \ldots, s_1(l, j))$ (possibly empty), where $s_1(l, j) \leq s_0(l, j)$ (without loss of generality, such expression is possible), we have

4) $\mu_2 \left(\cup_{s=1}^{s_1(l,j)} R_{ljs} - D_i \right) < \tau_{h(i)} / \sum_{l=1}^{l_0} j_0(l)$.

Denote, for each pair l, j, the set $\cup(\text{proj}_y(R) - \text{proj}_y((R)^\circ))$, where the union $\cup$ is over all cells R belonging to $\boldsymbol{R}_{m_0(i)}(I_{lj})$, by E_{lj}. Then, $E_{lj} = E_{lj'}$ for $j, j' \in \{1, 2, \ldots, j_0(l)\}$. Denote the common set by E_l.

Now put $Y_l = (Y - E_l) \cap \text{proj}_y((S_l)^\circ)$ for $l = 1, 2, \ldots, l_0$. Then, we have

$$Y_l \cap Y_{l'} = \emptyset \text{ for } l, \ l' \in \{1, 2, \ldots, l_0\} \text{ with } l \neq l';$$

$$\cup_{l=1}^{l_0} Y_l \subset Y \text{ and } \mu_1(Y - \cup_{l=1}^{l_0} Y_l) = 0.$$

In this case, as seen in [9, proof of Lemma 3], for each $l \in \{1, 2, \ldots, l_0\}$ there exists a finite sequence of intervals in $\text{proj}_y(R_0)$ written

$$J(y_v^l) \ (v = 1, 2, \ldots, v_0(l)),$$

having the following properties:

1*) $y_v^l \in Y_l$ for $v = 1, 2, \ldots, v_0(l)$;

2*) $y_v^l \in (J(y_v^l))^\circ$ for $v = 1, 2, \ldots, v_0(l)$;

3*) $\cup_{v=1}^{v_0(l)} J(y_v^l) \subset \text{proj}_y((S_l)^\circ)$;

4*) $\mu_1 \left(Y_l - \cup_{v=1}^{v_0(l)} J(y_v^l) \right) < \tau_i^* / l_0$;

5*) $norm(J(y_v^l)) < 1/m_i$ for $v = 1, 2, \ldots, v_0(l)$;

6*) $J(y_v^l) \cap J(y_{v'}^l) = \emptyset$ for $v \neq v'$;

7*) The both end-points of $J(y_v^l)$ belong to Y_l.

(Refore to [9, Remark 1, (1)] for the case $n_0 - 1 \geq 2$.)

By 3*), 4*) and (4.2), we have:

8*) $\mu_1(\mathrm{proj}_y(S) - \cup_{l=1}^{l_0} \cup_{v=1}^{v_0(l)} J(y_v^l)) < 2\tau_i^*$.

Because, $\mu_1(\mathrm{proj}_y(S) - \cup_{l=1}^{l_0} \cup_{v=1}^{v_0(l)} J(y_v^l)) = \sum_{l=1}^{l_0} \mu_1(\mathrm{proj}_y(S_l) - \cup_{v=1}^{v_0(l)} J(y_v^l)) \leq \sum_{l=1}^{l_0} \mu_1(\mathrm{proj}_y(S_l) - Y_l) + \sum_{l=1}^{l_0} \mu_1(Y_l - \cup_{v=1}^{v_0(l)} J(y_v^l)) < \mu_1(\mathrm{proj}_y(S) - \cup_{l=1}^{l_0} Y_l) + \tau_i^* \leq \mu_1(\mathrm{proj}_y(S) - Y) + \mu_1(Y - \cup_{l=1}^{l_0} Y_l) + \tau_i^* < 2\tau_i^*$.

In this case, as seen in [9, proof of Lemma 3], $J(y_v^l)$ $(v = 1, 2, \dots, v_0(l))$ can be chosen to have the following properties (α) and (β) in addition to the properties 1*)-7*) above:

(α) For each $l \in \{1, 2, \dots, l_0\}$, put

$$I_v^l = \mathrm{proj}_x(R_0) \times J(y_v^l) \text{ for } v = 1, 2, \dots, v_0(l).$$

Then, for every interval $I_v^l \cap R_{ljs}$ belonging to the family:

$$\big\{ I_v^l \cap R_{ljs} : (R_{ljs})^{y_v^l} \cap (A_i)^{y_v^l} \neq \emptyset, \\ \text{where } j \in \{1, 2, \dots, j_0(l)\} \text{ and } s \in \{1, 2, \dots, s_0(l, j)\} \big\} \tag{22°}$$

which is non-empty, we have

$$\mathrm{proj}_y(I_v^l \cap R_{ljs}) = J(y_v^l) \text{ for every } v \in \{1, 2, \dots, v_0(l)\}.$$

(β) For each $l \in \{1, 2, \dots, l_0\}$, put

$$I_{vj}^l = \mathrm{proj}_x(I_{lj}) \times J(y_v^l) \text{ for } v = 1, 2, \dots, v_0(l) \text{ and } j \in 1, 2, \dots, j_0(l),$$

and, in each I_{vj}^l, consider the family of all two-dimensional intervals I contained in I_{vj}^l, such that the both sides of I parallel to y-axis, say $ss(I)$, belong to $K, (I - ss(I)) \subset I_{vj}^l - K$ and $\mathrm{proj}_y(I) = \mathrm{proj}_y(I_{vj}^l)$, where K is the closed set consisting of the set $\cup(R_{ljs} \cap I_{vj}^l)$, $\cup$ is over all R_{ljs}, $s = 1, 2, \dots, s_0(l, j)$ with $(R_{ljs})^{y_v^l} \cap (A_i)^{y_v^l} \neq \emptyset$, and the both sides of I_{vj}^l parallel to y-axis. Denote the family by

$$L_{vjz}^l \ (z = 1, 2, \dots, z_0(l, v, j)).$$

For simplicity, for each pair l, v with $l \in \{1, 2, \dots, l_0\}$ and $v \in \{1, 2, \dots, v_0(l)\}$, denote the family

$$L_{vjz}^l \ (j = 1, 2, \dots, j_0(l),\ z = 1, 2, \dots, z_0(l, v, j))$$

by

$$L_{vw}^l \ (w = 1, 2, \dots, w_0(l, v)). \tag{23°}$$

Then we have

$$L_{vw}^l \cap A_i = \emptyset \text{ for } w = 1, 2, \dots, w_0(l, v);$$

and L^l_{vw} $(l=1,2,\dots,l_0,\ v=1,2,\dots,v_0(l),\ w=1,2,\dots,w_0(l,v))$ are mutually disjoint.

Next, for each $l\in\{1,2,\dots,l_0\}$, denote the family of intervals contiguous to the closed set consisting of $\cup_{v=1}^{v_0(l)}J(y^l_v)$ and the both end-points of $\mathrm{proj}_y(S_l)$ by

$$J^{*l}_u\ (u=1,2,\dots,u_0(l)).$$

(Refer to [9, Remark 1, (2)] for the case of $n_0-1\geq 2$.)

Put

$$I^{*l}_u=\mathrm{proj}_x(R_0)\times J^{*l}_u \text{ for } u=1,2,\dots,u_0(l);$$

$$I^{*l}_{uj}=I^{*l}_u\cap I_{lj} \text{ for } u=1,2,\dots,u_0(l) \text{ and } j=1,2,\dots,j_0(l). \qquad (24^\circ)$$

(i,1) Denote the family of intervals indicated in (α) (defined in (22°)):

$$\big\{I^l_v\cap R_{ljs}:(R_{ljs})^{y^l_v}\cap(A_i)^{y^l_v}\neq\emptyset,\ l\in\{1,2,\dots,l_0\},\ v\in\{1,2,\dots,v_0(l)\},$$

$$j\in\{1,2,\dots,j_0(l)\} \text{ and } s\in\{1,2,\dots,s_0(l,j)\}\big\}$$

by

$$R_s\ (s=1,2,\dots,s_0).$$

In this case without loss of generality, we can suppose that

$$R_s\cap D_i\neq\emptyset \text{ for } s\in\{1,2,\dots,s_1\} \text{ and } R_s\cap D_i=\emptyset \text{ for } s\in\{s_1+1,\dots,s_0\},$$

where $0\leq s_1\leq s_0$ (if $s_1=0$, then the former is empty; if $s_1=s_0$, then the latter is empty).

First, for R_s $(s=1,2,\dots,s_1)$, we have, as seen in [9, proof of Lemm 3, (i,2)],

1) $R_s\cap M_{m_{h(i)}}\neq\emptyset$ for $s=1,2,\dots,s_1$;

2) $$\begin{aligned}\mu_2\left(\cup_{s=1}^{s_1}R_s-M_{m_{h(i)}}\right)&\leq\mu_2\left(\cup_{s=1}^{s_1}R_s-D_i\right)\\&\leq\textstyle\sum_{l=1}^{l_0}\sum_{j=1}^{j_0(l)}\mu_2\left(\cup_{s=1}^{s_1(l,j)}R_{ljs}-D_i\right)\\&<\textstyle\sum_{l=1}^{l_0}\left(\left(\tau_{h(i)}/\sum_{l=1}^{l_0}j_0(l)\right)\times(j_0(l))\right)\\&=\tau_{h(i)}<\delta\left(m_{h(i)},\varepsilon_{h(i)}/2^{11}\right) \text{ by 4) above.}\end{aligned}$$

Further

3) $norm(R_s)<1/m_0(i)<1/m_i<1/m_{h(i)}$, since $m_0(i)>m_i$ and $i<h(i)$ by (21°) and (18°).

Hence, by the definition of (D) integrability we have

$$\left|\sum_{s=1}^{s_1}F(R_s)-\sum_{s=1}^{s_1}(L)\int_{R_s\cap F_{m_{h(i)}}}f\right|<\varepsilon_{h(i)}/2^{11}.$$

On the other hand, as in [9, proof of Lemma 3, (i,2)]

$$\left|\sum_{s=1}^{s_1}(L)\int_{R_s\cap(F_{m_{h(i)}}-D_i)} f\right| < \varepsilon_{h(i)}/2^8.$$

Hence

$$\left|\sum_{s=1}^{s_1}F(R_s)-\sum_{s=1}^{s_1}(L)\int_{R_s\cap D_i} f\right| < \varepsilon_i/2^{11}+\varepsilon_{h(i)}/2^8.$$

Next, consider for R_s $(s=s_1+1,\dots,s_0)$. Then, as seen in [9, proof of Lemma 3, (i,2)] we have

(1) $R_s\cap\overline{M}_{m_i}\neq\emptyset$ for $s=s_1+1,\dots,s_0$;

(2) $\mu_2\left(\cup_{s=s_1+1}^{s_0}R_s\right)<\delta(m_i,\varepsilon_i/2^7)$.

Further, we have

(3) $norm(R_s)<1/m_i$ for $s=s_1+1,\dots,s_0$.

Hence, by Lemma 6

$$\left|\sum_{s=s_1+1}^{s_0}F(R_s)-\sum_{s=s_1+1}^{s_0}(L)\int_{R_s\cap F_{m_i}} f\right| < 4\varepsilon_i/2^7=\varepsilon_i/2^5.$$

Therefore, as in [9, proof of Lemma 3, (i,2)], we obtain

$$\left|\sum_{s=1}^{s_0}F(R_s)-\sum_{s=1}^{s_0}(L)\int_{R_s\cap D_i} f\right| < \varepsilon_i/2^7+3\varepsilon_i/2^5.$$

(i,2) For the family indicated in (β) (defined in (23°)):

$$L^l_{vw}\ \ (v=1,2,\dots,v_0(l),\ w=1,2,\dots,w_0(l,v)):$$

Corresponding to each two-dimensional interval L^l_{vw}, consider the one-dimensional interval, say J^l_{vw}, determined uniquely by the following four conditions, by virtue of the assumption of (4.4) of the lemma:

1°) J^l_{vw} is contained in an interval, say J^{*l}_{vw}, which is one of the intervals contiguous to the closed set consisting of the set $(A_i)^{y^l_v}$, i.e., $(F_{n_im_i})^{y^l_v}$ and the both end-points of the interval $(R_0)^{y^l_v}$;

2°) One end-point of J^l_{vw} is one of the end-points of $(L^l_{vw})^{y^l_v}$;

3°) The other end-point of J^l_{vw} is the characteristic point of the interval J^{*l}_{vw} taken in 1°) above, say p^l_{vw};

4°) $J^l_{vw}\supset(L^l_{vw})^{y^l_v}$.

In this case, J^l_{vw} $(w=1,2,\dots,w_0(l,v))$ are classified into two parts: J^{l1}_{vw} $(w=1,2,\dots,w_1(l,v))$ and J^{l2}_{vw} $(w=1,2,\dots,w_2(l,v))$ so that each part consists of mutually disjoint intervals. Denote the interval associated with J^{l1}_{vw} in 1°) by J^{*l1}_{vw}. Similarly, we define J_{vw}^{*l2}. We denote the characteristic point of J_{vw}^{*l1} by p_{vw}^{l1}. Similarly, we define p_{vw}^{l2}. We denote the characteristic numbers of p_{vw}^{l1} and p_{vw}^{l2} by h_{vw}^{l1} and h_{vw}^{l2}, respectively. We have $n_i \leq h_{vw}^{l1} \leq m_i$ and $n_i \leq h_{vw}^{l2} \leq m_i$.

Put

$$H_{vw}^{l1} = J_{vw}^{l1} \times J(y_v^l) \text{ for } v = 1, 2, \ldots, v_0(l) \text{ and } w = 1, 2, \ldots, w_1(l, v).$$

For each $k \in N$ with $n_i \leq k \leq m_i$, denote the families of all intervals J_{vw}^{l1} and H_{vw}^{l1} for which $h_{vw}^{l1} = k$ by

$$J_{vw}^{l1k} \ (w = 1, 2, \ldots, w_1(l, v, k)) \text{ and } H_{vw}^{l1k} \ (w = 1, 2, \ldots, w_1(l, v, k)),$$

respectively. Then, H_{vw}^{l1k} $(l = 1, 2, \ldots, l_0,\ v = 1, 2, \ldots, v_0(l),\ w = 1, 2, \ldots, w_1(l, v, k))$ is an elemetary system in R_0 such that:

(1) $H_{vw}^{l1k} \cap \overline{M}_k \neq \emptyset$;

(2) $\mu_2\left(\cup_{l=1}^{l_0} \cup_{v=1}^{v_0(l)} \cup_{w=1}^{w_1(l,v,k)} H_{vw}^{l1k}\right) < \varepsilon_k^{**} \leq \delta(k, \varepsilon_k/2^{k+5})$ as seen in [9, proof of Lemma 3, (i,1)];

(3)
$$\begin{aligned} \mathit{norm}\ (H_{vw}^{l1k}) &= \max\{|J_{vw}^{l1k}|, \mathit{norm}(J(y_v^l))\} \\ &\leq \max\{\varepsilon_k^{**}, 1/m_i\} \text{ (by } h_{vw}^{l1} = k \text{ and } 5^*)) \\ &\leq 1/k \text{ (by } (6^\circ) \text{ and } k \leq m_i). \end{aligned}$$

Hence, by Lemma 6

$$\left| \sum_{l=1}^{l_0} \sum_{v=1}^{v_0(l)} \sum_{w=1}^{w_1(l,v,k)} F(H_{vw}^{l1k}) - \sum_{l=1}^{l_0} \sum_{v=1}^{v_0(l)} \sum_{w=1}^{w_1(l,v,k)} (L) \int_{H_{vw}^{l1k} \cap F_k} f \right| < 4(\varepsilon_k/2^{k+5}) = \varepsilon_k/2^{k+3}.$$

Further, as in [9, proof of Lemma 3, (i,1)]

$$\left| \sum_{l=1}^{l_0} \sum_{v=1}^{v_0(l)} \sum_{w=1}^{w_1(l,v,k)} (L) \int_{H_{vw}^{l1k} \cap F_k} f \right| < \varepsilon_k/2^{k+5}.$$

Therefore

$$\left| \sum_{k=n_i}^{m_i} \sum_{l=1}^{l_0} \sum_{v=1}^{v_0(l)} \sum_{w=1}^{w_1(l,v,k)} F(H_{vw}^{l1k}) \right| < \varepsilon_i/8.$$

Consequently

$$\left| \sum_{l=1}^{l_0} \sum_{v=1}^{v_0(l)} \sum_{w=1}^{w_1(l,v)} F(H_{vw}^{l1}) \right| < \varepsilon_i/8.$$

Similarly

$$\left|\sum_{l=1}^{l_0}\sum_{v=1}^{v_0(l)}\sum_{w=1}^{w_1(l,v)} F(H_{vw}^{l1} - L_{vw}^{l1})\right| < \varepsilon_i/8.$$

Hence

$$\left|\sum_{l=1}^{l_0}\sum_{v=1}^{v_0(l)}\sum_{w=1}^{w_1(l,v)} F(L_{vw}^{l1})\right| < \varepsilon_i/4.$$

Thus, since $L_{vw}^{l1} \cap D_i = \emptyset$,

$$\left|\sum_{l=1}^{l_0}\sum_{v=1}^{v_0(l)}\sum_{w=1}^{w_1(l,v)} \left(F(L_{vw}^{l1}) - (L)\int_{L_{vw}^{l1}\cap D_i} f\right)\right| < \varepsilon_i/4.$$

Similarly

$$\left|\sum_{l=1}^{l_0}\sum_{v=1}^{v_0(l)}\sum_{w=1}^{w_2(l,v)} \left(F(L_{vw}^{l2}) - (L)\int_{L_{vw}^{l2}\cap D_i} f\right)\right| < \varepsilon_i/4.$$

Therefore

$$\left|\sum_{l=1}^{l_0}\sum_{v=1}^{v_0(l)}\sum_{w=1}^{w_0(l,v)} \left(F(L_{vw}^{l}) - (L)\int_{L_{vw}^{l}\cap D_i} f\right)\right| < \varepsilon_i/2.$$

(i,3): For I_{uj}^{*l} $(l = 1,2,\dots,l_0,\ u = 1,2,\dots,u_0(l),\ j = 1,2,\dots,j_0(l))$ defined in (24°): For each pair l,u with $l \in \{1,2,\dots,l_0\}$ and $u \in \{1,2,\dots,u_0(l)\}$, denote the $(*)$-elementary system:

$$I_{uj}^{*l}\ (j = 1,2,\dots,j_0(l))$$

by S_u^l and consider the $(**)$-elementary system consisting of $(*)$-elementary systems

$$S_u^l\ (l = 1,2,\dots,l_0,\ u = 1,2,\dots,u_0(l)).$$

(Refer to [9, Remark 1, (3)] for the case of $n_0 - 1 \geq 2$.)

Then, as seen in [9, proof of Lemma 3]

there exists a $y_{lu} \in \mathrm{proj}_y(S_u^l) \cap \mathrm{proj}_y(B_{m_i})$ for which $(I_{uj}^{*l})^{y_{lu}} \cap (B_{m_i})^{y_{lu}} \neq \emptyset$ for $j = 1,2,\dots,j_0(l)$, for each pair l,u with $l \in \{1,2,\dots,l_0\}$ and $u \in \{1,2,\dots,u_0(l)\}$;

$\sum_{l=1}^{l_0}\sum_{u=1}^{u_0(l)} \left|\mathrm{proj}_y(S_u^l)\right| \leq \mu_1\left(\mathrm{proj}_y(S) - Y\right) + \mu_1\left(\cup_{l=1}^{l_0} Y_l - \cup_{l=1}^{l_0}\cup_{v=1}^{v_0(l)} J(y_v^l)\right)$ $< \tau_i^* + \tau_i^* \leq \tau_i < \rho^*(m_i, \varepsilon_i/2^4)$ by (4.2) and 4*);

and further, by (4.0)

$norm(\mathrm{proj}_y(S_u^l)) \leq norm(\mathrm{proj}_y(S_l)) < \tau_i^* < \tau_i < 1/m_i$ for every pair l,u with $l \in \{1,2,\dots,l_0\}$ and $u \in \{1,2,\dots,u_0(l)\}$.

(**Remark A**: When $n_0 = 2$, the assumption (4.0) is not needed. Because, by 8*), (20°) and (17°) we have $\mu_1(\mathrm{proj}_y(I^{*l}_{uj})) = \mu_1(J^{*l}_u) < 2\tau^*_i \le \tau_i < 1/m_i$.)

Therefore, by Lemma 7

$$\left|\sum_{l=1}^{l_0}\sum_{u=1}^{u_0(l)} F(S^l_u)\right| < \varepsilon_i/2^4.$$

On the other hand, since $\sum_{l=1}^{l_0}\sum_{u=1}^{u_0(l)} |S^l_u| \le \sum_{l=1}^{l_0}\sum_{u=1}^{u_0(l)} |\,\mathrm{proj}_y(S^l_u)| < 2\tau^*_i \le \eta(m_{h(i)}, \varepsilon_{h(i)}/2^5)$ and $D_i \subset F_{m_{h(i)}}$,

$$\left|\sum_{l=1}^{l_0}\sum_{u=1}^{u_0(l)} (L)\int_{S^l_u\cap D_i} f\right| < \varepsilon_{h(i)}/2^5 \le \varepsilon_i/2^5.$$

Therefore

$$\left|\sum_{l=1}^{l_0}\sum_{u=1}^{u_0(l)} F(S^l_u) - \sum_{l=1}^{l_0}\sum_{u=1}^{u_0(l)} (L)\int_{S^l_u\cap D_i} f\right| < \varepsilon_i/2^4 + \varepsilon_i/2^5.$$

Consequently, by(i,1), (i,2) and (i,3)

$$\left|F(S) - (L)\int_{s\cap D_i} f\right| < \varepsilon_i/2^7 + 3\varepsilon_i/2^5 + \varepsilon_i/2 + \varepsilon_i/2^4 + \varepsilon_i/2^5.$$

(ii) The case when $\mu_1(Y \cap \mathrm{proj}_y((S_l)^\circ)) = 0$ for $l = 1, 2, \dots, l_0$: As in [9, proof of Lemma 3], for every $l \in \{1, 2, \dots, l_0\}$, there exists a $y_l \in \mathrm{proj}_y(S_l) \cap \mathrm{proj}_y(B_{m_i})$ for which $(I_{lj})^{y_l} \cap (B_{m_i})^{y_l} \ne \emptyset$ for $j = 1, 2, \dots, j_0(l)$; $|\,\mathrm{proj}_y(S)| < \tau^*_i < \tau_i < \rho^*(m_i, \varepsilon_i/2^4)$; and by (4.0) $norm(\mathrm{proj}_y(S_l)) < \tau^*_i < \tau_i < 1/m_i$ for $l = 1, 2, \dots, l_0$.

(**Remark B**: When $n_0 = 2$, the condition (4.0) is not needed. Because, we have $\mu_1(Y \cap \mathrm{proj}_y((S_l)^\circ)) = 0$, and so $\mu_1(Y \cap \mathrm{proj}_y(S_l)) = 0$. Hence, by (4.2) $\mu_1(\mathrm{proj}_y(S_l)) < \tau^*_i < 1/m_i$. So $norm(\mathrm{proj}_y(S_l)) < 1/m_i$ for $l = 1, 2, \dots, l_0$.)

Hence, by Lemma 7

$$|F(S)| < \varepsilon_i/2^4.$$

Further, since $\mu_2(S) \le \mu_1(\mathrm{proj}_y(S)) < \tau^*_i < \eta(m_{h(i)}, \varepsilon_{h(i)}/2^5)$ and $D_i \subset F_{m_{h(i)}}$,

$$\left|(L)\int_{S\cap D_i} f\right| < \varepsilon_{h(i)}/2^5 \le \varepsilon_i/2^5.$$

Consequently

$$\left|F(S) - (L)\int_{S\cap D_i} f\right| < \varepsilon_i/2^4 + \varepsilon_i/2^5.$$

(iii) The case when $\mu_1(Y \cap \mathrm{proj}_y((S_l)^\circ)) > 0$ for some $l \in \{1, 2, \dots, l_0\}$ and $\mu_1(Y \cap \mathrm{proj}_y((S_l)^\circ)) = 0$ for some $l \in \{1, 2, \dots, l_0\}$: This case follows from the results of (i) and (ii).

By (i), (ii) and (iii) the proof is complete.

Proof of Lemma 9. The proof of Lemma 9 is complete by Remarks A and B with the proof of Lemma 8.

Theorem 4. Let $f(x_1, x_2, \dots, x_{n_0})$ be (D) integrable on the interval $R_0 = [a_1, b_1; a_2, b_2; \dots; a_{n_0}, b_{n_0}]$ in the n_0-dimensional Euclidean space E_{n_0} $(n_0 \geq 2)$. Then, the following two statements hold.

(1) Given any $n \in \{1, 2, \dots, n_0\}$, for almost all $q = (x_1, x_2, \dots, x_{n-1}, x_{n+1}, \dots, x_{n_0})$ in the $(n_0 - 1)$-dimensional interval $[a_1, b_1; a_2, b_2; \dots; a_{n-1}, b_{n-1}; a_{n+1}, b_{n+1}; \dots; a_{n_0}, b_{n_0}]$ the function $f(x_1, x_2, \dots, x_{n_0})$ considered as a function of x_n in the one-dimensional interval $[a_n, b_n]$ is (D) integrable on $[a_n, b_n]$.

(2) Corresponding to each $n \in \{1, 2, \dots, n_0\}$, there exists a nondecreasing sequence of closed sets D_i $(i = 1, 2, \dots)$ in R_0 such that $\mu_{n_0}(R_0 - \cup_{i=1}^{\infty} D_i) = 0$ and

$$(D) \int_{a_n}^{b_n} f(x_1, x_2, \dots, x_n, \dots, x_{n_0}) dx_n = \lim_{i \to \infty} (L) \int_{(D_i)^q} f(x_1, x_2, \dots, x_n, \dots, x_{n_0}) dx_n$$

for almost all $q = (x_1, x_2, \dots, x_{n-1}, x_{n+1} \dots, x_{n_0})$ in the $(n_0 - 1)$-dimensional interval $[a_1, b_1; a_2, b_2; \dots; a_{n-1}, b_{n-1}; a_{n+1}, b_{n+1}; \dots; a_{n_0}, b_{n_0}]$.

Proof. For simplicity, we prove only for the case when $n_0 = 2$ and $R_0 = [0, 1; 0, 1]$. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be the sequence of positive numbers given in (5°) such that $\varepsilon_n \downarrow 0$ and $\sum_{m=n+1}^{\infty} \varepsilon_m < \varepsilon_n$ for every $n \in N$. Let

$$A_i = F_{n_i m_i} \ (i = 1, 2, \dots) \text{ and } D_i = F_{n_i m_i} \cap F_{m_{h(i)}} \ (i = 1, 2, \dots)$$

$$\tau_i^* \ (i = 1, 2, \dots)$$

be the nondecreasing sequences of closed sets and the sequence of positive numbers defined as seen in the proof of Lemma 8 for the sequence $\{\varepsilon_n\}_{n=1}^{\infty}$. Let $Z = \text{proj}_y(R_0) - \cup_{i=1}^{\infty} \text{proj}_y(F_{n_i m_i})$ (defined in (8°)) as in (I).

As seen in [9, proof of Theorem 1], there exists a set X_0 of μ_1-measure zero with $X_0 \supset Z$ such that for every $y \in \text{proj}_y(R_0) - X_0$, we have:

(a) $(A_i)^y$ $(i = 1, 2, \dots)$ is a nondecreasing sequence of closed sets whose union is $(R_0)^y$; and

(b) $(D_i)^y$ $(i = 1, 2, \dots)$ is a nondecreasing sequence of closed sets such that $(D_i)^y \subset (A_i)^y$, $\mu_1((R_0)^y - \cup_{i=1}^{\infty}(D_i)^y) = 0$ and $f(x, y)$ is Lebesgue integrable on $(D_i)^y$ as a function of x for every $i \in N$.

Hence, by [9, Remark 2], in order that the function $f(x, y)$ is (D) integrable on [0, 1] as a function of x for almost all $y \in \text{proj}_y(R_0) - X_0$, it is sufficient to prove that, when we denote the set of all $y \in \text{proj}_y(R_0) - X_0$ for which one at least of the statements of (1) and (2) in [9, Lemma 5] is not true by Y^*, we have $\mu_1(Y^*) = 0$. In order to prove it, supposing $\mu_1(Y^*) > 0$, we lead a contradiction. For the proof, we can proceed in the same way as in the case of [9, Theorem 1], making the following alterations:

(1) We employ τ_i^* defined in (20°) of this paper instead of κ_i^* defined in [9, (14°)];

(2) We employ Lemma 8 of this paper instead of [9, Lemma 3];

(3) For each $s \in^* \{1, 2, \dots, s_0\}$ with $\mu_1(Z_s) \neq \emptyset$, we choose the one-dimensional elementary system K_l^s $(l = 1, 2, \dots, l_0(s))$ indicated in [9, proof of Theorem 1] so that it satisfies the condition

$$norm(K_l^s) < \tau_{i'(s)}^* \text{ for } l = 1, 2, \dots, l_0(s),$$

in addition to the following conditions indicated in [9, (23°) and (24°)]:

$(K_l^s)^\circ \cap Z_s \neq \emptyset$ for $l = 1, 2, \dots, l_0(s)$;

$\mu_1(Z_s - \cup_{l=1}^{l_0(s)} K_l^s) < k_0/2s_0$; and

$\mu_1(\cup_{l=1}^{l_0(s)} K_l^s - Z_s) < (1/s_0)(\min\{\delta, \tau_{i'(s)}^*\})$.

From this fact, each of the following two-dimensional (∗∗)-elementary systems:

$$S_l^s \ (s \in \triangle_i,\ l \in \{1, 2, \dots, l_0(s)\}) \text{ and } S_l^s \ (s \in \Lambda_i,\ l \in \{1, 2, \dots, l_0(s)\}),$$

considered in (A) and (B) as in [9, proof of Theorem 1], respectively, has the following property:

(4.0) $norm(\mathrm{proj}_y(S_l^s)) = norm(K_l^s) < \tau_i^*$.

By virtue of this fact, we can employ Lemma 8 of this paper instead of [9, Lemma 3].

Now, put

$$W_0 = X_0 \cup Y^*.$$

Then, $\mu_1(W_0) = 0$, and for every $y \in \mathrm{proj}_y(R_0) - W_0$, $f(x, y)$ is (D) integrable as a function of x on [0, 1], $\lim_{i\to\infty}(L)\int_{(D_i)^y} f(x, y)dx$ exists, and the limit value coincides with $(D)\int_0^1 f(x, y)dx$.

§2 Two-dimensional integration

Theorem 5 (Fubini's Theorem). Let $f(x, y)$ be (D) integrable on an interval $R_0 = [a, b; c, d]$ in the two-dimensional Euclidean space E_2. Then:

(1) For almost all $y \in [c, d]$, the function $f(x, y)$ considered as a function of x is (D) integrable on $[a, b]$;

(1′) For almost all $x \in [a, b]$, the function $f(x, y)$ considered as a function of y is (D) integrable on $[c, d]$;

(2) $(D)\int_a^b f(x, y)dx$ is (D) integrable on $[c, d]$;

(2′) $(D)\int_c^d f(x, y)dy$ is (D) integrable on $[a, b]$;

(3) $(D)\int_c^d \left((D)\int_a^b f(x, y)dx\right) dy = (D)\int_a^b \left((D)\int_c^d f(x, y)dy\right) dx = F([a, b; c, d])$.

Proof. (1) and (1′) are already proved in Theorem 4. Next, we prove only (2) and the first equality of (3) for the case of $R_0 = [0, 1; 0, 1]$. We omit the proof of the others, because the proof is similar. Put

$$f_i(y) = (L)\int_{(D_i)^y} f(x,y)dx \text{ for every } y \in \operatorname{proj}_y(R_0) - W_0, \text{ and}$$

$$= 0 \text{ for every } y \in W_0;$$

$$f(y) = (D)\int_0^1 f(x,y)dx \text{ for every } y \in \operatorname{proj}_y(R_0) - W_0, \text{ and}$$

$$= 0 \text{ for every } y \in W_0,$$

where $\{D_i\}_{i=1}^{\infty}$ is the sequence of closed sets and W_0 is the set of μ_1-measure zero, indicated in the proof of Theorem 4.

Since then $f(y) = \lim_{i\to\infty} f_i(y)$ on $\operatorname{proj}_y(R_0)$ and $f_i(y)$ is measurable for each $i \in N$, there exists a sequence of measurable sete M_k^* $(k = 0, 1, \dots)$ such that: $\cup_{k=0}^{\infty} M_k^* = \operatorname{proj}_y(R_0)$; $\mu_1(M_0^*) = 0$; and for each $k \in N$, $M_k^* \cap M_0^* = \emptyset$, $M_{k+1}^* \supset M_k^*$, M_k^* is a closed set, $f_i(y)$ converges uniformly to $f(y)$ on M_k^*, and $f(y)$ is Lebesgue integrable on M_k^*.

Let $\{B_k\}_{k=1}^{\infty}$ be the sequence of measurable sets indicated in Lemma 7 (defined in (11°)). Put

$Z_0 = \operatorname{proj}_y(R_0) - \cup_{k=1}^{\infty} \operatorname{proj}_y(D_k)$;

$L_k = \big((\operatorname{proj}_y(B_k) \cap Z_0) \cup \operatorname{proj}_y(D_k)\big) \cap (M_0^* \cup M_k^*)$ for $k = 1, 2, \dots$;

$N_k = \operatorname{proj}_y(D_k) \cap M_k^*$ for $k = 1, 2, \dots$.

Then, $\mu_1(Z_0) = 0$, L_k $(k = 1, 2, \dots)$ is a nondecreasing sequence of measurable sets whose union is $\operatorname{proj}_y(R_0)$ and N_k $(k = 1, 2, \dots)$ is a nondecreasing sequence of closed sets such that $N_k \subset L_k$ and $\mu_1(\operatorname{proj}_y(R_0) - \cup_{k=1}^{\infty} N_k) = 0$. Further $f(y)$ is Lebesgue integrable on N_k for each $k \in N$.

For $k \in N$ and $\varepsilon > 0$, take an $i_0(k,\varepsilon)$ so that $i_0(k,\varepsilon) \geq k$, $\varepsilon_{i_0(k,\varepsilon)} < \varepsilon/7$ and $|f(y) - f_{i_0(k,\varepsilon)}(y)| < \varepsilon/7$ for every $y \in M_k^*$. Let $\lambda(k,\varepsilon)$ be a positive number such that if $\mu_2(E) < \lambda(k,\varepsilon)$, then $\left|(L)\int_{E\cap D_{i_0(k,\varepsilon)}} f(x,y)d(x,y)\right| < \varepsilon/7$. Let $\lambda^*(k,\varepsilon)$ be a positive number such that if $\mu_1(E) < \lambda^*(k,\varepsilon)$, then $\left|(L)\int_{E\cap N_k} f(y)dy\right| < \varepsilon$. Put

$$\delta^*(k,\varepsilon) = (1/2)\min\{\tau^*_{i_0(k,\varepsilon)},\ \rho^*(k,\varepsilon/7),\ \lambda(k,\varepsilon),\ \lambda^*(k,\varepsilon/7)\}.$$

We denote, for any set $E \subset \operatorname{proj}_y(R_0)$, the set $\operatorname{proj}_x(R_0) \times E$ by E^*, and let $G(I)$ be the interval function in $\operatorname{proj}_y(R_0)$ defined by $G(I) = F((I)^*)$ for any interval $I \subset \operatorname{proj}_y(R_0)$, where F is the interval function indicated in the definition of (D) integrability.

Next, we prove that:

For $k \in N$ and $\varepsilon > 0$, if I_t $(t = 1, 2, \dots, t_0)$ is an elementary system in $\operatorname{proj}_y(R_0)$ such that

$$I_t \cap L_k \neq \emptyset \text{ for } t = 1, 2, \dots, t_0 \text{ and } \mu_1\left(\cup_{t=1}^{t_0} I_t - L_k\right) < \delta^*(k,\varepsilon),$$

then

$$\left|\sum_{t=1}^{t_0} G(I_t) - \sum_{t=1}^{t_0} (L)\int_{I_t\cap N_k} f(y)dy\right| < \varepsilon.$$

For I_t $(t = 1, 2, \ldots, t_0)$, denote by I_{1t} $(t = 1, 2, \ldots, t_1)$ the family of all intervals I_t for which $I_t \cap \mathrm{proj}_y(D_k) \neq \emptyset$, where $t \in \{1, 2, \ldots, t_0\}$; and by I_{2t} $(t = 1, 2, \ldots, t_2)$ the others. Let $\{A_k\}_{k=1}^{\infty}$ $(k = 1, 2, \ldots)$ be the sequence of closed sets indicated in the proof of Theorem 4.

(i) For I_{1t} $(t = 1, 2, \ldots, t_1)$: By Proposition 13, there exists an elementary system H_{1t} $(t = 1, 2, \ldots, t_1)$ in $\mathrm{proj}_y(R_0)$ such that $(H_{1t})^\circ \supset I_{1t}$ for $t = 1, 2, \ldots, t_1$, $\mu_1(\cup_{t=1}^{t_1} H_{1t} - \cup_{t=1}^{t_1} I_{1t}) < \delta^*(k, \varepsilon)$, and

$$\left|\sum_{t=1}^{t_1} G(H_{1t}) - \sum_{t=1}^{t_1} G(I_{1t})\right| < \varepsilon/7.$$

In this case, for the $(**)$-elementary system $(H_{1t})^*$ $(t = 1, 2, \ldots, t_1)$ in R_0, we have $\mathrm{proj}_y((H_{1t})^*) = H_{1t}$ and $\mathrm{proj}_y(((H_{1t})^*)^\circ) = (H_{1t})^\circ$. Further, as seen in [9, proof of Theorem 2], putting $Y = \cup_{t=1}^{t_1} (H_{1t})^\circ \cap \mathrm{proj}_y(A_{i_0(k,\varepsilon)})$, we have $\mu_1(\cup_{t=1}^{t_1} H_{1t} - Y) < \tau^*_{i_0(k,\varepsilon)}$, and $Y \cap (H_{1t})^\circ = (H_{1t})^\circ \cap \mathrm{proj}_y(A_{i_0(k,\varepsilon)}) \neq \emptyset$ for $t = 1, 2, \ldots, t_1$. Hence, by Lemma 9

$$\left|\sum_{t=1}^{t_1} F((H_{1t})^*) - \sum_{t=1}^{t_1} (L)\int_{(H_{1t})^*\cap D_{i_0(k,\varepsilon)}} f(x, y)d(x, y)\right| < \varepsilon_{i_0(k,\varepsilon)} < \varepsilon/7.$$

And so

$$\left|\sum_{t=1}^{t_1} G(H_{1t}) - \sum_{t=1}^{t_1} (L)\int_{H_{1t}} f_{i_0(k,\varepsilon)}(y)dy\right| < \varepsilon/7.$$

Thus, as seen in [9, proof of Theorem 2], the following holds:

$$\left|\sum_{t=1}^{t_1} G(I_{1t}) - \sum_{t=1}^{t_1} (L)\int_{I_{1t}\cap N_k} f(y)d(y)\right| < 5\varepsilon/7.$$

(ii) For I_{2t} $(t = 1, 2, \ldots, t_2)$: For the $(**)$-elementary system $(I_{2t})^*$ $(t = 1, 2, \ldots, t_2)$, as seen in [9, proof of Theorem 2], we have $(I_{2t})^* \cap B_k \neq \emptyset$ for $t = 1, 2, \ldots, t_2$, and $\mu_1(\mathrm{proj}_y(\cup_{t=1}^{t_2} (I_{2t})^*)) < \delta^*(k, \varepsilon) < \rho^*(k, \varepsilon/7)$. Further, $norm(\mathrm{proj}_y((I_{2t})^*)) = \mu_1(I_{2t}) < \delta^*(k, \varepsilon) < \tau^*_{i_0(k,\varepsilon)} \leq \tau^*_k < \tau_k < 1/m_k$ and so $norm(\mathrm{proj}_y((I_{2t})^*)) < 1/k$ by (7°) for $t = 1, 2, \ldots, t_2$. Hence, by Lemma 7

$$\left|\sum_{t=1}^{t_2} F((I_{2t})^*)\right| < \varepsilon/7, \text{and so } \left|\sum_{t=1}^{t_2} G(I_{2t})\right| < \varepsilon/7.$$

Therefore, as seen in [9, proof of Theorem 2] we obtain

$$\left|\sum_{t=1}^{t_2} G(I_{2t}) - \sum_{t=1}^{t_2} (L)\int_{I_{2t}\cap N_k} f(y)d(y)\right| < 2\varepsilon/7.$$

Thus, by (i) and (ii)

$$\left|\sum_{t=1}^{t_0} G(I_t) - \sum_{t=1}^{t_0} (L) \int_{I_t \cap N_k} f(y) d(y)\right| < \varepsilon.$$

Consequently, $f(y)$ is (D_0) integrable on $\text{proj}_y(R_0)$ and so (D) integrable on $\text{proj}_y(R_0)$, and the (D) integral of $f(y)$ on $\text{proj}_y(R_0)$ coincides with $F(R_0)$.

By Proposition 11, (2) and Proposition 12, as a corollary of Theorem 5 we obtain:

Theorem 6 (Fubini's Theorem). Let $f(x, y)$ be strongly (LA) integrable on an interval $R_0 = [a, b; c, d]$ in the two-dimensional Euclidean space E_2. Then:

(1) For almost all $y \in [c, d]$, the function $f(x, y)$ considered as a function of x is strongly (LA) integrable on $[a, b]$;

(1′) For almost all $x \in [a, b]$, the function $f(x, y)$ considered as a function of y is strongly (LA) integrable on $[c, d]$;

(2) $(SLA) \int_a^b f(x, y) dx$ is strongly (LA) integrable on $[c, d]$;

(2′) $(SLA) \int_c^d f(x, y) dy$ is strongly (LA) integrable on $[a, b]$;

(3) $$(SLA) \int_c^d \Big((SLA) \int_a^b f(x, y) dx\Big) dy = (SLA) \int_a^b \Big((SLA) \int_c^d f(x, y) dy\Big) dx = (SLA) \int_{R_0} f(x, y) d(x, y).$$

As a corollary of Theorems 2 and 6, by Proposition 6 we obtain:

Theorem 7. If a finitely additive interval function $F(I)$ in an interval $R_0 = [a, b; c, d]$ in the two-dimensional Euclidean space E_2 is derivable in the strong sense at every point of R_0, then:

(1) For almost all $y \in [c, d]$, the function $F'_s(x, y)$ considered as a function of x is special Denjoy integrable on $[a, b]$;

(1′) For almost all $x \in [a, b]$, the function $F'_s(x, y)$ considered as a function of y is special Denjoy integrable on $[c, d]$;

(2) $(\mathcal{D}) \int_a^b F'_s(x, y) dx$ is special Denjoy integrable on $[c, d]$;

(2′) $(\mathcal{D}) \int_c^d F'_s(x, y) dy$ is special Denjoy integrable on $[a, b]$;

(3) $$F(R_0) \Big(= (SLA) \int_{R_0} F'_s(x, y) d(x, y)\Big) = (\mathcal{D}) \int_c^d \Big((\mathcal{D}) \int_a^b F'_s(x, y) dx\Big) dy = (\mathcal{D}) \int_a^b \Big((\mathcal{D}) \int_c^d F'_s(x, y) dy\Big) dx,$$

where $(\mathcal{D}) \int$ denotes the special Denjoy integral.

Problem. Does Fubini's theorem hold for the strong (LA) integral in the n_0-dimensional case when $n_0 \geq 3$?

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