ON GENERALIZED FRACTIONAL INTEGRAL OPERATORS

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Abstract. We prove the boundedness of the generalized fractional integral operators and their modified versions on Morrey spaces and on Campanato spaces respectively. Our approach involves the Hardy-Littlewood maximal function and Young functions.

1. Introduction

Let $\mathbf{R}^+ := (0,\infty)$. Associated to a function $\rho : \mathbf{R}^+ \to \mathbf{R}^+$, we define the mapping $f \mapsto T_\rho f$ by

$$T_\rho f(x) := \int_{\mathbb{R}^n} f(y)\, \frac{\rho(|x-y|)}{|x-y|^n}\, dy,$$

for any suitable function f on $\mathbb{R}^n$. We also define its modified version $\widetilde{T}_\rho$ by

$$\widetilde{T}_\rho f(x) := \int_{\mathbb{R}^n} f(y) \left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n} \right) dy,$$

where B_0 is the unit ball around the origin and χ_{B_0} is the characteristic function of B_0. For example, if $\rho(t) = t^\alpha,\ 0 < \alpha < n$, then $T_\rho = I_\alpha$ — the fractional integral operator or the Riesz potential. Hence T_ρ may be viewed as a generalization of the fractional integral operator.

Next, for $1 \le p < \infty$ and a suitable function $\phi : \mathbf{R}^+ \to \mathbf{R}^+$, we define the generalized Morrey space $\mathcal{M}_{p,\phi} = \mathcal{M}_{p,\phi}(\mathbb{R}^n)$ to be the set of all functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{\mathcal{M}_{p,\phi}} := \sup_B \frac{1}{\phi(B)} \left(\frac{1}{|B|} \int_B |f(y)|^p dy \right)^{1/p} < \infty,$$

and the generalized Campanato space $\mathcal{L}_{p,\phi} = \mathcal{L}_{p,\phi}(\mathbb{R}^n)$ to be the set of all functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{\mathcal{L}_{p,\phi}} := \sup_B \frac{1}{\phi(B)} \left(\frac{1}{|B|} \int_B |f(y) - f_B|^p dy \right)^{1/p} < \infty.$$

Here the supremums are taken over all open balls $B = B(a,r)$ in $\mathbb{R}^n$, $|B|$ denotes the Lebesgue measure of B in $\mathbb{R}^n$, $\phi(B) = \phi(r)$, and $f_B := \frac{1}{|B|}\int_B f(y)dy$. For $\mathcal{M}_{p,\phi}$, the function $\phi(r)$ is usually required to be nonincreasing and $r^n\phi^p(r)$ to be nondecreasing. For $\mathcal{L}_{p,\phi}$, it is $\frac{\phi(r)}{r}$ that is required to be nonincreasing.

One may observe that f belongs to $\mathcal{L}_{p,\phi}$ if there exist a constant $C < \infty$ and, for every ball B, a constant $c_B < \infty$ such that

$$\frac{1}{\phi(B)} \left(\frac{1}{|B|} \int_B |f(y) - c_B|^p dy \right)^{1/p} < C,$$

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for we then have $\|f\|_{\mathcal{L}_{p,\phi}} < 2C$. Accordingly, $\mathcal{M}_{p,\phi} \subseteq \mathcal{L}_{p,\phi}$. Further, if $1 \le p \le q < \infty$, then $\mathcal{M}_{p,\phi} \supseteq \mathcal{M}_{q,\phi}$ and $\mathcal{L}_{p,\phi} \supseteq \mathcal{L}_{q,\phi}$. Unlike BMO (the space of Bounded Mean Oscillation functions), the Campanato space $\mathcal{L}_{p,\phi}$ is generally dependent of the exponent p (see [1] or [14]). For certain functions ϕ, $\mathcal{M}_{p,\phi}$ and $\mathcal{L}_{p,\phi}$ reduce to some classical spaces. For a brief history of these spaces, see [10], where further references are listed. For recent applications, see e.g. [6].

In [8, 9], Nakai showed that T_ρ is bounded from $\mathcal{M}_{1,\phi}$ to $\mathcal{M}_{1,\psi}$, while $\widetilde{T}_\rho$ is bounded from $\mathcal{L}_{1,\phi}$ to $\mathcal{L}_{1,\psi}$, under some appropriate conditions on ρ, ϕ and ψ. In [3], Eridani showed that, for $1 < p < \infty$, T_ρ is bounded from $\mathcal{M}_{p,\phi}$ to $\mathcal{M}_{p,\psi}$, while $\widetilde{T}_\rho$ is bounded from $\mathcal{M}_{p,\phi}$ to $\mathcal{L}_{p,\psi}$, under similar conditions on ρ, ϕ and ψ. In this paper, we prove that, under some other conditions on ρ, ϕ and ψ, the operator T_ρ is bounded from $\mathcal{M}_{p,\phi}$ to $\mathcal{M}_{q,\psi}$, while $\widetilde{T}_\rho$ is bounded from $\mathcal{L}_{p,\phi}$ to $\mathcal{L}_{q,\psi}$, for $1 < p \le q < \infty$.

Related results may be found in a recent work of Sugano and Tanaka [12].

2. Basic assumptions and facts

Let us begin with a few assumptions, particularly on the associated function ρ, and some relevant facts that follow. Hereafter, C, C_i, C_p and $C_{p,q}$ denote positive constants, which are not necessarily the same from line to line.

In the definition of T_ρ, we always assume that ρ satisfies the following conditions:

(2.1) $\int_0^1 \frac{\rho(t)}{t}\,dt < \infty$;

(2.2) $\frac{1}{2} \le \frac{r}{s} \le 2 \Rightarrow \frac{1}{C_1} \le \frac{\rho(r)}{\rho(s)} \le C_1$.

For $\widetilde{T}_\rho$, we assume that ρ also satisfies two additional conditions, namely:

(2.3) $\int_r^\infty \frac{\rho(t)}{t^2}\,dt \le C_2 \frac{\rho(r)}{r}$ for all $r > 0$;

(2.4) $\frac{1}{2} \le \frac{r}{s} \le 2 \Rightarrow \left|\frac{\rho(r)}{r^n} - \frac{\rho(s)}{s^n}\right| \le C_3 |r - s| \frac{\rho(s)}{s^{n+1}}$.

For example, the function $\rho(r) = r^\alpha$, $0 < \alpha < n$, satisfies (2.1), (2.2) and (2.4). If $0 < \alpha < 1$, then $\rho(r) = r^\alpha$ also satisfies (2.3).

A function ρ satisfying (2.2) is said to satisfy *the doubling condition* (with *a doubling constant* C_1). If ρ satisfies the doubling condition, then for every integer k and $r > 0$ we have

$$\int_{2^k r}^{2^{k+1} r} \frac{\rho(t)}{t}\,dt \sim \rho(2^k r).$$

Further, it follows from the doubling condition that

$$\rho(r) \le C \int_0^r \frac{\rho(t)}{t}\,dt,$$

for every $r > 0$. Next, if ρ satisfies (2.1)–(2.4), then we have Nakai's lemma which states that

$$\int_{\mathbb{R}^n} \left(\frac{\rho(|x_1 - y|)}{|x_1 - y|^n} - \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} \right) dy = 0$$

for every choice of x_1 and x_2 (see [8]). For such a function ρ, the operator $\widetilde{T}_\rho$ maps a constant to a constant, and hence it is well-defined from one generalized Campanato space to another.

In the next section, we shall involve the so-called Hardy-Littlewood maximal operator M, which is defined by

$$Mf(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy.$$

A classical result for M is that it is bounded on L^p for $1 < p \leq \infty$ (see e.g. [13]). Now, if ϕ satisfies the doubling condition and

(2.5) $\int_r^\infty \frac{\phi(t)^p}{t} dt \leq C\phi(r)^p$ for all $r > 0$,

for some $1 < p < \infty$, then there exists $C_p > 0$ such that

$$\|Mf\|_{\mathcal{M}_{p,\phi}} \leq C_p \|f\|_{\mathcal{M}_{p,\phi}},$$

that is, M is bounded on $\mathcal{M}_{p,\phi}$ (see [7]).

We shall also involve Young functions and Orlicz spaces in our discussion. A function $\Phi : [0,\infty] \to [0,\infty]$ is called a *Young* function if Φ is convex, $\lim_{r\to 0^+} \Phi(r) = \Phi(0) = 0$ and $\lim_{r\to\infty} \Phi(r) = \Phi(\infty) = \infty$. A Young function is always nondecreasing. For a Young function Φ, we define $\Phi^{-1}(r) = \inf\{s : \Phi(s) > r\}$ (with $\inf \emptyset = \infty$). If Φ is continuous and bijective, then Φ^{-1} is the usual inverse function. If a Young function Φ satisfies

(2.6) $0 < \Phi(r) < \infty$ for $0 < r < \infty$,

then Φ is continuous and bijective from $[0,\infty)$ to itself. In this case, the inverse function Φ^{-1} is increasing, continuous and concave, and hence satisfies the doubling condition.

For a Young function Φ, we define the Orlicz space $L^\Phi = L^\Phi(\mathbb{R}^n)$ to be the set of all locally integrable function f on $\mathbb{R}^n$ for which $\int_{\mathbb{R}^n} \Phi\big(\frac{|f(x)|}{\epsilon}\big)\, dx < \infty$ for some $\epsilon > 0$. We equip L^Φ with the norm

$$\|f\|_{L^\Phi} := \inf\left\{\epsilon > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\epsilon}\right) dx \leq 1\right\}.$$

Note that for $\Phi(r) = r^p$, $1 \leq p < \infty$, we have $L^\Phi = L^p$. For further properties of Young functions and Orlicz spaces, see e.g. [11]. For their relevance with our subject, see [8, 9].

One more terminology. A function $\theta : \mathbf{R}^+ \to \mathbf{R}^+$ is said to be *almost decreasing* if there exists a constant $C > 0$ such that $\theta(r) \geq C\,\theta(s)$ for $r \leq s$. *Almost increasing* functions can be defined analogously.

3. The boundedness of T_ρ on Morrey spaces

We shall here consider the generalized fractional integral operator T_ρ. For $1 < p < q < \infty$, it is well-known that the fractional integral operator I_α is bounded from L^p to L^q provided that $\alpha/n = 1/p - 1/q$ (see e.g. [13], p. 354). More generally, I_α is bounded from the Morrey space $L^{p,\lambda}$ to $L^{q,\mu}$ where $\alpha/n = 1/p - 1/q$, $0 \leq \lambda < n - \alpha p$ and $p\mu = q\lambda$. (In our notation, $L^{p,\lambda} = \mathcal{M}_{p,\phi}$ with $\phi(r) = r^{(\lambda-n)/p}$.) This result is due to Spanne (see [10], Theorem 5.4) and is reproved by Chiarenza and Frasca [2]. (Actually, Chiarenza and Frasca obtained a stronger result stating that I_α is bounded from $L^{p,\lambda}$ to $L^{q,\lambda}$ where $\alpha/(n-\lambda) = 1/p - 1/q$ and $0 < \lambda < n - \alpha p$, from which Spanne's result follows as a corollary. Their proofs are valid for the case $\lambda = 0$.) The classical result can be recovered from Spanne's by taking $\lambda = 0$ (because $L^{p,0} = L^p$). A further generalization of the above result is obtained by Nakai [7], who showed that I_α is bounded from $\mathcal{M}_{p,\phi}$ to $\mathcal{M}_{q,\psi}$ for appropriate functions ϕ and $\psi(r) = r^\alpha \phi(r)$. Here Spanne's result can be recovered from Nakai's by taking $\phi(r) = r^{(\lambda-n)/p}$ with $0 \leq \lambda < n - \alpha p$ and $\alpha/n = 1/p - 1/q$.

For T_ρ, we have the results of Nakai [9] and Eridani [3] mentioned earlier. While T_ρ is a generalization of I_α, these results for T_ρ cannot, unfortunately, be viewed as a natural generalization of those for I_α (in the sense that we cannot recover the $L^p - L^q$ beoundedness of I_α from them). Recently, Eridani and Gunawan [4] obtains a generalization of Chiarenza-Frasca's result, which has been reformulated by Gunawan [5] as follows. Notice that Chiarenza-Frasca's result can be recovered by taking $\rho(r) = r^\alpha$ and $\phi(r) = r^{(\lambda-n)/p}$ with $0 \le \lambda < n$ and $\alpha/(n-\lambda) = 1/p - 1/q$.

Theorem 3.1. [4, 5] *Suppose that ρ and ϕ satisfies the doubling condition. Suppose also that ϕ is surjective and satisfies the inequality* (2.5) *and*

$$\phi(r)\int_0^r \frac{\rho(t)}{t}dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t}dt \le C\phi(r)^{p/q}, \quad \text{for all } r > 0,$$

for $1 < p < q < \infty$. Then there exists $C_{p,q} > 0$ such that

$$\|T_\rho f\|_{\mathcal{M}_{q,\phi^{p/q}}} \le C_{p,q}\|f\|_{\mathcal{M}_{p,\phi}}$$

that is, T_ρ is bounded from $\mathcal{M}_{p,\phi}$ to $\mathcal{M}_{q,\phi^{p/q}}$.

Sketch of Proof. The idea is to split the integral into two parts, namely

$$T_\rho f(x) = \int_{|x-y|<R} f(y)\,\frac{\rho(|x-y|)}{|x-y|^n}\,dy + \int_{|x-y|\ge R} f(y)\,\frac{\rho(|x-y|)}{|x-y|^n}\,dy = I_1(x) + I_2(x).$$

Then we estimate each part, by decomposing the integral further, diadically. For $I_1(x)$, we use the hypotheses on ρ and ϕ and the property of the Hardy-Littlewood maximal operator M to get

$$|I_1(x)| \le C\,Mf(x)\,\phi(R)^{(p-q)/q}.$$

For $I_2(x)$, we use the hypotheses on ρ and ϕ and the fact that $f \in \mathcal{M}_{p,\phi}$ to obtain

$$|I_2(x)| \le C\,\|f\|_{\mathcal{M}_{p,\phi}}\phi(R)^{p/q}.$$

By the surjectivity of ϕ, we can choose $R > 0$ such that $\phi(R) = Mf(x).\|f\|^{-1}_{\mathcal{M}_{p,\phi}}$, assuming that f is not identically 0 and that $Mf(x) < \infty$ for every $x \in \mathbb{R}^n$. With this value of $\phi(R)$, the two estimates equal and hence, for every $x \in \mathbb{R}^n$, we have

$$|T_\rho f(x)|^q \le C\,Mf(x)^p\|f\|^{q-p}_{\mathcal{M}_{p,\phi}}.$$

The desired inequality then follows from this and the fact that the maximal operator M is bounded on $\mathcal{M}_{p,\phi}$. (QED)

Our new result for T_ρ is the following theorem, which may be considered as a generalization of Spanne's result.

Theorem 3.2. *Suppose that ρ satisfies* (2.1) *and* (2.2). *Suppose further that $\frac{\rho(r)}{r^n}$ and $r^{-n/p}\int_0^r \frac{\rho(t)}{t}\,dt$ are almost decreasing, $\int_r^\infty \frac{\rho(t)t^{-n/p}}{t}\,dt \le C\,r^{-n/p}\int_0^r \frac{\rho(t)}{t}\,dt$, and there exist Young functions Φ_1 satisfying* (2.6) *and Φ_2 such that*

$$r^{-n/p}\int_0^r \frac{\rho(t)}{t}\,dt \sim \Phi_1^{-1}(r^{-n}) \quad \text{and} \quad \Phi_1^{-1}(r^{-n})\Phi_2^{-1}(r^{-n}) \sim r^{-n/q}$$

for $1 < p \le q < \infty$. If ϕ satisfies the doubling condition and

$$\phi(r)\int_0^r \frac{\rho(t)}{t}\,dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t}\,dt \le C\,\psi(r), \quad \text{for all } r > 0,$$

then T_ρ is bounded from $\mathcal{M}_{p,\phi}$ to $\mathcal{M}_{q,\psi}$.

Proof. Let $B = B(a,r)$ be any ball in $\mathbb{R}^n$ and $\tilde{B} = B(a,2r)$. For every $x \in \mathbb{R}^n$, write

$$T_\rho f(x) = \int_{\tilde{B}} f(y)\,\frac{\rho(|x-y|)}{|x-y|^n}\,dy + \int_{\tilde{B}^{\mathrm{C}}} f(y)\,\frac{\rho(|x-y|)}{|x-y|^n}\,dy = I_B^1(x) + I_B^2(x).$$

To estimate I_B^1, we set $\tilde{f} = f\chi_{\tilde{B}}$. Then we have

$$\left(\int_B |I_B^1(x)|^q\,dx\right)^{1/q} = \left(\int_{\mathbb{R}^n} |T_\rho\tilde{f}(x)\chi_B(x)|^q\,dx\right)^{1/q} \le C\,\|T_\rho\tilde{f}\|_{L^{\Phi_1}}\|\chi_B\|_{L^{\Phi_2}}$$

(see [11]). But T_ρ is bounded from L^p to L^{Φ_1} (see Corollary 3.2 of [8]) and $\|\chi_B\|_{L^{\Phi_2}} \le \left(\Phi_2^{-1}(|B|^{-1})\right)^{-1}$. Hence we obtain

$$\begin{aligned}\left(\int_B |I_B^1(x)|^q\,dx\right)^{1/q} &\le C\|\tilde{f}\|_{L^p}\left(\Phi_2^{-1}(|B|^{-1})\right)^{-1}\\ &\le C\,r^{n/p}\phi(r)\,\|f\|_{\mathcal{M}_{p,\phi}}\left(\Phi_1^{-1}(r^{-n})r^{n/q}\right)\\ &\le C\,r^{n/q}\phi(r)\,\|f\|_{\mathcal{M}_{p,\phi}}\int_0^r \frac{\rho(t)}{t}\,dt\\ &\le C\,r^{n/q}\psi(r)\|f\|_{\mathcal{M}_{p,\phi}}.\end{aligned}$$

Now we estimate I_B^2. Observe that for every $x \in B$ we have

$$|I_B^2(x)| \le \int_{|x-y|\ge r} |f(y)|\,\frac{\rho(|x-y|)}{|x-y|^n}\,dy.$$

Hence, as in [3], we obtain

$$|I_B^2(x)| \le C\,\|f\|_{\mathcal{M}_{p,\phi}}\int_r^\infty \frac{\rho(t)\phi(t)}{t}\,dt \le C\,\psi(r)\|f\|_{\mathcal{M}_{p,\phi}},$$

whence

$$\left(\int_B |I_B^2(x)|^q\,dx\right)^{1/q} \le C\,r^{n/q}\psi(r)\|f\|_{\mathcal{M}_{p,\phi}}.$$

Combining the two estimates, we get the desired inequality for T_ρ. (QED)

4. The boundedness of $\widetilde{T}_\rho$ on Campanato spaces

We now turn to the modified fractional integral operator $\widetilde{T}_\rho$. For $\rho(r) = r^\alpha$, the operator $\widetilde{T}_\rho = \widetilde{I}_\alpha$ is well-defined for $0 < \alpha < n+1$ and is known to be bounded from L^p to BMO when $p > 1$ and $\alpha = n/p$, from L^p to Lip_β when $p > 1$ and $0 < \alpha - n/p = \beta < 1$, from BMO to Lip_α when $0 < \alpha < 1$, and from Lip_β to Lip_γ when $0 < \alpha + \beta = \gamma < 1$.

For a general function ρ, Nakai [8, 9] proved that $\widetilde{T}_\rho$ is bounded from $\mathcal{L}_{1,\phi}$ to $\mathcal{L}_{1,\psi}$ for appropriate functions ϕ and ψ. For $\phi(r) = r^\beta$ with $0 \le \beta \le 1$, the space $\mathcal{L}_{1,\phi}$ reduces to BMO (when $\beta = 0$) or Lip_β (when $0 < \beta \le 1$). In this case, Nakai's result covers the BMO–Lip_α and Lip_β–Lip_γ results for $\widetilde{I}_\alpha$. For $\phi(r) = r^\beta$ with $-n/p \le \beta < 0,\ 1 < p < \infty$, we have Eridani's result [3] which covers the other results for $\widetilde{I}_\alpha$. The following theorem is an extension of Eridani's.

Theorem 4.1. *Suppose that ρ satisfies* (2.1)–(2.4)*, and that ϕ satisfies the doubling condition and $\int_1^\infty \frac{\phi(t)}{t}\,dt < \infty$. If*

$$\int_r^\infty \frac{\phi(t)}{t}\,dt \int_0^r \frac{\rho(t)}{t}\,dt + r\int_r^\infty \frac{\rho(t)\phi(t)}{t^2}\,dt \le C\psi(r) \quad \text{for all } r > 0,$$

then $\widetilde{T}_\rho$ *is bounded from* $\mathcal{L}_{p,\phi}$ *to* $\mathcal{L}_{p,\psi}$ *for* $1 < p < \infty$.

Proof. Let $f \in \mathcal{L}_{p,\phi}$. For any ball $B = B(a,r)$ in $\mathbb{R}^n$, let $\tilde{B} = B(a, 2r)$ and, for every $x \in B$, write

$$\begin{aligned}
I_B(x) &:= \int_{\mathbb{R}^n} (f(y) - f_{\tilde{B}})\left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^n}\right)dy \\
I_B^1(x) &:= \int_{\tilde{B}} (f(y) - f_{\tilde{B}})\frac{\rho(|x-y|)}{|x-y|^n}\,dy \\
I_B^2(x) &:= \int_{\tilde{B}^{\complement}} (f(y) - f_{\tilde{B}})\left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)}{|a-y|^n}\right)dy \\
C_B^1 &:= \int_{\mathbb{R}^n} (f(y) - f_{\tilde{B}})\left(\frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n}\right)dy \\
C_B^2 &:= \int_{\mathbb{R}^n} f_{\tilde{B}}\left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n}\right)dy.
\end{aligned}$$

Then clearly

$$\widetilde{T}_\rho f(x) - (C_B^1 + C_B^2) = I_B(x) = I_B^1(x) + I_B^2(x),$$

and one may observe that C_B^1 and C_B^2 are well-defined constants (see [8]).

To estimate I_B^1, write $\tilde{f} := (f - f_{\tilde{B}})\chi_{\tilde{B}}$ and $\tilde{\phi}(r) := \int_r^\infty \frac{\phi(t)}{t}\,dt$. Then, as in [3], we have

$$|I_B^1(x)| \le \int_{\tilde{B}} |\tilde{f}(y)|\frac{\rho(|x-y|)}{|x-y|^n}\,dy \le C\,M\tilde{f}(x)\int_0^r \frac{\rho(t)}{t}\,dt \le C\,\frac{\psi(r)}{\tilde{\phi}(r)}M\tilde{f}(x).$$

By L^p boundedness of M and Fact 6.2 (see Appendices), we obtain

$$\begin{aligned}
\frac{1}{\psi(r)}\left(\frac{1}{|B|}\int_B |I_B^1(x)|^p dx\right)^{1/p} &\le \frac{C}{\tilde{\phi}(r)|B|^{1/p}}\left(\int_B [M\tilde{f}(x)]^p dx\right)^{1/p} \le \frac{C_p}{\tilde{\phi}(r)|B|^{1/p}}\,\|\tilde{f}\|_{L^p} \\
&\le \frac{C_p}{\tilde{\phi}(r)|B|^{1/p}}\left(\|(f-\sigma(f))\chi_{\tilde{B}}\|_{L^p} + |\tilde{B}|^{1/p}|f_{\tilde{B}} - \sigma(f)|\right) \\
&\le C_p\left(\|f-\sigma(f)\|_{\mathcal{M}_{p,\tilde{\phi}}} + \|f\|_{\mathcal{L}_{p,\phi}}\right) \le C_p\,\|f\|_{\mathcal{L}_{p,\phi}},
\end{aligned}$$

where $\sigma(f) = \lim_{r\to\infty} f_{B(0,r)}$.

It then remains to estimate I_B^2. By (2.2) and (2.4), we have

$$\begin{aligned}
|I_B^2(x)| &\le \int_{\tilde{B}^{\complement}} |f(y) - f_{\tilde{B}}|\left|\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y-a|)}{|y-a|^n}\right|dy \\
&\le C|x-a|\int_{|y-a|\ge 2r} |f(y) - f_{\tilde{B}}|\frac{\rho(|y-a|)}{|y-a|^{n+1}}\,dy \\
&= C|x-a|\sum_{k=2}^\infty \int_{2^{k-1}r\le|y-a|<2^k r} \frac{|f(y) - f_{\tilde{B}}|\rho(|y-a|)}{|y-a|^{n+1}}\,dy \\
&\le C|x-a|\sum_{k=2}^\infty \frac{\rho(2^k r)}{(2^k r)^{n+1}}\int_{|y-a|<2^k r} |f(y) - f_{\tilde{B}}|\,dy \\
&\le C|x-a|\sum_{k=2}^\infty \frac{\rho(2^k r)}{2^k r}\left(\frac{1}{(2^k r)^n}\int_{|y-a|<2^k r} |f(y) - f_{\tilde{B}}|^p\,dy\right)^{1/p}.
\end{aligned}$$

But, for each $k \geq 2$, we have

$$\left(\frac{1}{|B(a,2^k r)|}\int_{B(a,2^k r)} |f(y) - f_{\tilde{B}}|^p dy\right)^{1/p} \leq C\,\|f\|_{\mathcal{L}_{p,\phi}} \int_{2r}^{2^k r} \frac{\phi(s)}{s}\,ds$$

(see Fact 6.1 in Appendices). Hence, by (2.2), (2.3), and our assumption on ϕ and ψ, we obtain

$$\begin{aligned}
|I_B^2(x)| &\leq C\,|x-a|\|f\|_{\mathcal{L}_{p,\phi}} \sum_{k=2}^{\infty} \frac{\rho(2^k r)}{2^k r} \int_{2r}^{2^k r} \frac{\phi(s)}{s}\,ds \\
&\leq C\,|x-a|\|f\|_{\mathcal{L}_{p,\phi}} \sum_{k=2}^{\infty} \int_{2^k r}^{2^{k+1} r} \frac{\rho(t)}{t^2}\left(\int_{2r}^{t} \frac{\phi(s)}{s}\,ds\right) dt \\
&\leq C\,|x-a|\|f\|_{\mathcal{L}_{p,\phi}} \int_{2r}^{\infty} \frac{\rho(t)}{t^2}\left(\int_{2r}^{t} \frac{\phi(s)}{s}\,ds\right) dt \\
&= C\,|x-a|\|f\|_{\mathcal{L}_{p,\phi}} \int_{2r}^{\infty} \left(\int_{s}^{\infty} \frac{\rho(t)}{t^2}\,dt\right) \frac{\phi(s)}{s}\,ds \\
&\leq C\,r\,\|f\|_{\mathcal{L}_{p,\phi}} \int_{2r}^{\infty} \frac{\rho(s)\phi(s)}{s^2}\,ds \leq C\,\psi(r)\,\|f\|_{\mathcal{L}_{p,\phi}},
\end{aligned}$$

whence

$$\frac{1}{\psi(r)}\left(\frac{1}{|B|}\int_B |I_B^2(x)|^p dx\right)^{1/p} \leq C\,\|f\|_{\mathcal{L}_{p,\phi}}.$$

This completes the proof. (QED)

The results for T_ρ indicate that the modified fractional integral operator $\widetilde{T}_\rho$ must also be bounded from $\mathcal{L}_{p,\phi}$ to $\mathcal{L}_{q,\psi}$ for appropriate functions ϕ and ψ. Indeed, we have the following analog of Theorem 3.2 for $\widetilde{T}_\rho$.

Theorem 4.2. *Suppose that ρ satisfies (2.1) – (2.4). Suppose further that $\frac{\rho(r)}{r^n}$ and $r^{-n/p}\int_0^r \frac{\rho(t)}{t}\,dt$ are almost decreasing, $\int_r^\infty \frac{\rho(t)t^{-n/p}}{t}\,dt \leq C\,r^{-n/p}\int_0^r \frac{\rho(t)}{t}\,dt$, and there exist Young functions Φ_1 satisfying (2.6) and Φ_2 such that*

$$r^{-n/p}\int_0^r \frac{\rho(t)}{t}\,dt \sim \Phi_1^{-1}(r^{-n}) \quad \text{and} \quad \Phi_1^{-1}(r^{-n})\Phi_2^{-1}(r^{-n}) \sim r^{-n/q}$$

for $1 < p \leq q < \infty$. If ϕ satisfies the doubling condition and

$$\phi(r)\int_0^r \frac{\rho(t)}{t}\,dt + r\int_r^\infty \frac{\rho(t)\phi(t)}{t^2}\,dt \leq C\,\psi(r), \quad \text{for all } r > 0,$$

then $\widetilde{T}_\rho$ is bounded from $\mathcal{L}_{p,\phi}$ to $\mathcal{L}_{q,\psi}$.

Proof. Let $f \in \mathcal{L}_{p,\phi}$. For any ball $B = B(a,r)$ in $\mathbb{R}^n$, let $\tilde{B} = B(a,2r)$ and define I_B, I_B^1, I_B^2, C_B^1 and C_B^2 as in the proof of Theorem 4.1.

To estimate I_B^1, write $\tilde{f} := (f - f_{\tilde{B}})\chi_{\tilde{B}}$ as before. Then $I_B^1 = T_\rho \tilde{f}$ (it is T_ρ, not $\widetilde{T}_\rho$), and hence (as in the proof of Theorem 3.2) we have

$$\left(\int_B |I_B^1(x)|^q\,dx\right)^{1/q} \leq C\,r^{n/q}\psi(r)\|f\|_{\mathcal{L}_{p,\phi}}.$$

Meanwhile, we have the same estimate for I_B^2 as in the proof of Theorem 4.1, whence

$$\left(\int_B |I_B^2(x)|^q\,dx\right)^{1/q} \le C\,r^{n/q}\psi(r)\|f\|_{\mathcal{L}_{p,\phi}}.$$

The desired inequality for $\widetilde{T}_\rho$ follows immediately from these two estimates. (QED)

5. Examples

Let ℓ be a continuous function on $(0,\infty)$ such that

$$\ell(r) = \begin{cases} 1/(\log 1/r) & \text{for small } r>0, \\ \log r & \text{for large } r>0. \end{cases}$$

We assume that ℓ is Lipschitz continuous on every closed and bounded interval contained in $(0,\infty)$. Then $\ell(r) \sim \ell(r^n) \sim 1/\ell(1/r^n)$. Let

$$(5.1)\qquad 1<p<\infty,\ 0<\alpha<n/p,\ \beta\ge 0 \quad\text{and}\quad \rho(r)=r^\alpha\ell^\beta(r).$$

Then ρ satisfies the assumption in Theorem 3.2. Moreover, if $0<\alpha<1$, then ρ satisfies the assumption in Theorem 4.2. In particular, one may observe that

$$\int_0^r \frac{\rho(t)}{t}\,dt \sim r^\alpha\ell^\beta(r).$$

Example 5.1. Take $\phi(r)=r^{-n/p}\ell(r)^{\beta q/(p-q)}$ where $1/q=1/p-\alpha/n$. Then $\phi(r)^{(p-q)/q} = \rho(r)$ and

$$\int_r^\infty \frac{\rho(t)\phi(t)}{t}\,dt \sim \phi(r)^{p/q}.$$

From Theorem 3.1 it follows that T_ρ is bounded from $\mathcal{M}_{p,\phi}$ to $\mathcal{M}_{q,\phi^{p/q}}$.

Now, for $\beta>0$, let Φ_i $(i=1,2)$ be Young functions and

$$\Phi_1(s)\sim s^q\ell^\beta(s) \quad \text{for } s>0, \quad 1/q=1/p-\alpha/n,$$
$$\Phi_2(s) = \begin{cases} 1/\exp(1/s^{1/\beta}) & \text{for small } s>0, \\ \exp(s^{1/\beta}) & \text{for large } s>0. \end{cases}$$

Then

$${\Phi_1}^{-1}(r)\sim r^{1/q}/\ell^\beta(r), \quad {\Phi_2}^{-1}(r)\sim \ell^\beta(r),$$

and

$$r^{-n/p}\int_0^r \frac{\rho(t)}{t}\,dt \sim {\Phi_1}^{-1}\left(\frac{1}{r^n}\right) \sim \left(\frac{1}{r^n}\right)^{1/q}\ell^\beta(r),$$
$${\Phi_1}^{-1}(r){\Phi_2}^{-1}(r)\sim r^{1/q}.$$

For $\beta=0$, let

$$\Phi_1(s)=s^q,\ 1/q=1/p-\alpha/n, \quad \Phi_2(s) = \begin{cases} 0 & \text{for } 0\le s<1, \\ +\infty & \text{for } s\ge 1. \end{cases}$$

Then

$${\Phi_1}^{-1}(r)\sim r^{1/q}, \quad {\Phi_2}^{-1}(r)\equiv 1.$$

Example 5.2. Under the condition (5.1), let $\phi(r)r^{n/p}$ be almost increasing and $\phi(r)r^{\alpha+\epsilon}$ be almost decreasing for some $\epsilon > 0$. Then

$$\int_r^\infty \frac{\rho(t)\phi(t)}{t}\,dt \sim \rho(r)\phi(r).$$

From Theorem 3.2 it follows that T_ρ is bounded from $\mathcal{M}_{p,\phi}$ to $\mathcal{M}_{q,\psi}$ for $\psi(r) = \rho(r)\phi(r)$. (In the case $\beta = 0$, this boundedness also follows from Theorem 3 in [7].)

Example 5.3. Under the condition (5.1) and $0 < \alpha < 1$, let $\phi(r)r^{n/p}$ be almost increasing and $\phi(r)r^{\alpha-1+\epsilon}$ be almost decreasing for some $\epsilon > 0$. Then

$$r\int_r^\infty \frac{\rho(t)\phi(t)}{t^2}\,dt \sim \rho(r)\phi(r).$$

From Theorem 4.2 it follows that $\widetilde{T}_\rho$ is bounded from $\mathcal{L}_{p,\phi}$ to $\mathcal{L}_{q,\psi}$ for $\psi(r) = \rho(r)\phi(r)$. (If $\psi(r)$ is almost increasing, then this boundedness also follows from Theorem 3.6 in [9] since $\mathcal{L}_{p,\phi} \subset \mathcal{L}_{1,\phi}$ and $\mathcal{L}_{q,\psi} = \mathcal{L}_{1,\psi}$.)

Let us now consider the case where

$$1 < p = q < \infty,\ \beta > 0 \quad \text{and} \quad \rho(r) = \begin{cases} (\log 1/r)^{-\beta-1} & \text{for small } r > 0, \\ (\log r)^{\beta-1} & \text{for large } r > 0. \end{cases} \tag{5.2}$$

We assume that ρ is Lipschitz continuous on every closed and bounded interval contained in $(0,\infty)$. Then $\int_0^r \frac{\rho(t)}{t}\,dt \sim \ell^\beta(r)$ and ρ satisfies the assumptions in Theorem 3.2 and in Theorem 4.2. Now let Φ_i $(i = 1,2)$ be Young functions and

$$\Phi_1(s) \sim s^p \ell^\beta(s), \quad \Phi_2(s) = \begin{cases} 1/\exp(1/s^{1/\beta}) & \text{for small } s > 0, \\ \exp(s^{1/\beta}) & \text{for large } s > 0. \end{cases}$$

Then

$${\Phi_1}^{-1}(r) \sim r^{1/p}\ell^{-\beta}(r), \quad {\Phi_2}^{-1}(r) \sim \ell^\beta(r),$$

and

$$r^{-n/p}\int_0^r \frac{\rho(t)}{t}\,dt \sim {\Phi_1}^{-1}\left(\frac{1}{r^n}\right) \sim \left(\frac{1}{r^n}\right)^{1/p} \ell^\beta(r),$$
$${\Phi_1}^{-1}(r){\Phi_2}^{-1}(r) \sim r^{1/p}.$$

Example 5.4. Under the condition (5.2), let $\phi(r) = r^\delta \ell^\gamma(r)$, for $-n/p < \delta < 0$ and $-\infty < \gamma < +\infty$, or for $\delta = -n/p$ and $0 \le \gamma < +\infty$. Then

$$\int_r^\infty \frac{\rho(t)\phi(t)}{t}\,dt \sim \rho(r)\phi(r) \le C\phi(r)\int_0^r \frac{\rho(t)}{t}\,dt \sim r^\delta \ell^{\beta+\gamma}(r).$$

From Theorem 3.2 it follows that T_ρ is bounded from $\mathcal{M}_{p,\phi}$ to $\mathcal{M}_{p,\psi}$ for $\psi(r) = r^\delta \ell^{\beta+\gamma}(r)$. (This boundedness also follows from Theorem 1 in [3].)

Example 5.5. Under the condition (5.2), let $\phi(r) = r^\delta \ell^\gamma(r)$, for $-n/p < \delta < 1$ and $-\infty < \gamma < +\infty$, or for $\delta = -n/p$ and $0 \le \gamma < +\infty$. Then

$$r\int_r^\infty \frac{\rho(t)\phi(t)}{t^2}\,dt \sim \rho(r)\phi(r) \le C\phi(r)\int_0^r \frac{\rho(t)}{t}\,dt \sim r^\delta \ell^{\beta+\gamma}(r).$$

From Theorem 4.2 it follows that $\widetilde{T}_\rho$ is bounded from $\mathcal{L}_{p,\phi}$ to $\mathcal{L}_{p,\psi}$ for $\psi(r) = r^\delta \ell^{\beta+\gamma}(r)$. (If $\delta < 0$, then $\int_r^\infty \frac{\phi(t)}{t}\,dt \sim \phi(r)$ and so this boundedness also follows from Theorem 4.1. If $\delta > 0$, or if $\delta = 0$ and $\beta + \gamma \ge 0$, then this boundedness also follows from Theorem 3.6 in [9] since $\mathcal{L}_{p,\phi} \subset \mathcal{L}_{1,\phi}$ and $\mathcal{L}_{p,\psi} = \mathcal{L}_{1,\psi}$.)

6. Appendices

Here we present facts that we used earlier in the proof of Theorem 4.1 (and implicitly in the proof of Theorem 4.2).

Fact 6.1. *If $f \in \mathcal{L}_{p,\phi}$ for some $1 \le p < \infty$ and ϕ satisfies the doubling condition, then for any ball $B = B(a,r)$ in $\mathbb{R}^n$ and $k = 1, 2, 3, \ldots$, we have*

$$\left(\frac{1}{|B(a,2^k r)|} \int_{B(a,2^k r)} |f(y) - f_B|^p dy \right)^{1/p} \le C \, \|f\|_{\mathcal{L}_{p,\phi}} \int_r^{2^k r} \frac{\phi(t)}{t} \, dt,$$

where $C > 0$ is dependent only on n and the doubling constant of ϕ.

Proof. By Minkowski's inequality, we have

$$\begin{aligned} &\left(\frac{1}{|B(a,2^k r)|} \int_{B(a,2^k r)} |f(y) - f_B|^p dy \right)^{1/p} \\ &\le \left(\frac{1}{|B(a,2^k r)|} \int_{B(a,2^k r)} |f(y) - f_{B(a,2^k r)}|^p dy \right)^{1/p} + \sum_{j=0}^{k-1} |f_{B(a,2^j r)} - f_{B(a,2^{j+1} r)}|. \end{aligned}$$

But, for each $j = 0, \ldots, k-1$, one may observe that

$$\begin{aligned} |f_{B(a,2^j r)} - f_{B(a,2^{j+1} r)}| &\le \frac{1}{|B(a,2^j r)|} \int_{B(a,2^j r)} |f(y) - f_{B(a,2^{j+1} r)}| \, dy \\ &\le 2^n \left(\frac{1}{|B(a,2^{j+1} r)|} \int_{B(a,2^{j+1} r)} |f(y) - f_{B(a,2^{j+1} r)}|^p dy \right)^{1/p} \le C \, \phi(2^{j+1} r) \|f\|_{\mathcal{L}_{p,\phi}}. \end{aligned}$$

Summing up, we get

$$\begin{aligned} \frac{1}{|B(a,2^k r)|} \int_{B(a,2^k r)} |f(y) - f_B| \, dy &\le C \, \|f\|_{\mathcal{L}_{p,\phi}} \sum_{j=0}^{k-1} \phi(2^{j+1} r) \\ &\le C \, \|f\|_{\mathcal{L}_{p,\phi}} \sum_{j=0}^{k-1} \int_{2^j r}^{2^{j+1} r} \frac{\phi(t)}{t} \, dt = C \, \|f\|_{\mathcal{L}_{p,\phi}} \int_r^{2^k r} \frac{\phi(t)}{t} \, dt, \end{aligned}$$

since ϕ satisfies the doubling condition. (QED)

Fact 6.1 can actually be generalized as follows.

Fact 6.1′. *Let $f \in \mathcal{L}_{p,\phi}$ for some $1 \le p < \infty$ and ϕ satisfy the doubling condition. If $B(a,r) \subset B(b,s)$ in $\mathbb{R}^n$, then*

$$\left| f_{B(a,r)} - f_{B(b,s)} \right| \le C \, \|f\|_{\mathcal{L}_{p,\phi}} \int_r^{2s} \frac{\phi(t)}{t} \, dt,$$

where $C > 0$ is dependent only on n and the doubling constant of ϕ.

Proof. Indeed, if $2^{-k-1}s \le r < 2^{-k}s$, then, choosing balls B_j $(j = 0, 1, \cdots k)$ so that the radius of B_j is $2^{-j}s$ and $B(a,r) \subset B_k \subset B_{k-1} \subset \cdots \subset B_0 = B(b,s)$, we have

$$\left|f_{B(a,r)} - f_{B(b,s)}\right| \le \left|f_{B(a,r)} - f_{B_k}\right| + \sum_{j=0}^{k-1} |f_{B_{j+1}} - f_{B_j}| \le C\|f\|_{\mathcal{L}_{p,\phi}} \sum_{j=0}^{k} \phi(2^{-j}s) \le C\|f\|_{\mathcal{L}_{p,\phi}} \int_{2^{-k}s}^{2s} \frac{\phi(t)}{t}\, dt. \text{ (QED)}$$

Fact 6.2. *Let* $1 \le p < \infty$, ϕ *satisfy the doubling condition and* $\int_1^\infty \frac{\phi(t)}{t}\, dt < \infty$. *If* $f \in \mathcal{L}_{p,\phi}$, *then* $f_{B(0,r)}$ *converges as* r *tends to infinity and*

$$\|f - \lim_{r\to\infty} f_{B(0,r)}\|_{\mathcal{M}_{p,\tilde{\phi}}} \le C\|f\|_{\mathcal{L}_{p,\phi}},$$

where $\tilde{\phi}(r) = \int_r^\infty \frac{\phi(t)}{t}\, dt$ *and* $C > 0$ *is dependent only on* n *and the doubling constant of* ϕ.

Proof. From Fact 6.1′ it follows that there exists a constant $\sigma(f)$, independent of $a \in \mathbb{R}^n$, such that

$$\lim_{r\to\infty} f_{B(a,r)} = \sigma(f),$$

and

$$\left|f_{B(a,r)} - \sigma(f)\right| \le C\|f\|_{\mathcal{L}_{p,\phi}} \int_r^\infty \frac{\phi(t)}{t}\, dt.$$

Hence we have, for all $B = B(a,r)$,

$$\left(\frac{1}{|B|}\int_B |f(x) - \sigma(f)|^p dx\right)^{1/p} \le \left(\frac{1}{|B|}\int_B |f(x) - f_B|^p dx\right)^{1/p} + |f_B - \sigma(f)| \le \|f\|_{\mathcal{L}_{p,\phi}}\phi(r) + C\|f\|_{\mathcal{L}_{p,\phi}}\tilde{\phi}(r) \le C\|f\|_{\mathcal{L}_{p,\phi}}\tilde{\phi}(r). \text{ (QED)}$$

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