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TWO-PLAYER GAMES OF "SCORE SHOWDOWN"

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ABSTRACT. There are some games widely played in the routine world of gambles, roulette, quiz show and the sports excercises. The object of the game is to get the highest score among all of the players in the game, from one or two chances of sampling. The two-player sequential-move, imperfect-information games with the three kids of score functions are investigated, and the optimal strategies for the two players and the winning probabilities they can get in the optimal play are derived.

1 The Game of "Score Showdown". Consider the two players I and II (sometimes they are denoted by 1 and 2, respectively). Let $X_{ij}(i,j=1,2)$ be the random variable (r.v.) observed by player i at the j-th observation. We assume that X_{ij} s are *i.i.d.*, each with uniform distribution on $[0,1]$. The game is played in the two stages:

In the first stage, I observes that $X_{11}=x$ and chooses one of the either A_1 (*i.e.*, I accepts x) or R_1 (*i.e.*, I rejects x, and resamples a new r.v. X_{12}). The observed value x and I's choice of either A_1 or R_1 *are informed to* II. But the observed value of X_{12} *is not informed to* II.

In the second stage, after knowing I's choice of $x\&(A_1$ or $R_1)$, II observes that $X_{21}=y$ and chooses either one of A_2 (*i.e.*, II accepts y) or R_2 (*i.e.*, II rejects y and resamples a new r.v. X_{22}).

Let, for $i=1,2$,

$$S_i(X_{i1},X_{i2})=\left\{\begin{array}{c}X_{i1}\\X_{i2}\end{array}\right\},\ \text{ if } X_{i1} \text{ is} \left\{\begin{array}{c}\text{accepted}\\\text{rejected}\end{array}\right\} \text{by player } i \tag{1.1}$$

which we call the *score* for player i.

After the second stage is over, the showdown is made, the scores are compared, and the player with the higher score than opponent's becomes the winner. Each player aims to maximize the probability of his (or her) winning. We assume that both players are intelligent, and each player should prepare for the optimal strategy employed by his opponent.

The solutions to the game with the score function (1.1) and the related simultaneous-move game are given in Section 2. The games with other scores "Competing Average" and "Showcase Showdown" are analyzed in Sections 3 and 4. Some interesting sequential-move games, other than those treated in the present paper, which are often seen in card games like LaRelance poker, TV show like Price is Right, and competition by sports players, are discussed in Ref.[1, 2, 3, 4, 5, and 7].

2 Keep-or-Exchange. Let $W_i(i=1,2)$ be the event that player i wins. To find the players' optimal strategies we must derive them in reverse order. Define state $\left\{\begin{array}{c}(y|x,A_1)\\(y|x,R_1)\end{array}\right\}$

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for II, to mean that I $\left\{ \begin{array}{c} \text{accepted} \\ \text{rejected} \end{array} \right\} X_{11} = x$ in the first stage and II has just observed $X_{21} = y$ in the second stage. Then we have

(2.1) $$p_{2A}(y|x, A_1) = P\{W_2|\text{II accepts } X_{21} = y \text{ in state } (y|x, A_1)\} = I(y > x),$$

(2.2) $$\begin{aligned} p_{2R}(y|x, A_1) &= P\{W_2|\text{II rejects } X_{21} = y \text{ in state } (y|x, A_1)\} \\ &= P(X_{22} > x) = \bar{x} \equiv 1 - x, \quad \text{indep. of } y, \end{aligned}$$

(2.3) $$\begin{aligned} p_{2A}(y|x, R_1) &= P\{W_2|\text{II accepts } X_{21} = y \text{ in state } (y|x, R_1)\} \\ &= P(X_{12} < y) = y, \quad \text{indep. of } x, \end{aligned}$$

and

(2.4) $$\begin{aligned} p_{2R}(y|x, R_1) &= P\{W_2|\text{II rejects } X_{21} = y \text{ in state } (y|x, R_1)\} \\ &= P(X_{12} < X_{22}) = \frac{1}{2}, \quad \text{indep. of } x \text{ and } y, \end{aligned}$$

Theorem 1 *The solution to the two-player game with the score function* (1.1) *is as follows. The optimal strategy for* I *in the first stage is given by:*

(2.5) $$\textit{Accept (Reject)} X_{11} = x, \quad \textit{if } x > (<) \sqrt{3/8} \approx 0.6124.$$

The optimal strategy for II *in the second stage is given by*

(2.6) $$\textit{Accept (Reject)} X_{21} = y, \quad \textit{if } y > (<) \left\{ \begin{array}{c} x \\ 1/2 \end{array} \right\} \quad \textit{in state} \quad \left\{ \begin{array}{c} (y|x, A_1) \\ (y|x, R_1) \end{array} \right\}.$$

The optimal values are

(2.7) $$\begin{aligned} P(W_1) &= \frac{1}{3}\left\{1 + 2\,(3/8)^{3/2}\right\} \approx 0.4864 \\ P(W_2) &= 1 - P(W_1) = \frac{2}{3}\left\{1 - (3/8)^{3/2}\right\} \approx 0.5136. \end{aligned}$$

Proof. From (2.1)-(2.2), we find that

(2.8) $$\begin{aligned} P(W_2|x, A_1) &\equiv P(W_2|\text{ I accepted } X_{11} = x \text{ in the first stage}) \\ &= \int_0^1 \{p_{2A}(y|x, A_1) \vee p_{2R}(y|x, A_1)\}\, dy \quad (*) \\ &= \int_0^1 \{I(x < y) \vee \bar{x}\}\, dy = x\bar{x} + \bar{x} = 1 - x^2, \end{aligned}$$

and computing the maximum in $(*)$ gives the first half of (2.6).

Also, from (2.3)-(2.4), we have

(2.9) $$\begin{aligned} P(W_2|x, R_1) &\equiv P(W_2|\text{ I rejected } X_{11} = x \text{ in the first stage}) \\ &= \int_0^1 \{p_{2A}(y|x, R_1) \vee p_{2R}(y|x, R_1)\}\, dy \quad (**) \\ &= \int_0^1 \left(y \vee \frac{1}{2}\right) dy = \frac{5}{8}, \quad \text{indep. of } x, \end{aligned}$$

where computation of the maximum in $(**)$ gives the second half of (2.6).

It is clear that draw of the game cannot occur with positive probability, and therefore, by (2.8)-(2.9),

$$\begin{aligned} p_{1A}(x) &\equiv P\left[W_1 | \{\text{I accepted } X_{11}=x\} \cap \{\text{II behaves optimally in the second stage}\}\right] \\ &= 1-P(W_2|x,A_1)=x^2, \\ p_{1R}(x) &\equiv P\left[W_1 | \{\text{I rejected } X_{11}=x\} \cap \{\text{II behaves optimally in the second stage}\}\right] \\ &= 1-P(W_2|x,R_1)=3/8, \end{aligned}$$

and hence

$$P(W_1)=\int_0^1 \{p_{1A}(x)\vee p_{1R}(x)\}dx=\int_0^1\left(x^2\vee\frac{3}{8}\right)dx=\frac{1}{3}\left\{1+2\left(\frac{3}{8}\right)^{3/2}\right\}$$

where computation of the maximum in the second expression leads to (2.5). □

Theorem 1 shows that II, the second-mover, has an advantage over I, the first-mover. In the simultaneous-move version of the game, the unfair information acquisition by the players disappears. I (II) privately observes $X_{11}=x(X_{21}=y)$ and chooses either one of A_1 or R_1 (A_2 or R_2). The observed value and choice by each player are not known by the opponent. Suppose that players' strategies have the form of

$$\text{I accepts (rejects) } X_{11}=x, \text{ if } x>(<)a,$$

$$\text{II accepts (rejects) } X_{21}=y, \text{ if } y>(<)b,$$

Let $M(a,b)\equiv P\{W_1|\text{I chooses } a \text{ and II chooses } b\}$. We want to solve the zero-sum game with payoff function $M(a,b)$ on the unit square. Surprisingly the solution is given by the famous golden bisection number.

Theorem 2 *Solution to the two-player simultaneous-move version of the "Score Showdown" game with score* (1.1) *is as follows; The game has the unique saddle point* (g,g) *and the saddle value* $M(g,g)=\frac{1}{2}$, *where* $g=\frac{1}{2}(\sqrt{5}-1)\approx 0.61803$.

Proof. $M(a,b)=P(X_{11}<a,X_{21}<b,X_{12}>X_{22})+P(X_{11}<a,X_{21}>b,X_{12}>X_{21})+P(X_{11}>a,X_{21}>b,X_{11}>X_{21})+P(X_{11}>a,X_{21}<b,X_{11}>X_{22})$ and the first, second, third and fourth terms in the r.h.s. are $\frac{1}{2}ab$, $\frac{1}{2}a\bar{b}^2$, $\frac{1}{2}\bar{b}^2I(a\le b)+(\bar{a}\bar{b}-\frac{1}{2}\bar{a}^2)I(a>b)$ and $\frac{1}{2}(1-a^2)b$, respectively. Thus we have

$$M(a,b)=\frac{1}{2}\{a+b-ab(a+\bar{b})\}+\left\{\begin{array}{c}\frac{1}{2}\bar{b}^2\\ \bar{a}\bar{b}-\frac{1}{2}\bar{a}^2\end{array}\right\},\ \text{if } a\left\{\begin{array}{c}\le\\ >\end{array}\right\}b. \tag{2.10}$$

One can easily assertain from this equation, that $M(a,b)+M(b,a)=1,\forall(a,b)$, and hence $M(b,b)\equiv\frac{1}{2},\forall b\in(0,1)$.

Now for any fixed $b\in(0,1), M(a,b)$ given by (2.10) is a concave quadratic function of a, satisfying

$$M(0,b)=M(1,b)=\frac{1}{2}(1-b\bar{b})<\frac{1}{2}=M(b,b),$$

$$\left.\frac{\partial M}{\partial a}\right|_{a=0}=\frac{1}{2}(1-b\bar{b})>0,\quad \left.\frac{\partial M}{\partial a}\right|_{a=1}=-\frac{1}{2}(1+b\bar{b})<0$$

and

$$\left.\frac{\partial M}{\partial a}\right|_{a=b-0}=\left.\frac{\partial M}{\partial a}\right|_{a=b+0}=\frac{1}{2}(1-b-b^2).$$

Consequently, for $b = \frac{1}{2}(\sqrt{5}-1)(\equiv g$, golden bisection number), $\max_a M(a,g) = M(g,g) = \frac{1}{2}$, since $g + g^2 = 1$.

Also (2.10) gives

$$M(g,b) = \frac{1}{2}\left\{g + b - gb(g+\bar{b})\right\} + \left\{ \begin{array}{c} \frac{1}{2}\bar{b}^2 \\ \bar{g}\bar{b} - \frac{1}{2}\bar{g}^2 \end{array} \right\}, \text{ if } g \left\{ \begin{array}{c} \leq \\ > \end{array} \right\} b,$$

which is a convex quadratic function of b, satisfying

$$M(g,0) = M(g,1) = \frac{1}{2}\ (1 - g\bar{g}) < \frac{1}{2} = M(g,g),$$

$$\left.\frac{\partial M}{\partial b}\right|_{b=0} = -g < 0 < 2g = \left.\frac{\partial M}{\partial b}\right|_{b=1},$$

and

$$\left.\frac{\partial M}{\partial b}\right|_{b=g-0} = \left.\frac{\partial M}{\partial b}\right|_{b=g+0} = g^2 + g - 1 = 0.$$

Hence we have

$$\min_b M(g,b) = M(b,b) = \frac{1}{2}.$$

This completes the proof of the theorem. □

Remark 1. The game (2.10) on the unit square is a typical example of the concave-convex game (see. Ref. [6 ; Section 2.5]). Eq. (2.10) gives

$$\left\{ \begin{array}{l} \frac{\partial M}{\partial a} = \frac{1}{2}(1 - b + b^2) - ab + (b-a)I(a > b), \\ \frac{\partial M}{\partial b} = ab - \frac{1}{2}(1 + a + a^2) + bI(a \leq b) + aI(a > b). \end{array} \right.$$

For $a > b$, the conditions $\frac{\partial M}{\partial a} = 0$ and $\frac{\partial M}{\partial b} = 0$ give

$$(*)\ \ a = \frac{1 + b + b^2}{2(1+b)} \quad \text{and} \quad (**)\ \ b = \frac{1 - a + a^2}{2a}$$

respectively. The simultaneous equation $(*)$ and $(**)$ has a unique root $(a,b) = (g,g)$. (See Figure 1).

For $a < b$, the result remains unchanged, if a and b are interchanged.

3 Competing Average.

We consider in this section the case where the score is defined by

$$(3.1) \qquad S_i(X_{i1}, X_{i2}) = \left\{ \begin{array}{l} X_{i1}, \\ \frac{1}{2}(X_{i1} + X_{i2}), \end{array} \right. \quad \text{if } \left\{ \begin{array}{l} X_{i1} \text{ is accepted}, \\ X_{i2} \text{ is resampled}. \end{array} \right.$$

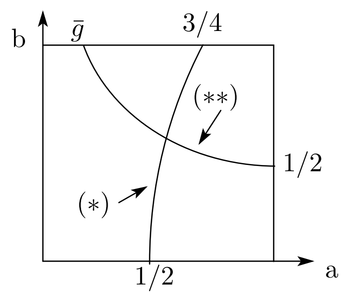

Figure 1. The unique root

We use R(=resample), instead of R(=reject) used in the previous sections. Definition of states $(y|x, A_1)$, *etc.* are the same as in Section 2. Then we have

$$(3.2) \qquad p_{2A}(y|x, A_1) = I(y > x),$$

$$(3.3)\qquad p_{2R}(y|x,A_1) = P(y+X_{22}>2x) = (1-2x+y)I(0<2x-y<1)+I(2x-y\le 0),$$

$$(3.4)\quad p_{2A}(y|x,R_1)=P(2y>x+X_{12})=(2y-x)I(0<2y-x<1)+I(2y-x\ge 1),$$

and

$$(3.5)\qquad p_{2R}(y|x,R_1) = P\left\{\frac{1}{2}(y+X_{22})>\frac{1}{2}(x+X_{12})\right\}=P(X_{22}-X_{12}>x-y) = I(x\le y)\left\{1-\frac{1}{2}(x+\bar{y})^2\right\}+I(x>y)\frac{1}{2}(\bar{x}+y)^2.$$

On the basis of (3.2)-(3.5), we prove the following

Theorem 3 *The solution to the two-player game with the score function* (3.1) *is as follows. The optimal strategy for I in the first stage is given by*

$$(3.6)\qquad \text{Accept (Resample) } X_{11}=x,\ \text{ if } x>(<)x_0\approx 0.549.$$

where x_0 *is the unique root in* $(\frac{1}{2},1)$ *of the equation*

$$(3.7)\qquad 7/(2x)-4\sqrt{2x}=6-(15/2)x+x^2.$$

The optimal strategy for II in the second stage is given by

$$(3.8)\quad \text{Accept (Resample) } X_{21}=y,\ \text{ if } y>(<)\left\{\begin{array}{c} x \\ y_0(x)\end{array}\right\}\ \text{ in state } \left\{\begin{array}{c}(y|x,A_1)\\(y|x,R_1)\end{array}\right\},$$

where

$$(3.9)\qquad y_0(x)=\left(\sqrt{2\bar{x}}-\bar{x}\right)I(x<\frac{1}{2})+(1+x-\sqrt{2x})I(x\ge\frac{1}{2}).$$

The optimal values for the players are $P(W_1)\approx 0.490$ *and* $P(W_2)\approx 0.510$.

Proof. We have from (3.2)-(3.3)

$$(3.10)\qquad P(W_2|x,A_1) \equiv P\{W_2|\text{I accepted } X_{11}=x\} = \int_0^1\{p_{2A}(y|x,A_1)\vee p_{2R}(y|x,A_1)\}\,dy = \begin{cases}\displaystyle\int_0^x(1-2x+y)dy+\bar{x}=1-\frac{3}{2}x^2, & \text{if } x\le\frac{1}{2},\\ \displaystyle\int_{2x-1}^x(1-2x+y)dy+\bar{x}=\frac{1}{2}\bar{x}(3-x), & \text{if } x\ge\frac{1}{2}.\end{cases}$$

The computation involved here is shown by Figure 2 in state $(y|x,A_1)$ and it leads to the first half of the statement (3.8).

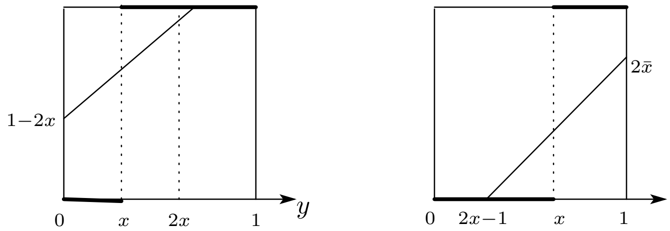

Figure 2a. Case $x\in(0,\frac{1}{2})$ Figure 2b. Case $x\in(\frac{1}{2},1)$

The function $P(W_2|x, A_1)$ is decreasing and concave (convex) for $x < (>)\frac{1}{2}$ with the values, $1, 5/8$, and 0 at $x = 0, 1/2$ and 1, respectively.

We make similar calculations to the above. From (3.4)-(3.5),

$$(3.11)\qquad \begin{aligned} P(W_2|x, R_1) &\equiv P\{W_2|\text{I resampled } X_{12} \text{ in addition to } X_{11} = x\} \\ &= \int_0^1 \{p_{2A}(y|x, R_1) \vee p_{2R}(y|x, R_1)\}\, dy \\ &= \begin{cases} \displaystyle\int_0^x \frac{1}{2}(\bar{x}+y)^2 dy + \int_x^{y_0} \left\{1 - \frac{1}{2}(x+\bar{y})^2\right\} dy \\ \qquad\qquad \displaystyle + \int_{y_0}^{\frac{1}{2}(1+x)} (2y - x)dy + \frac{1}{2}\bar{x}, & \text{if } x \le \frac{1}{2}, \\ \displaystyle\int_0^{y_0} \frac{1}{2}(\bar{x}+y)^2 dy + \int_{y_0}^{\frac{1}{2}(1+x)} (2y-x)dy + \frac{1}{2}\bar{x}, & \text{if } x \ge \frac{1}{2}. \end{cases} \end{aligned}$$

where $y_0 = y_0(x)$ is a unique root of the equation $2y - x = 1 - \frac{1}{2}(x+\bar{y})^2$, *i.e.*, $y_0(x) = \sqrt{2\bar{x}} - \bar{x}$, if $x \le \frac{1}{2}$; and $2y - x = \frac{1}{2}(\bar{x}+y)^2$ *i.e.*, $y_0 = 1 - (\sqrt{2x} - x)$, if $x \ge \frac{1}{2}$. Computation involved here may be clear by Figure 3. The function $y_0 = y_0(x)$ is shown by Figure 4. Thus we have the statement in the second half of (3.8). And so, we obtain, after simplifying,

$$(3.12)\qquad P(W_2|x, R_1) = \begin{cases} y_0(2 - \sqrt{2\bar{x}}) + \frac{1}{6}(2 - \sqrt{2\bar{x}})^3 \\ \qquad - \frac{1}{6}\bar{x}^3 + \frac{1}{4}(3 - 6x - x^2), & \text{if } x \le \frac{1}{2}, \\ \frac{1}{6}y_0\left\{(1+x)\sqrt{2x} + \bar{x}^2\right\} + \frac{1}{4}(3+x)\bar{x}, & \text{if } x \ge \frac{1}{2}. \end{cases}$$

One can find, by computer calculation or direct differentiation, that this function is decreasing in $[0, 1]$, with values $\frac{1}{12}(8\sqrt{2} - 1) \approx 0.8595$, $\frac{7}{12}$ and $\frac{2}{3}(\sqrt{2} - 1) \approx 0.2761$, at $x = 0, 1/2$ and 1, respectively.

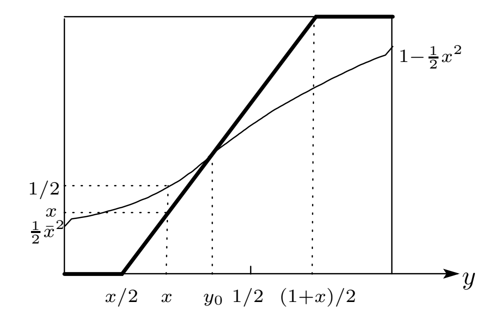

Figure 3a. Case $x \in (0, \frac{1}{2})$

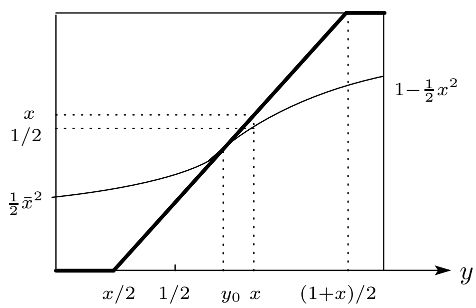

Figure 3b. Case $x \in (\frac{1}{2}, 1)$

Since

$$P(W_1|x, A_1) \vee P(W_1|x, R_1) = 1 - \{P(W_2|x, A_1) \wedge P(W_2|x, R_1)\},$$

the optimal policy for I in the first stage is;

$$\text{Accept (Resample) } X_{11} = x, \text{ if } x > (<) x_0,$$

where x_0 is a unique root in $(1/2, 1)$ of the equation $P(W_2|x, A_1) = P(W_2|x, R_1)$, *i.e.*, $\frac{1}{6}y_0\left\{(1+x)\sqrt{2x} + \bar{x}^2\right\} + \left(\frac{3+x}{4}\right)\bar{x} = \frac{1}{2}\bar{x}(3-x)$ (by (3.10)-(3.12)). By substituting $y_0 = 1 + x - \sqrt{2x}$ into this equation and simplifying, we obtain (3.6) (See Figure 5).

It remains to derive the optimal values for the players.

By (3.6), (3.9), (3.10) and(3.12), it follows that

$$\begin{aligned} P(W_2) &= \int_0^{x_0} P(W_2|x, R_1)dx + \int_{x_0}^1 P(W_2|x, A_1)dx \\ &= \int_0^{\frac{1}{2}} \left[(\sqrt{2\bar{x}} - \bar{x})(2 - \sqrt{2\bar{x}}) + \frac{1}{6}(2 - \sqrt{2\bar{x}})^3 - \frac{1}{6}\bar{x}^3 + \frac{1}{4}(3 - 6x - x^2)\right] dx \\ &\quad + \int_{\frac{1}{2}}^{x_0} \left[\frac{1}{6}(1 + x - \sqrt{2x})\left\{(1+x)\sqrt{2x} + \bar{x}^2\right\} + \frac{1}{4}(3+x)\bar{x}\right] dx + \int_{x_0}^1 \frac{1}{2}\bar{x}(3-x)dx \end{aligned}$$

where the three integrals are computed with the result

$$\approx 0.36514 + 0.02784 + 0.11699 \approx 0.50998$$

This completes the proof of Theorem 3. □

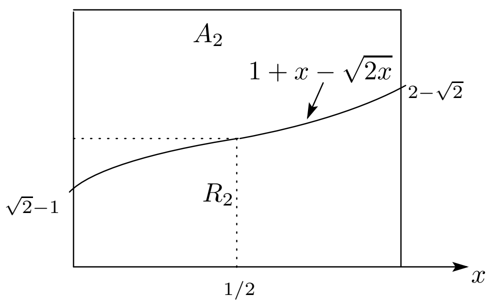

Figure 4. The function $y_0(x)$ in state $(y|x, R_1)$

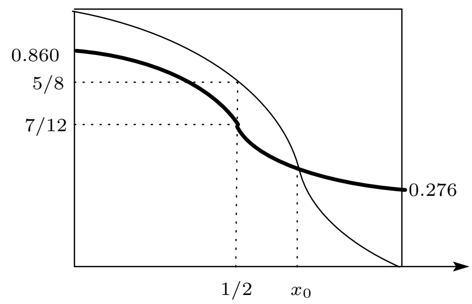

Figure 5. Determination of x_0

Remark2. Theorem 3 shows that the disadvantage for the first-mover is very small. We give here a numerical example to Theorem 3. If $X_{11} = x = 0.482$, in the first stage, then I announces 0.482 and R_1 (since $x < x_0 \approx 0.549$), and resamples X_{12}. In the second stage, if II observes $X_{21} = y = 0.713$, then II accepts it (since $y > y_0(x) = \sqrt{2\bar{x}} - \bar{x} \approx 0.298$ at $x = 0.482$). After the showdown is made, I (II) is the winner, if $\frac{1}{2}(0.482 + X_{12}) > (<) 0.713$.

4 Showcase Showdown. Consider the case where the score is defined by

$$(4.1) \qquad S_i(X_{i1}, X_{i2}) = \begin{cases} X_{i1}, \\ (X_{i1} + X_{i2})I(X_{i1} + X_{i2} < 1), \end{cases} \text{if } \begin{cases} X_{i1} \text{ is accepted,} \\ X_{i2} \text{ is resampled.} \end{cases}$$

The problem becomes a little bit difficult, since draw (*i.e.*, both players have zero scores) can occur with positive probability.

We define states $(y|x, A_1)$, *etc*, as the same as in Section 3. We have

$$(4.2) \qquad p_{2A}(y|x, A_1) = I(x < y),$$

$$(4.3) \qquad p_{2R}(y|x, A_1) = P(x < y + X_{22} < 1) = \bar{y}I(x \le y) + \bar{x}I(x > y),$$

$$(4.4) \qquad p_{2A}(y|x, R_1) = P(x + X_{12} > 1 \text{ or } x + X_{12} < y) = yI(x \le y) + xI(x > y),$$

$$\begin{aligned} (4.5) \quad p_{2R}(y|x, R_1) &= P\left\{(x + X_{12})I(x + X_{12} < 1) < (y + X_{22})I(y + X_{22} < 1)\right\} \\ &= I(x \le y)\frac{1}{2}(1 - y^2) + I(x > y)\left(\frac{1}{2}\bar{x}^2 + x\bar{y}\right). \end{aligned}$$

and therefore, from (4.2)-(4.3),

$$(4.6)\qquad P(W_2|x,A_1)=\int_0^1\{p_{2A}(y|x,A_1)\vee p_{2R}(y|x,A_1)\}dy=x\bar{x}+\bar{x}=1-x^2.$$

The optimal strategy for II in state $(y|x,A_1)$ is;

$$(4.7)\qquad \text{Accept (Resample)}\ \ X_{21}=y,\ \text{ if }\ y>(<)x$$

(See Figure 4). Also from (4.4)-(4.5)

$$(4.8)\qquad P(W_2|x,R_1)=\int_0^1\{p_{2A}(y|x,R_1)\vee p_{2R}(y|x,R_1)\}dx$$

$$=\begin{cases}\displaystyle\int_0^x\left(\frac{1}{2}\bar{x}^2+x\bar{y}\right)dy+\int_x^{y_0}\frac{1}{2}(1-y^2)dy+\int_{y_0}^1 ydy, & \text{if } x\le\sqrt{2}-1\\[2ex] \displaystyle\int_0^{y_0}\left(\frac{1}{2}\bar{x}^2+x\bar{y}\right)dy+\int_{y_0}^x xdy+\int_x^1 ydy, & \text{if } x\ge\sqrt{2}-1\end{cases}$$

$$=\begin{cases}\dfrac{1}{6}x^3+\dfrac{1}{3}(2\sqrt{2}-1), & \text{if } x\le\sqrt{2}-1\\[2ex] \bar{x}^4/(8x)+\dfrac{1}{2}x^2+\dfrac{1}{2}, & \text{if } x\ge\sqrt{2}-1,\end{cases}$$

where y_0 is a unique root of the equation $y=\frac{1}{2}(1-y^2)$, *i.e.*, $y_0=\sqrt{2}-1$, if $x\le\sqrt{2}-1$; and $x=\frac{1}{2}\bar{x}^2+x\bar{y}$, *i.e.*, $y_0=\bar{x}^2/(2x)=\frac{1}{2}(x+x^{-1})-1$, if $x\ge\sqrt{2}-1$. Computation involved here for deriving the second expression in the r.h.s. of (4.8) may be clear as shown by Figure 6. The function $y_0=y_0(x)$ is shown by Figure 7.

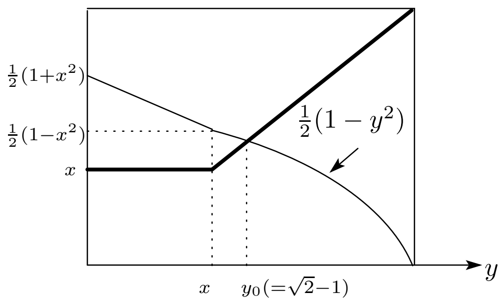

Figure 6a. Case $x\in(0,\sqrt{2}-1)$

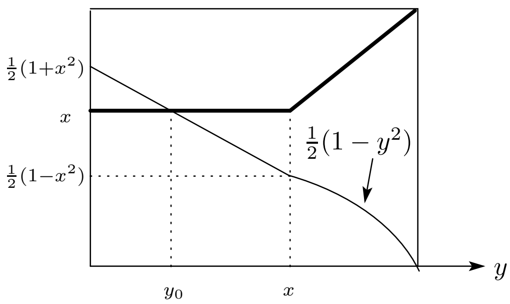

Figure 6b. Case $x\in(\sqrt{2}-1,1)$

The optimal strategy for II in state $(y|x,R_1)$ is ;

$$(4.9)\qquad \text{Accept (Resample)}\ \ X_{21}=y,\ \text{ if }\ y>(<)y_0(x).$$

Now, by using (4.7)-(4.9) we can find the probability of draw, denoted by $P(D)$.

$$(4.10)\qquad P(D|x,A_1)\ =\ P[D|\{\text{I accepts }\ X_{11}=x\}\cap\{\text{II behaves optimally in the second stage}\}]=0,$$

$$(4.11)\quad P(D|x,R_1)\ =\ P[D|\{\text{I resamples }\ X_{12},\text{in addition to }\ X_{11}=x\}\cap\{\text{II behaves optimally in the second stage}\}],$$

$$= P\{x + X_{12} > 1, X_{21} < y_0(x), X_{21} + X_{22} > 1\} = 1 - x(y_0(x))^2$$
$$= \begin{cases} \frac{1}{2}(3-2\sqrt{2})x, \\ \bar{x}^4/(8x), \end{cases} \quad \text{if } x \begin{cases} < \\ > \end{cases} \sqrt{2}-1$$

and so, it follows from (4.6) and (4.8) that

$$p_{1A}(x) = 1 - P(W_2|x, A_1) - P(D|x, A_1) = x^2. \tag{4.12}$$

$$p_{1R}(x) = 1 - P(W_2|x, R_1) - P(D|x, R_1) \tag{4.13}$$
$$= 1 - \begin{cases} \frac{1}{6}x^3 + \frac{1}{3}(2\sqrt{2}-1) \\ \bar{x}^4/(8x) + \frac{1}{2}x^2 + \frac{1}{2} \end{cases} - \begin{cases} \frac{1}{2}(3-2\sqrt{2})x \\ \bar{x}^4/(8x), \end{cases} \quad \text{if } x \begin{cases} < \\ > \end{cases} \sqrt{2}-1$$
$$= \begin{cases} -\frac{1}{6}x^3 - \frac{1}{2}(3-2\sqrt{2})x + \frac{2}{3}(2-\sqrt{2}), \\ -\bar{x}^4/(4x) - \frac{1}{2}x^2 + \frac{1}{2}, \end{cases} \quad \text{if } x \begin{cases} < \\ > \end{cases} \sqrt{2}-1$$

The two functions above in the different parts are both concave and decreasing, and they are continuously connected at $x = \sqrt{2}-1$ as shown by Figure 8.

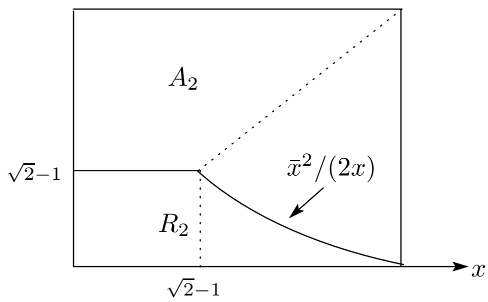

Figure 7. $y_0(x)$ in state $(y|x, R_1)$

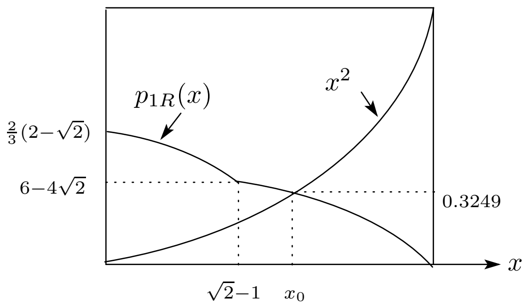

Figure 8. Determination of x_0

From (4.12)-(4.13), we have

$$P(W_1) = \int_0^1 \{p_{1A}(x) \vee p_{1R}(x)\}dx \tag{4.14}$$
$$= \int_0^{\sqrt{2}-1} \left\{-\frac{1}{6}x^3 - \frac{1}{2}(3-2\sqrt{2})x + \frac{2}{3}(2-\sqrt{2})\right\} dx$$
$$+ \int_{\sqrt{2}-1}^{x_0} \left\{-\bar{x}^4/(4x) - \frac{1}{2}x^2 + \frac{1}{2}\right\} dx + \int_{x_0}^1 x^2 dx,$$

where $x_0 \approx 0.570$ is a unique root in $(\sqrt{2}-1, 1)$ of the equation $p_{1A}(x) = p_{1R}(x)$, *i.e.*,

$$\bar{x}^4 = 2x(1-3x^2). \tag{4.15}$$

The optimal strategy for I in the first stage is;

$$\text{Accept (Resample) } X_{11} = x, \text{ if } x > (<) x_0. \tag{4.16}$$

By substituting the value $x_0 = 0.570$ into (4.14) and computing, we finally get $P(W_1) \approx 0.4768$, in which the integrand of second integral in (4.14) was approximated by the closest linear function. This result combined with (4.10)-(4.11) gives

$$P(D) = \int_0^{x_0} P(D|x, R_1)dx = \int_0^{\sqrt{2}-1} \frac{1}{2}(3-2\sqrt{2})x dx \tag{4.17}$$
$$+ \int_{\sqrt{2}-1}^{x_0} \bar{x}^4/(8x) dx \approx 0.0108.$$

Here, in computing the second integral we used the above approximation again, considering

$$\bar{x}^4/(8x) = -\frac{1}{2}\left\{-\bar{x}^4/(4x) - \frac{1}{2}x^2 + \frac{1}{2}\right\} - \frac{1}{4}x^2 + \frac{1}{4}.$$

Then at last we have, from (4.14) and (4.17),

$$P(W_2) = 1 - P(W_1) - P(D) \approx 1 - 0.4768 - 0.0108 = 0.5124. \tag{4.18}$$

Summarizing the whole arguments above we obtain

Theorem 4 *The solution to the two-player game with the score* (4.1) *is as follows. The optimal strategy for I in the first stage is given by ;*

$$\text{Accept (Resample) } X_{11} = x, \text{ if } x > (<) x_0 \approx 0.570,$$

where x_0 *is the unique root in* $(\sqrt{2}-1, 1)$ *of the equation* $\bar{x}^4 = 2x(1-3x^2)$.

The optimal strategy for II in the second stage is given by ;

$$\text{Accept (Resample) } X_{21} = y, \text{ if } y > (<) \begin{cases} x \\ y_0(x), \end{cases} \text{ in state } \begin{cases} (y|x, A_1) \\ (y|x, R_1), \end{cases}$$

where $y_0(x) = (\sqrt{2}-1)I(x \le \sqrt{2}-1) + \left(\bar{x}^2/(2x)\right) I(x > \sqrt{2}-1)$.

The players' optimal values and the probability of draw under the optimal play are

$$P(W_1) \approx 0.4768, P(W_2) \approx 0.5124 \text{ and } P(D) \approx 0.0108.$$

The exprecit expressions of $P(W_1)$ *and* $P(D)$ *are given by* (4.14) *and* (4.17) *respectively.*

5 Final Remarks.

Remark 3. Consider the one-player version of the games, where player aims to maximize his (or her) expected score. Let a^* and $M(a^*)$ be the optimal threshold number and the expected score obtained by using it, respectively. Then it is easy to obtain the table below.

	Keep-or Exchange Theorem 1&2	Competing Average Theorem 3	Showcase Showdown Theorem 4
a^*	1/2 (0.6124 in (2.5))	1/2 (0.549 in (3.6))	$\sqrt{2}-1$ (0.570 in(4.16))
$M(a^*)$	5/8	9/16	$\frac{1}{3}(2\sqrt{2}-1) \approx 0.60904$

(*c.f.* The corresponding number x_0 in the two-player cases are given in the parentheses. The player behaves much easier than in the two-player cases, since he hasn't the second-mover.)

Remark4. There remain some interesting problems to be solved.

1°) Simultaneous-move versions of Competing Average and Showcase Showdown. Do there exist interesting threshold values like g (golden bisection number) as in Keep-or-Exchange?

2°) Less-information case. If the first-mover opens his $X_{11} = x$ to both of himself and II, but his choice of A and R kept private, then how do the solutions change?

3°) Dependent-observation case. If X_{i1} and $X_{i2} (i = 1, 2)$ are dependent r.v.s, for example, a family with density

$$f(x_1, x_2) = 1 + \theta(2x_1 - 1)(2x_2 - 1), \quad \forall (x_1, x_2) \in [0, 1]^2, \forall |\theta| \leq 1,$$

then how do the solutions change?

And furthermore

4°) Extension to the three-player games should be investigated. Intuitively, it would seem that, the last-mover has an advantage over the middle-mover, and the middle-mover, in turn, has an advantage over the first-mover. Is this intuition always correct? We present some results in the forth-coming paper.

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