# EXTENDED COMPLEMENTARY DOMAIN OF THE FURUTA INEQUALITY

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Abstract. We discuss monotone properties of Furuta type inequalities: Let $A \geq B \geq 0$ and $0 \leq t \leq p$. Then for each $t \in [0,\ 1]$, $A^t \ \sharp_{\frac{1-t}{p-t}}\ B^p$ is increasing for $p \geq 1$, and if $A^p \geq B^p$ for some fixed $p > 0$, then $A^t \ \sharp_{\frac{1-t}{p-t}}\ B^p$ is increasing for $t \in [0,1]$. Moreover, if $A^t \geq B^t$ for some fixed $t > 0$, then the following inequalities hold;

$$for \ \ t \leq \beta \leq p, \ \ (A^t \ \natural_{\frac{\beta-t}{p-t}}\ B^p)^{\frac{\delta}{\beta}} \leq B^\delta \leq A^t \ \sharp_{\frac{\delta-t}{p-t}}\ B^p,$$

$$for \ \ p \leq \delta \leq \beta, \ \ (A^t \ \natural_{\frac{\beta-t}{p-t}}\ B^p)^{\frac{\delta}{\beta}} \leq A^t \ \natural_{\frac{\delta-t}{p-t}}\ B^p.$$

Consequently, it improves a recent result of Yang and Zuo.

**1. Introduction.** Recently, C.Yang and H.Zuo [25] have shown monotone properties of Furuta type inequalities on complementary domain. They point out a fresh aspect of operator means, however their results are depending on [18] too much and the proofs are confused a little. The $\alpha$-power mean of $A$ and $B$ introduced by Kubo-Ando [24] is given by

$$A \ \sharp_\alpha \ B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}} \quad \text{for} \ \ 0 \leq \alpha \leq 1. \tag{1}$$

We can regard (1) as a path connecting $A(= A \ \sharp_0 \ B)$ and $B(= A \ \sharp_1 \ B)$, (cf.[17],[21]). We use the notation $\natural_r$ to distinguish from $\alpha$-power mean $\sharp_\alpha$ ($\alpha \in [0,1]$) as follows:

$$A \ \natural_r \ B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r A^{\frac{1}{2}} \quad \text{for} \ \ r \notin [0,1]$$

Throughout this note, $A$ and $B$ are positive operators on a Hilbert space. For convenience, we denote $A \geq 0$ (resp. $A > 0$) if $A$ is a positive (resp. positive invertible) operator.

The main theorem of [25] is the following:

**Theorem A.** *Let $A \geq B \geq 0$ and fix $0 < p_0 < 1$. Then $A^t \ \sharp_{\frac{p_0-t}{p-t}}\ B^p$ is an increasing function for $t \in [0,p_0]$ and $p \in [p_0,1]$.*

This result shows the monotonicity of the paths $A^t \ \sharp_{\frac{p_0-t}{p-t}}\ B^p$ from $A^t$ to $B^p$ at the fixed point $p_0$.

The Furuta inequality [13] (cf.[14]) can be written by the form of $\alpha$-power mean as follows ([2],[16]).

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**Furuta inequality:** *If $A \geq B \geq 0$, then*

(F) $$A^{-t} \sharp_{\frac{1+t}{p+t}} B^p \leq A \quad \text{and} \quad B \leq B^{-t} \sharp_{\frac{1+t}{p+t}} A^p$$

*holds for $t \geq 0$ and $1 \leq p$.*

From this formulation, $A^{-t} \sharp_\alpha B^p$ is a path from $A^{-t}$ to $B^p$ and (F) is understood as the comparison between $A$ and the operator value $A^{-t} \sharp_{\frac{1+t}{p+t}} B^p$ at the internally dividing point 1 of $[-t, p]$ with the ratio $\frac{1+t}{p+t}$.

The Furuta inequality (F) is an extension of the Löwner-Heinz inequality:

(LH) $$If \quad A \geq B \geq 0, \quad \text{then} \quad A^\alpha \geq B^\alpha \quad \text{for} \quad 0 \leq \alpha \leq 1.$$

In [16], we had rearranged (F) more precisely in one line by the form of $\alpha$-power mean (cf.[3]).

**Satellite theorem of Furuta inequality:** *If $A \geq B \geq 0$, then*

(SF) $$A^{-t} \sharp_{\frac{1+t}{p+t}} B^p \leq B \leq A \leq B^{-t} \sharp_{\frac{1+t}{p+t}} A^p$$

*holds for all $t \geq 0$ and $1 \leq p$.*

Whether similar relations to (F) hold for the case $t \leq 0$ is a subject easily drawn ([4],[5],[6],[18]). In this context, Theorem A gives some answers to this. The following is a further generalization of these.

**Theorem 1.** *Let $A \geq B > 0$ and $0 \leq t \leq 1 \leq p$. Then*

(1) $$A^t \sharp_{\frac{1-t}{p-t}} B^p \quad is \quad increasing \quad for \quad p \geq 1.$$

(2) $$If \quad A^p \geq B^p, \quad then \quad A^t \sharp_{\frac{1-t}{p-t}} B^p \quad is \quad increasing \quad for \quad t \in [0, 1].$$

**Proof.** (1) For $1 \leq p_1 \leq p_2$,

$$A^t \sharp_{\frac{1-t}{p_2-t}} B^{p_2} = A^t \sharp_{\frac{1-t}{p_1-t}} (A^t \sharp_{\frac{p_1-t}{p_2-t}} B^{p_2}) \geq A^t \sharp_{\frac{1-t}{p_1-t}} (B^t \sharp_{\frac{p_1-t}{p_2-t}} B^{p_2}) = A^t \sharp_{\frac{1-t}{p_1-t}} B^{p_1}.$$

(2) For $0 \leq t_1 \leq t_2 \leq 1 \leq p$ with $t_2 - t_1 \leq p - t_2$,

$$\begin{aligned} &A^{t_2} \sharp_{\frac{1-t_2}{p-t_2}} B^p = B^p \sharp_{\frac{p-1}{p-t_2}} A^{t_2} = B^p \sharp_{\frac{p-1}{p-t_1}} (B^p \natural_{\frac{p-t_1}{p-t_2}} A^{t_2}) \\ &= B^p \sharp_{\frac{p-1}{p-t_1}} (A^{t_2} \natural_{\frac{t_1-t_2}{p-t_2}} B^p) = B^p \sharp_{\frac{p-1}{p-t_1}} A^{t_2}(A^{-t_2} \sharp_{\frac{t_2-t_1}{p-t_2}} B^{-p})A^{t_2} \\ &\geq B^p \sharp_{\frac{p-1}{p-t_1}} A^{t_2}(A^{-t_2} \sharp_{\frac{t_2-t_1}{p-t_2}} A^{-p})A^{t_2} = B^p \sharp_{\frac{p-1}{p-t_1}} A^{t_1} = A^{t_1} \sharp_{\frac{1-t_1}{p-t_1}} B^p. \end{aligned}$$

As an application, we have the following result which is an extension of Theorem A. As a matter of fact, the range of $p$ is extended:

**Corollary** *Let $A \geq B \geq 0$ and fix $0 < p_0 < 1$. Then $A^t \sharp_{\frac{p_0-t}{p-t}} B^p$ is an increasing function for $t \in [0, p_0]$ and $p \geq p_0$.*

**Proof.** Since $A \geq B \geq 0$, $A_1 = A^{p_0} \geq B_1 = B^{p_0}$ for fixed $p_0 \in (0,1)$ by (LH). By Theorem 1, $A_1^t \sharp_{\frac{1-t}{p-t}} B_1^p$ is increasing for $p \geq 1$ and $t \in [0,1]$. Putting $t_1 = p_0 t,\ p_1 = p_0 p$, we have $A_1^t \sharp_{\frac{1-t}{p-t}} B_1^p = A^{t_1} \sharp_{\frac{p_0-t_1}{p_1-t_1}} B^{p_1}$ is increasing for $p_1 \geq p_0$ and $t_1 \in [0, p_0]$.

**2. Extended complementary domain.** As a generalized form of (F), Furuta had shown the following grand Furuta inequality [15], which is a parameteric one interpolating (F) and Ando-Hiai inequality equivalent to their majorization theorem [1]. We cite it here by the satellite form with the $\alpha$-power mean ([8],[20]).

If $A \geq B > 0$, then for $r \geq 0$ and $0 \leq t \leq 1 \leq p \leq \beta$,
(GF)
$$A^{-r} \sharp_{\frac{1+r}{\beta+r}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}} \leq B \leq A \leq (B^t \natural_{\frac{\beta-t}{p-t}} A^p)^{\frac{1}{\beta}} \leq B^{-r} \sharp_{\frac{1+r}{\beta+r}} (B^t \natural_{\frac{\beta-t}{p-t}} A^p).$$

The following has shown in [10] which is the key point for our proof of the grand Furuta inequality, and (2) has shown in [8].

**Theorem B.** *Let $A \geq B \geq 0$ and $0 \leq t \leq 1 \leq p \leq \beta$. Then the following* (1) *and* (2) *holds.*

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}} \leq B \leq A \tag{1}$$

$$(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leq B^p \tag{2}$$

*Additionally, both $(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}}$ and $(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}}$ are decreasing for $\beta (\geq p)$.*

As complements of the Furuta inequality, we had investigated the part $A^t \natural_{\frac{\beta-t}{p-t}} B^p$ of (GF) and had the following results in [18](cf.[4],[5],[6],[19]).

**Theorem C.** *If $A \geq B > 0$, then*

$$A^t \natural_{\frac{2p-t}{p-t}} B^p \leq B^{2p} \leq A^{2p} \quad for \quad 0 \leq t < p \leq \frac{1}{2}, \tag{1}$$

$$A^t \natural_{\frac{1-t}{p-t}} B^p \leq B \leq A \quad for \quad 0 \leq t < p \leq 1 \quad and \quad p \geq \frac{1}{2}. \tag{2}$$

We here remark that $A^t \sharp_\alpha B^p$ is a path connecting $A^t$ with $B^p$ and $A^t \natural_{\frac{\beta-t}{p-t}} B^p$ for $\beta \geq p$ is regarded as the value of the path at the external point $\beta$ of $[t, p]$ with ratio $\frac{\beta-t}{p-t}$. From the viewpoint of this, Theorem C says that $A^t \natural_{\frac{\beta-t}{p-t}} B^p$ is comparable with $A^\beta$ and $B^\beta$, at the points $\beta = 1$ and $\beta = 2p$, respectively.

C.Yang and H.Zuo [25] have tried to see (GF) for the case $r \leq 0$. So we also follow it from the same view point as theirs. The next is an application of Theorem B.

**Theorem 2.** *Let $A \geq B \geq 0$ and $0 \leq t \leq 1 \leq p$. Then for each $1 \geq r \geq 0$, $A^r \sharp_{\frac{1-r}{p-r}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}}$ is decreasing for $\beta \geq p$ and*

$$A^r \sharp_{\frac{1-r}{p-r}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}} \leq A^r \sharp_{\frac{1-r}{\beta-r}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)$$

*holds for* $\beta \geq p$. *If* $A^p \geq B^p$, *then* $A^r \sharp_{\frac{1-r}{p-r}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}}$ *is increasing for* $r \in [0,\ 1]$.

**Proof.** Theorem B assures $B_1 = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}} \leq A$ and $(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}}$ is decreasing for $\beta \geq p$. Hence so is $A^r \sharp_{\frac{1-r}{p-r}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}}$. The desired inequality holds from

$$A^r \sharp_{\frac{1-r}{\beta-r}} B_1^\beta = A^r \sharp_{\frac{1-r}{p-r}} (A^r \sharp_{\frac{p-r}{\beta-r}} B_1^\beta) \geq A^r \sharp_{\frac{1-r}{p-r}} (B_1^r \sharp_{\frac{p-r}{\beta-r}} B_1^\beta) = A^r \sharp_{\frac{1-r}{p-r}} B_1^p.$$

Next, if $A^p \geq B^p$, then $A^p \geq B^p \geq B_1^p$ by Theorem B (2), so that $A^r \sharp_{\frac{1-r}{p-r}} B_1^p = A^r \sharp_{\frac{1-r}{p-r}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{p}{\beta}}$ is increasing for $r \in [0,\ 1]$ by Theorem 1.

Now, our discussions of the above are done under the assumptions of $A^p \geq B^p$ or $A \geq B$ for $p \geq 1 \geq t > 0$, so we next assume $A^t \geq B^t$ as in below.
For this, we need the following key fact, which was shown in the proof of Theorem B ([9],[22],[23]).

**Lemma.** *Let* $A,\ B > 0$ *and* $0 \leq t < p \leq \beta \leq 2p - t$. *If* $A^t \geq B^t$, *then*

$$A^t \natural_{\frac{\beta-t}{p-t}} B^p \leq B^\beta.$$

**Proof.** We give a proof for convenience. Since $A \natural_r B = A(A^{-1} \natural_{-r} B^{-1})A$, we have

$$A^t \natural_{\frac{\beta-t}{p-t}} B^p = B^p \natural_{\frac{p-\beta}{p-t}} A^t = B^p(B^{-p} \sharp_{\frac{\beta-p}{p-t}} A^{-t})B^p \leq B^p(B^{-p} \sharp_{\frac{\beta-p}{p-t}} B^{-t})B^p = B^\beta.$$

In [19], we had compared the order among the paths represented by $\alpha$-mean, which is our view point of the Furuta type inequalities. The following is inspired by Theorem A.

**Theorem 3.** *Let* $A,\ B > 0$ *and* $0 \leq t < p \leq \beta$. *If* $A^t \geq B^t$, *then the following inequalities hold.*

$$\text{(1)} \qquad For\ \ t \leq \delta \leq p,\ \ (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \leq B^\delta \leq A^t \sharp_{\frac{\delta-t}{p-t}} B^p.$$

$$\text{(2)} \qquad For\ \ p \leq \delta \leq \beta,\ \ (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \leq A^t \natural_{\frac{\delta-t}{p-t}} B^p.$$

**Proof.** (1) Since $A^t \sharp_{\frac{\delta-t}{p-t}} B^p \geq B^t \sharp_{\frac{\delta-t}{p-t}} B^p = B^\delta$, the right side of (1) holds. To show the first inequality, we have only to use Lemma inductively. For $p \leq \beta_1 \leq 2p - t$, Lemma says $A^t \natural_{\frac{\beta_1-t}{p-t}} B^p \leq B^{\beta_1}$ and we have $(A^t \natural_{\frac{\beta_1-t}{p-t}} B^p)^{\frac{\delta}{\beta_1}} \leq B^\delta$ by (LH). That is, (1) is proved for the case $p \leq \beta \leq 2p - t$.
If $\beta > 2p - t$, then we take $\beta_1 < \beta_2 < \ldots < \beta_{k-1} < \beta_k = \beta$ such that $\beta_{i+1} < 2\beta_i - t$ for $i = 1, \ldots, k-1$ and put $B_i = (A^t \natural_{\frac{\beta_i-t}{p-t}} B^p)^{\frac{1}{\beta_i}}$ for $i = 1, \ldots, k$. Thus it suffices to show that $B_k^\delta \leq B^\delta$. For this, we prove $B_k^\delta \leq B_{k-1}^\delta \leq \ldots \leq B_1^\delta \leq B^\delta$ by induction. In the preceding discussion, it is shown that $B_1^\delta \leq B^\delta$. So we assume that $B_i^\delta \leq B^\delta$ for some $i$. Since $B_i^t \leq B^t \leq A^t$ and $\beta_i \leq \beta_{i+1} \leq 2\beta_i - t$, it follows from Lemma that $A^t \natural_{\frac{\beta_{i+1}-t}{\beta_i-t}} B_i^{\beta_i} \leq B_i^{\beta_{i+1}}$. Therefore we have

$$B_{i+1}^{\beta_{i+1}} = A^t \natural_{\frac{\beta_{i+1}-t}{p-t}} B^p = A^t \natural_{\frac{\beta_{i+1}-t}{\beta_i-t}} (A^t \natural_{\frac{\beta_i-t}{p-t}} B^p) = A^t \natural_{\frac{\beta_{i+1}-t}{\beta_i-t}} B_i^{\beta_i} \leq B_i^{\beta_{i+1}}$$

and so $B_{i+1}^{\delta} \le B_i^{\delta}$ by (LH). By the induction, we have $B_k^{\delta} \le B_{k-1}^{\delta} \le \ldots \le B_1^{\delta} \le B^{\delta}$.

Proof of (2) is similar to (1), but we need two steps to catch $\beta$. First of all, we show that if we put $D = (A^t \natural_{\frac{\delta-t}{p-t}} B^p)^{\frac{1}{\delta}}$, then $D^t \le A^t$. If $p \le \delta \le 2p - t$, then

$$D^{\delta} = A^t \natural_{\frac{\delta-t}{p-t}} B^p \le B^{\delta}$$

by Lemma and so $D^t \le B^t \le A^t$ by (LH) since $0 \le t < p \le \delta$. In the case $2p - t < \delta \le \beta$, we devide $[p, \delta]$ into $p = \delta_0 < \delta_1 < \cdots < \delta_l = \delta$ suth that $\delta_{i+1} \le 2\delta_i - t$ $(i = 0, \cdots, l-1)$ and put $D_i = (A^t \natural_{\frac{\delta_i-t}{p-t}} B^p)^{\frac{1}{\delta_i}}$ $(i = 0, \cdots, l)$. Then, as in the proof of (1), we have $D_{i+1}^{\delta_{i+1}} \le D_i^{\delta_{i+1}}$ and so $D^t = D_l^t \le D_{l-1}^t \le \ldots \le D_0^t = B^t \le A^t$.

Since we could prove $D^t \le A^t$, we next apply it to Lemma. First, if $\delta \le \beta \le 2\delta - t$, then

$$A^t \natural_{\frac{\beta-t}{p-t}} B^p = A^t \natural_{\frac{\beta-t}{\delta-t}} (A^t \natural_{\frac{\delta-t}{p-t}} B^p) = A^t \natural_{\frac{\beta-t}{\delta-t}} D^{\delta} \le D^{\beta} = (A^t \natural_{\frac{\delta-t}{p-t}} B^p)^{\frac{\beta}{\delta}}.$$

Hence we have the conclusion $(A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\delta}{\beta}} \le A^t \natural_{\frac{\delta-t}{p-t}} B^p (\le B^{\delta})$ by (LH). On the other hand, if $2\delta - t < \beta$, we use the same method as in (1) again: We take $\delta = \beta_0 < \beta_1 < \ldots < \beta_{k-1} < \beta_k = \beta$ such that $\beta_{i+1} \le 2\beta_i - t$ for $i = 1, \ldots, k-1$ and put $B_i = (A^t \natural_{\frac{\beta_i-t}{p-t}} B^p)^{\frac{1}{\beta_i}}$ $(i = 0, \cdots, k)$. Thus we obtain $B_{i+1}^{\beta_{i+1}} \le B_i^{\beta_{i+1}}$ and so $B_{i+1}^{\delta} \le B_i^{\delta}$, $(i = 1, \ldots, k-1)$. That is, we have the coclusion;

$$(A^t \natural_{\frac{\beta-t}{\delta-t}} B^p)^{\frac{\delta}{\beta}} \le \ldots \le (A^t \natural_{\frac{\beta_1-t}{\delta-t}} B^p)^{\frac{\delta}{\beta_1}} \le A^t \natural_{\frac{\delta-t}{p-t}} B^p.$$

**3. Chaotic order.** In Theorem 3, we consider operator inequalities of Furuta type under the assumption $A^t \ge B^t$ for some $t$ with $0 < t \le p$. We note that the chaotic order $\log A \ge \log B$, simply denoted by $A \gg B$, is regarded as the case $t = 0$ in the sense that $\log X = \lim_{t \to +0} \frac{X^t - I}{t}$ (cf.[7]). In this section, we discuss them under the chaotic order $A \gg B$. Some useful characterizations of the chaotic order have given in [23], (cf., [22]) as follows:

**Theorem D.** *If $A \gg B$, then the following* (1) *and* (2) *hold.*

(1) $$A^{-t} \sharp_{\frac{\delta+t}{p+t}} B^p \le B^{\delta} \text{ and } A^{\delta} \le B^{-t} \sharp_{\frac{\delta+t}{p+t}} A^p \text{ for } t \ge 0 \text{ and } 0 \le \delta \le p.$$

(2) $$A^{-t} \sharp_{\frac{\alpha+t}{p+t}} B^p \le A^{\alpha} \text{ and } B^{\alpha} \le B^{-t} \sharp_{\frac{\alpha+t}{p+t}} A^p \text{ for } -t \le \alpha \le 0 \text{ and } 0 \le p.$$

Concerning to Theorem D, Fujii and Kim [12] point out the next characterization of chaotic order(cf.[23]).

**Theorem E.** *Let $A$, $B > 0$. Then the following are equivalent.*

(1) $$A \gg B$$

(2) $$A^{-t} \natural_{\frac{\beta+t}{p+t}} B^p \ge B^{\beta} \text{ for } 0 \le p \le \beta \le 2p, \ t \ge 0$$

(3) $$A^{\beta} \ge B^{-t} \natural_{\frac{\beta+t}{p+t}} A^p \text{ for } 0 \le p \le \beta \le 2p, \ t \ge 0$$

We give additions to Theorem E in parallel with Theorem D.
Under the assumptions $-2t \le \alpha \le -t,\ p \ge 0$, the following (4) and (5) are also equivalent to $A \gg B$.

$$(4) \quad A^{-t} \natural_{\frac{\alpha+t}{p+t}} B^p \ge A^\alpha \qquad\qquad (5) \quad B^\alpha \ge B^{-t} \natural_{\frac{\alpha+t}{p+t}} A^p$$

For convenience, we prove them. Since $-t \le \alpha + t \le 0$,

$$A^{-t} \natural_{\frac{\alpha+t}{p+t}} B^p = A^{-t}(A^t \sharp_{\frac{-\alpha-t}{p+t}} B^{-p})A^{-t} = A^{-t}(A^t \sharp_{\frac{-\alpha-t}{t}} (A^t \sharp_{\frac{t}{p+t}} B^{-p}))A^{-t}$$
$$= A^{-t}(A^t \sharp_{\frac{-\alpha-t}{t}} (A^{-t} \sharp_{\frac{t}{p+t}} B^p)^{-1})A^{-t} \ge A^{-t}(A^t \sharp_{\frac{-\alpha-t}{t}} I)A^{-t} = A^\alpha.$$

In Theorem 3, we can move $\beta(\ge p)$ freely, but now we must put some restriction on $\beta$ under the assumption of chaotic order $A^0 \ge B^0$.

**Theorem 4.** *Let $A,\ B > 0$ and $0 \le t \le p \le \beta \le 2p$. If $A \gg B$, then the following* (1) *and* (2) *hold.*
(1) *For $0 \le \delta \le p$*

$$A^{-t} \sharp_{\frac{\delta+t}{p+t}} B^p \le B^\delta \le (A^{-t} \natural_{\frac{\beta+t}{p+t}} B^p)^{\frac{\delta}{\beta}} \quad \text{and} \quad (B^{-t} \natural_{\frac{\beta+t}{p+t}} A^p)^{\frac{\delta}{\beta}} \le A^\delta \le B^{-t} \sharp_{\frac{\delta+t}{p+t}} A^p$$

(2) *For $-t \le \gamma \le 0$*

$$A^{-t} \sharp_{\frac{\gamma+t}{p+t}} B^p \le A^\gamma \le (B^{-t} \natural_{\frac{\beta+t}{p+t}} A^p)^{\frac{\gamma}{\beta}} \quad \text{and} \quad (A^{-t} \natural_{\frac{\beta+t}{p+t}} B^p)^{\frac{\gamma}{\beta}} \le B^\gamma \le B^{-t} \sharp_{\frac{\gamma+t}{p+t}} A^p$$

**Proof.** (1) $A^{-t} \sharp_{\frac{\delta+t}{p+t}} B^p \le B^\delta$ and $A^\delta \le B^{-t} \sharp_{\frac{\delta+t}{p+t}} A^p$ are obtained by Theorem D and the rest ones follow from Theorem E (2) and (LH) because $0 \le \frac{\delta}{\beta} \le 1$. (2) also follows from Theorem D (2), Theorem E (3) and (LH) since $-1 \le \frac{\gamma}{\beta} \le 0$.

**Remark.** Finally we mention that $A \ge B > 0$ does not imply

$$B^{-t} \natural_{\frac{\beta+t}{p+t}} A^p \le A^\beta$$

for $0 \le t \le p \le \beta$, in general. For example, we choose $t = 1,\ p = 2,\ \beta = 6$, clearly $\beta \not\le 2p$. Then the inequality

$$B^{-1} \natural_{\frac{7}{3}} A^2 \ge A^2B^2A^2$$

is assured by

$$B^{-1} \natural_{\frac{7}{3}} A^2 = A^2 \natural_{-\frac{4}{3}} B^{-1} = A^2(A^{-2} \natural_{\frac{4}{3}} B)A^2 = A^2(B \natural_{-\frac{1}{3}} A^{-2})A^2$$
$$= \quad A^2B(B^{-1} \sharp_{\frac{1}{3}} A^2)BA^2 \ge A^2B(A^{-1} \sharp_{\frac{1}{3}} A^2)BA^2 = A^2B^2A^2.$$

Now suppose that it holds for every pair $A \ge B > 0$, i.e., $B^{-1} \natural_{\frac{7}{3}} A^2 \le A^6$. Then we have $A^6 \ge A^2B^2A^2$, which says that $A \ge B > 0$ implies $A^2 \ge B^2$, a contradiction.

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