SPECTRAL PROPERTIES OF LOG-HYPONORMAL OPERATORS

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ABSTRACT. Let $T \in B(\mathcal{H})$ be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. T is said to be log-hyponormal if T is invertible and $\log(TT^*) \le \log(T^*T)$. In this paper we show that log-hyponormal operators have several spectral properties similar to p-hyponormal operators.

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1. Introduction. Recently, K. Tanahashi ([24,25]) studied log-hyponormal operators and showed that Putnam's inequality holds for log-hyponormal operators. In this paper, we show that log-hyponormal operators are isoloid and Weyl's theorem holds for such operators. Also we introduce the symbols of log-hyponormal operators and give an application.

Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. For an operator $T \in B(\mathcal{H})$, T is said to be log-hyponormal if T is invertible and satisfies $\log(TT^*) \le \log(T^*T)$. T is said to be p-hyponormal for $0 < p \le 1$ if $(TT^*)^p \le (T^*T)^p$. If $p = 1$, T is called a hyponormal operator, and if $p = \frac{1}{2}$, T is called a semi-hyponormal operator. The Löwner-Heinz inequality [18, 19] implies that if $0 \le B \le A$, then $B^\alpha \le A^\alpha$ for $0 < \alpha < 1$. Hence p-hyponormal operators are q-hyponormal for $0 < q \le p$. Let T be an invertible p-hyponormal operator. Then $(TT^*)^q \le (T^*T)^q$ for all $0 < q \le p$. Hence

$$\frac{(TT^*)^q - I}{q} \le \frac{(T^*T)^q - I}{q},$$

and, by letting $q \to +0$, we have

$$\log(TT^*) \le \log(T^*T).$$

Thus invertible p-hyponormal operators are log-hyponormal. Putnam's inequality of p-hyponormal operator T is the following :

$$\left\| \frac{(T^*T)^p - (TT^*)^p}{p} \right\| \le \frac{1}{\pi} \int_{\sigma(T)} r^{2p-1} dr d\theta.$$

Putnam [23] proved the case $p = 1$, Xia [26] proved the case $\frac{1}{2} \le p < 1$ and Chō, Itoh [8] proved the case $0 < p < \frac{1}{2}$. If T is invertible, then, by similar arguments as before, we have the following inequality.

$$\| \log(T^*T) - \log(TT^*) \| \le \frac{1}{\pi} \int_{\sigma(T)} r^{-1} dr d\theta.$$

Tanahashi [25] proved that this inequality holds for log-hyponormal operators.

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Thus we may consider that log-hyponormal operators are p-hyponormal operators with $p = 0$. Tanahashi [24] proved that there is a log-hyponormal operator which is not p-hyponormal for any $p > 0$. Hyponormal operators have been studied by many authors and it is known that hyponormal operators have many interesting properties similar to normal operators (cf. [22]). Semi-hyponormal operator was defined by Xia [27] and p-hyponormal operator for $0 < p < \frac{1}{2}$ was defined by Aluthge [1, 2]. f-hyponormal operator for an operator monotone function f was defined by Fujii, Himeji and Matsumoto [13]. Inspiring by the results due to Ando [3] and Chō and Itoh [8], Tanahashi [23, 24] studied log-hyponormal operator. (See [1, 2, 6, 7, 8, 9, 12, 13, 21, 27] for properties of p-hyponormal operators.)

Let $T = U|T|$ be the polar decomposition of a log-hyponormal T. Then the operator U is unitary since T is invertible. For an operator T we denote the spectrum, the approximate point spectrum, the kernel and the range of T by $\sigma(T)$, $\sigma_a(T)$, $N(T)$ and $R(T)$, respectively. A point z is in the set $\sigma_{na}(T)$ (the normal approximate point spectrum) if there exists a sequence $\{x_n\}$ of unit vectors such that $(T \Leftrightarrow z)x_n \to 0$ and $(T \Leftrightarrow z)^* x_n \to 0$ as $n \to \infty$. It is clear that if T is normal, then $\sigma(T) = \sigma_{na}(T)$.

2. Spectral properties. Aluthge [1] studied the Aluthge transformation $\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ of p-hyponormal operator T with a polar decomposition $T = U|T|$ where $0 < p \leq 1$ and U is unitary. He showed that the Aluthge transformation $\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ is hyponormal if $\frac{1}{2} \leq p \leq 1$ and $\left(p + \frac{1}{2}\right)$-hyponormal if $0 < p \leq \frac{1}{2}$. The following theorem on the Aluthge transformation of log-hyponormal operator was proved in [24], but we will give another proof.

Theorem 1 (Proposition 4 of [24]). Let $T = U|T|$ be log-hyponormal. For $0 < s, t$, let $S = |T|^s U |T|^t$ and $p = \dfrac{\min\{s,t\}}{s+t}$. Then

$$(SS^*)^p \ \leq \ |T|^{2p} \ \leq \ (S^*S)^p,$$

that is, S is p-hyponormal.

For the proof of this theorem, we need the following results.

Theorem A (the Furuta inequality [15]). Let $0 < p, q, r \in R$ and $A, B \in B(\mathcal{H})$ satisfy $0 \leq B \leq A$. If $p + 2r \leq (1+2r)q$ and $1 \leq q$, then $B^{\frac{p+2r}{q}} \leq (B^r A^p B^r)^{\frac{1}{q}}$ and $(A^r B^p A^r)^{\frac{1}{q}} \leq A^{\frac{p+2r}{q}}$.

Theorem B (Theorem of [14]). Let $A, B \in B(\mathcal{H})$ be invertible positive operators with $\log B \leq \log A$. Then for every $\delta \in (0,1)$ there exists $\alpha \in (0,1)$ such that $B^\alpha \leq (e^\delta A)^\alpha$.

Lemma 2. Let $T = U|T|$ be log-hyponormal. For $0 < t < 1$, let $S = |T|^t U |T|^{1-t}$ and $p = \min\{t, 1 \Leftrightarrow t\}$. Then S is p-hyponormal.

Proof. For the completeness, we give a proof. By Theorem B, for every $\delta \in (0,1)$ there exists $\alpha \in (0,1)$ such that $(e^\delta |T|)^\alpha \geq |T^*|^\alpha = U|T|^\alpha U^*$. Hence we have

$$e^{2\alpha\delta} U^* |T|^\alpha U \ \geq \ e^{\alpha\delta} |T|^\alpha \ \geq \ U |T|^\alpha U^*.$$

Let $A = e^{2\alpha\delta}U^*|T|^\alpha U$, $B = e^{\alpha\delta}|T|^\alpha$ and $C = U|T|^\alpha U^*$. Since $\frac{2t}{\alpha} + 2\frac{1 \Leftrightarrow t}{\alpha} = \frac{2}{\alpha}$ and $(1 + 2 \cdot \frac{1 \Leftrightarrow t}{\alpha})\frac{1}{p} = \frac{1}{p} + \frac{1 \Leftrightarrow t}{p} \cdot \frac{2}{\alpha}$, from $0 < p \leq 1 \Leftrightarrow t$, we have

$$\frac{2t}{\alpha} + 2 \cdot \frac{1 \Leftrightarrow t}{\alpha} \leq (1 + 2 \cdot \frac{1 \Leftrightarrow t}{\alpha})\frac{1}{p}.$$

By Furuta inequality,

$$(1) \qquad (S^*S)^p = (e^{-\delta(1-t)}B^{\frac{1-t}{\alpha}}e^{-4\delta t}A^{\frac{2t}{\alpha}}e^{-\delta(1-t)}B^{\frac{1-t}{\alpha}})^p$$

$$(2) \qquad = e^{-2\delta(1-t)p}(B^{\frac{1-t}{\alpha}}A^{\frac{2t}{\alpha}}B^{\frac{1-t}{\alpha}})^p \geq e^{-2\delta(1-t)p}B^{\frac{2p}{\alpha}}.$$

Similarly, we have, by Furuta inequality, $(SS^*)^p \leq e^{-2\delta tp}B^{\frac{2p}{\alpha}}$. Hence we have $(S^*S)^p \geq e^{2\delta tp}|T|^{2p}$ and $(SS^*)^p \leq e^{2\delta tp}|T|^{2p}$. Since $\delta \in (0,1)$ is arbitrary, we have $(S^*S)^p \geq |T|^{2p} \geq (SS^*)^p$. That is, S is p-hyponormal.

Proof of Theorem 1. Let $r = s+t > 0$. Then $U|T|^r$ is log-hyponormal and $(|T|^r)^{\frac{s}{r}}U(|T|^r)^{1-\frac{s}{r}} = |T|^sU|T|^{r-s} = |T|^sU|T|^t$. Since $p = \min\{\frac{s}{s+t}, \frac{t}{s+t}\} = \min\{\frac{s}{r}, 1 \Leftrightarrow \frac{s}{r}\}$, we have that $|T|^sU|T|^t$ is p-hyponormal by Lemma 2.

Theorem 3. Let $T = U|T|$ be log-hyponormal. For real numbers s and t, let $S = |T|^sU|T|^t$. Then

$$\sigma(S) = \{ r^{s+t}e^{i\theta} : re^{i\theta} \in \sigma(T)\}.$$

Proof. Since $|T|^s$ is normal, by Lemma 3 of [10] it holds that $\sigma(S) = \sigma(|T|^sU|T|^t) = \sigma(U|T|^{s+t})$. Hence we only have to prove that $\sigma(U|T|^s) = \{ r^se^{i\theta} : re^{i\theta} \in \sigma(T)\}$ for any real number s.

First we show the case $s > 0$. Let $0 \leq t \leq 1$, $T(t) = U|T|^{1-t+st}$ and $\tau_t(re^{i\theta}) = r^{1-t+st}e^{i\theta}$ for $re^{i\theta} \in C$. Then $T(t)$ is log-hyponormal, $T(0) = T$ and $T(1) = S$. We can prove that $\sigma_{na}(U|T|^{1-t+st}) = \{r^{1-t+st}e^{i\theta} \ |re^{i\theta} \in \sigma_{na}(T)\}$ by similar arguments in Lemma 4 of [25]. Since $\sigma_a(T(t)) = \sigma_{na}(T(t))$ by Lemma 3 of [25], we have that $\sigma(T(t)) = \tau_t(\sigma(T(0)))$ by Lemma I.3.1 of [26]. Hence $\sigma(U|T|^s) = \{ r^se^{i\theta} : re^{i\theta} \in \sigma(T)\}$.

Let $s = 0$. Then we have to prove that $\sigma(U) = \{ e^{i\theta} : re^{i\theta} \in \sigma(T)\}$. By taking cT ($0 < c$) insead of T, we may assume that $0 < \log|T|$. Let $S = U\log|T|$. Then S is semi-hyponormal and $|S| = \log|T|$. Since $\sigma(U) = \{e^{i\theta} \mid re^{i\theta} \in \sigma_a(S)\}$ by Theorem 2.3.3 of [26] and $\sigma_a(S) = \{(\log\rho)e^{i\theta} \mid \rho e^{i\theta} \in \sigma_a(T)\}$ by Lemma 6 of [25], we have that $\sigma(U) = \{e^{i\theta} \mid re^{i\theta} \in \sigma_a(T)\}$. Since the boundary of $\sigma(T)$ is contained in $\sigma_a(T)$, this implies that $\sigma(U) = \{e^{i\theta} \mid re^{i\theta} \in \sigma(T)\}$.

Next let $s < 0$. Since $\sigma(U|T|^s) = \sigma(|T|^sU) = \sigma((U^*|T|^{-s})^{-1})$ and $\sigma(U^*|T|^{-s}) = \sigma(|T|^{-s}U^*) = \sigma((U|T|^{-s})^*)$, we have $\sigma(U|T|^s) = \{ r^se^{i\theta} : re^{i\theta} \in \sigma(T)\}$. So the proof is complete.

Theorem 4. Let $T = U|T|$ be log-hyponormal. For $s, t > 0$, let $S = |T|^sU|T|^t$. Then $Tx = re^{i\theta}x$ if and only if $Sx = r^{s+t}e^{i\theta}x$.

Proof. Let $Tx = re^{i\theta}x$. Then, by Theorem 11 of [24], it holds that

$$Ux = e^{i\theta}x \text{ and } |T|x = rx.$$

Hence it is clear that $Sx = r^{s+t}e^{i\theta}x$. Next let $Sx = r^{s+t}e^{i\theta}x$. Then $U|T|^t x = r^{s+t}e^{i\theta}|T|^{-s}x$ and hence it holds that $U|T|^{s+t}(|T|^{-s}x) = r^{s+t}e^{i\theta}(|T|^{-s}x)$. Since $U|T|^{s+t}$ is log-hyponormal, also by Theorem 11 of [24] we have

$$U|T|^{-s}x = e^{i\theta}|T|^{-s}x \text{ and } |T|^{t+s}|T|^{-s}x = r|T|^{-s}x.$$

Therefore, it follows $|T|x = rx$ and $Ux = e^{i\theta}x$.
So the proof is complete.

Theorem 5. Let T be log-hyponormal. If z is an isolated point of $\sigma(T)$, then z is an eigen-value of T. That is, T is isoloid.

Proof. Let $S = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. Since $\sigma(T) = \sigma(S)$, z is an isolated point of $\sigma(S)$. Since, by Theorem 1, S is semi-hyponormal, z is an eigen-value of S. Hence we have $Sx = zx$ for some vector x. By Theorem 4, we have $Tx = zx$. Hence z is an eigen-value of T.

Next we show that Weyl's theorem holds for log-hyponormal operators. For an operator T, let $\pi_{00}(T)$ denote the set of all isolated eigenvalues of finite multiplicity of T and let $\omega(T)$ denote the Weyl spectrum of T, i.e., $\omega(T) = \cap\{\sigma(T+K) : K \text{ is compact}\}$. Baxley [4] introduced the following two conditions:

C-1: If $\{z_n\}$ is an infinite sequence of distinct points of the set of all eigenvalues of finite multiplicity of T and $\{x_n\}$ is any sequence of corresponding normalized eigen-vectors, then the sequence $\{x_n\}$ does not converge.

C-2: If $z \in \pi_{00}(T)$, then $T \Leftrightarrow zI$ has closed range and $\dim N(T \Leftrightarrow zI) = \dim(R(T \Leftrightarrow zI)^{\perp})$.

Baxley [4] proved that if T satisfies the conditions C-1 and C-2, then

$$\omega(T) \;=\; \sigma(T) \;\Leftrightarrow\; \pi_{00}(T),$$

that is, Weyl's theorem holds for T.

Theorem 6. Let T be log-hyponormal. Then T satisfies the conditions C-1 and C-2.

Proof. From Theorem 11 of [18] it holds that if $Tx = zx$, then $T^*x = \bar{z}x$. It follows that if $Tx = ax$ and $Ty = by$ $(a \neq b)$, then $(x, y) = 0$. Therefore, it is clear that T satisfies C-1. Let $z \in \pi_{00}(T)$. Then $N(T \Leftrightarrow zI)$ is the reducing subspace of U and $|T|$. Hence we decompose

$$T = T_1 \bigoplus T_2 \;\text{ on }\; N(T \Leftrightarrow zI) \bigoplus N(T \Leftrightarrow zI)^{\perp}.$$

Assume that $z \in \sigma(T_2)$. Since $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$, z is an isolated point of $\sigma(T_2)$. By Theorem 4, we have a contradiction. Hence $z \notin \sigma(T_2)$ and $R(T_2 \Leftrightarrow zI) = N(T \Leftrightarrow zI)^{\perp}$. Since $R(T \Leftrightarrow zI) = R(T_2 \Leftrightarrow zI)$, $R(T \Leftrightarrow zI)$ is closed and $\dim N(T \Leftrightarrow zI) = \dim(R(T \Leftrightarrow zI)^{\perp})$. So the proof is complete.

Hence we have the following result:

Theorem 7. Let T be log-hyponormal. Then Weyl's theorem holds for T, that is,

$$\omega(T) \;=\; \sigma(T) \;\Leftrightarrow\; \pi_{00}(T).$$

Theorem 8. Let $T = U|T|$ be log-hyponormal. And, for $s, t > 0$, let $S = |T|^s U|T|^t$. Then

$$\omega(S) \ = \ \{ \ r^{s+t}e^{i\theta} \ : \ re^{i\theta} \in \omega(T) \ \}.$$

Proof. By Theorem 7, Weyl's theorem holds for T and by Theorem 0 of [9]. Also, Weyl's theorem holds for S because S is p-hyponormal, where $p = \dfrac{\min\{s,t\}}{s+t}$. By Theorem 3

$$\sigma(S) = \sigma(U|T|^{s+t}) = \{ \ r^{s+t}e^{i\theta} \ : \ re^{i\theta} \in \sigma(T)\}.$$

Since by Theorem 4 it holds that $\pi_{00}(S) = \{ \ r^{s+t}e^{i\theta} \ : \ re^{i\theta} \in \pi_{00}(T)\}$, it follows that

$$\omega(S) \ = \ \{ \ r^{s+t}e^{i\theta} \ : \ re^{i\theta} \in \omega(T) \ \}.$$

Also we have the following spectral mapping theorem of Weyl spectrum. Proof is similar to the proof of [12]. But for the completeness we will give a proof.

Theorem 9. Let T be log-hyponormal and f be a holomorphic function defined in a neighborhood of $\sigma(T)$. Then $f(\omega(T)) = \ \omega(f(T)) = \sigma(f(T)) \Leftrightarrow \pi_{00}(f(T))$.

Proof. Let $\mathcal{F}_0$ be the set of all Fredholm operators of index zero. Let $p(z)$ be an arbitrary polynomial. Let $\lambda \in C$ and

$$\lambda \Leftrightarrow p(z) = a_0 \prod_{j=1}^{n} (\lambda_j \Leftrightarrow z).$$

Since T is log-hyponormal, it holds $\mathrm{ind}(T \Leftrightarrow z) \le 0$ as $N(T \Leftrightarrow z) \subset N((T \Leftrightarrow z)^*)$ by Lemma 3 of [24]. Moreover the commutativity of factors in polynomials, Corollary 1.3.4 of [5] and Theorem 3.2.7 of [5],

$$\lambda \notin \omega(p(T)) \iff \lambda \Leftrightarrow p(T) = a_0 \prod_{j=1}^{n} (\lambda_j \Leftrightarrow T) \ \in \mathcal{F}_0 \iff \lambda_j \Leftrightarrow T \ \in \mathcal{F}_0 \ \text{ for all } j = 1, ..., n$$

$$\iff \lambda_j \notin \omega(T) \ \text{ for all } j = 1, ..., n \iff \lambda \notin p(\omega(T)).$$

Hence

$$\omega(p(T)) = p(\omega(T)).$$

Therefore, it holds that

$$f(\omega(T)) \ = \ \omega(f(T))$$

for a holomorphic function defined in a neighborhood of $\sigma(T)$ by Theorem 2 of [20]. Since T is isoloid by Theorem 5, we have

$$f(\sigma(T) \Leftrightarrow \pi_{00}(T)) \ = \ \sigma(f(T)) \Leftrightarrow \pi_{00}(f(T)).$$

by Lemma of [17]. Hence the proof is complete.

3. Polar symbols of log-hyponormal operators. First for a pair of operators (T, S), a point $(z, w) \in C^2$ is in the joint approximate point spectrum $\sigma_\pi(T, S)$ if there exists a sequence $\{x_m\}$ such that

$$(T \Leftrightarrow z)x_m \ \to \ 0 \text{ and } (S \Leftrightarrow w)x_m \ \to \ 0 \text{ as } m \ \to \ \infty.$$

For a unitary operator U we define operators $\mathcal{S}_U^{\pm}(T)$ by

$$\mathcal{S}_U^{\pm}(T) = \text{s - }\lim_{n \to \pm\infty} U^{-n} T U^{n}.$$

If $T = U|T|$ is p-hyponormal with unitary U, then $\mathcal{S}_U^{\pm}(|T|^{2p})$ exist (cf. [26]). The operators $\mathcal{S}_U^{\pm}(|T|^{2p})$ are called the polar symbols of $|T|$. Let $T_{(k)} = U\{k\mathcal{S}_U^{+}(|T|^{2p}) + (1 \Leftrightarrow k)\mathcal{S}_U^{-}(|T|^{2p})\}^{\frac{1}{2p}}$ for every $0 \le k \le 1$. Then it holds that $\sigma(T) = \bigcup_{0 \le k \le 1} \sigma(T_{(k)})$ (cf. [7]). Next let $T = U|T|$ be log-hyponormal. We can find out $c > 0$ such that $\log|cT| \ge 0$. Since then $S = U\log|cT|$ is semi-hyponormal, there exist the polar symbols $\mathcal{S}_U^{\pm}(|cT|)$. And it holds that $\mathcal{S}_U^{\pm}(\log|cT|) = \log c + \mathcal{S}_U^{\pm}(\log|T|)$. For $0 \le k \le 1$, the operator T_k is defined by

$$T_k \;=\; U\exp\{k\mathcal{S}_U^{+}(\log|T|) \;+\; (1 \Leftrightarrow k)\mathcal{S}_U^{-}(\log|T|)\}.$$

The operator T_k is called the generalized polar symbol of T. Then we have the following

Theorem 10. Let $T = U|T|$ be log-hyponormal. Then

$$\sigma(T) \;=\; \bigcup_{0 \le k \le 1} \sigma(T_k).$$

Proof. Let $\log|T| \ge 0$ and let $S = U\log|T|$. Since then S is semi-hyponormal, by Theorem 4.4.1 of [26] it holds that

$$\sigma(S) \;=\; \bigcup_{0 \le k \le 1} \sigma(S_{(k)}),$$

where $S_{(k)} = U\{k\mathcal{S}_U^{+}(|S|) + (1 \Leftrightarrow k)\mathcal{S}_U^{-}(|S|)\} = U\{k\mathcal{S}_U^{+}(\log|T|) + (1 \Leftrightarrow k)\mathcal{S}_U^{-}(\log|T|)\}$. Let $re^{i\theta} \in C$ with $r \ne 0$. Since then, for every $k \in [0,1]$, $S_{(k)}$ is normal,

$$re^{i\theta} \in \sigma(T) \;\Leftrightarrow\; (\log r)e^{i\theta} \in \sigma(S) \;\text{ by Lemma 6 of [25]}$$

$$\Leftrightarrow\; \exists k \in [0,1]\;;\; (\log r)e^{i\theta} \in \sigma(S_{(k)}) \;=\; \sigma_{na}(S_{(k)}) \;\Leftrightarrow\; \exists k \in [0,1]\;;\; (e^{i\theta}, \log r) \in \sigma_{\pi}(U, |S_{(k)}|)$$

$$\Leftrightarrow\; \exists k \in [0,1]\;;\; (e^{i\theta}, r) \in \sigma_{\pi}(U, |T_k|) \;\Leftrightarrow\; \exists k \in [0,1]\;;\; re^{i\theta} \in \sigma(T_k).$$

Since T and T_k are invertible for every k $(0 \le k \le 1)$, we have

$$\sigma(T) = \bigcup_{0 \le k \le 1} \sigma(T_k).$$

If $\log|T| \le 0$, then we choose $c > 0$ such that $\log|cT| \ge 0$. Then

$$(cT)_k = \exp\{k\mathcal{S}_U^{+}(\log|cT|) + (1 \Leftrightarrow k)\mathcal{S}_U^{-}(\log|cT|)\}$$

$$= c \cdot \exp\{k\mathcal{S}_U^{+}(\log|T|) + (1 \Leftrightarrow k)\mathcal{S}_U^{-}(\log|T|)\} \;=\; c \cdot T_k.$$

Hence the proof is complete.

Finally, we give an application of the theorem above. Let $T = U|T|$ be log-hyponormal. For $0 \le k \le 1$, let $R_k = \exp\{k\mathcal{S}_U^{+}(\log|T|) + (1 \Leftrightarrow k)\mathcal{S}_U^{-}(\log|T|)\}$. Then U and R_k commute. Hence $\sigma_{\pi}(U, R_k)$ is non-empty for every k $(0 \le k \le 1)$. The Xia spectrum $\sigma_X(U, |T|)$ of a log-hyponormal operator $T = U|T|$ is defined by

$$\sigma_X(U, |T|) \;=\; \bigcup_{0 \le k \le 1} \sigma_{\pi}(U, R_k).$$

Then we have the following theorem.

Theorem 11. Let $T = U|T|$ be log-hyponormal. Then

$$re^{i\theta} \in \sigma(T) \text{ if and only if } (e^{i\theta}, r) \in \sigma_X(U, |T|).$$

Proof. By Theorem 10 we have $\sigma(T) = \bigcup_{0 \le k \le 1} \sigma(T_k)$. Since $T_k = UR_k$ is a normal operator, by the spectral mapping theorem of the joint approximate point spectrum (cf. [6],[11],[25]) we have

$$re^{i\theta} \in \sigma(UR_k) \text{ if and only if } (e^{i\theta}, r) \in \sigma_\pi(U, R_k).$$

Hence we have that $re^{i\theta} \in \sigma(T)$ if and only if $(e^{i\theta}, r) \in \sigma_X(U, |T|)$. So the proof is complete.

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