

SIMULTANEOUS UNITARIZABILITY AND SIMILARITY PROBLEM

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ABSTRACT. Let A be a unital C^* -algebra for which the similarity problem can be solved. If α is an action of a discrete amenable group G on A , then the similarity problem can be also solved for the C^* -crossed product $A \rtimes_{\alpha} G$. We apply this idea to determine the simultaneous unitarizability for bounded linear operators.

1. INTRODUCTION

Let x be a bounded linear operator on a Hilbert space \mathcal{H} . We call x unitarizable if there exists an invertible operator $S \in B(\mathcal{H})$ such that SxS^{-1} is unitary, where $B(\mathcal{H})$ is the set of all bounded linear operators on \mathcal{H} . The following fact is well-known:

$$x \in B(\mathcal{H}) \text{ is unitarizable if and only if } \sup\{\|x^n\| \mid n \in \mathbb{Z}\} < \infty,$$

as the result of B. Sz.-Nagy [8]. This result was extended by J. Dixmier [5] as the following form:

Proposition 1. *Let $\varphi : G \rightarrow B(\mathcal{H})$ be a continuous representation ($g \mapsto \varphi(g)\xi$ continuous for any $\xi \in \mathcal{H}$) of a locally compact, amenable group G . If φ is uniformly bounded, i.e.*

$$\sup_{g \in G} \|\varphi(g)\| < \infty,$$

then there exists an invertible operator $S \in B(\mathcal{H})$ such that

$$\pi : g \in G \longrightarrow S\varphi(g)S^{-1} \in B(\mathcal{H})$$

is a unitary representation of G on \mathcal{H} .

With related to the above statement, we treat the problem: does there exist, for a family \mathcal{F} of unitarizable operators, an invertible operator $S \in B(\mathcal{H})$ such that SxS^{-1} is unitary for all $x \in \mathcal{F}$. If there exists such an operator S , we call \mathcal{F} simultaneously unitarizable.

The above Dixmier's result for an amenable group is a powerful tool to determine the simultaneous unitarizability for a family of unitarizable operators. In this paper, we extend this idea to unitarizability for some C^* -crossed products by amenable groups.

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2. SIMULTANEOUS UNITARIZABILITY

For a unital C^* -algebra A , we denote $\mathcal{U}(A)$ by all unitary elements in A . The first half part of the proof for the following statement is essentially the same as the proof of Dixmier's result ([4], [7]).

Theorem 2. *Let $\psi : A \rightarrow B(\mathcal{H})$ be a bounded representation of a unital C^* -algebra A and let $\varphi : G \rightarrow B(\mathcal{H})$ be a uniformly bounded, continuous representation of a locally compact, amenable group G such that*

$$\varphi(g)\psi(\mathcal{U}(A))\varphi(g^{-1}) \subset \mathcal{U}(B(\mathcal{H})) \quad \text{for all } g \in G.$$

Then there exists an invertible operator $S \in B(\mathcal{H})$ such that $S\psi(\cdot)S^{-1}$ is a $$ -representation and $S\varphi(\cdot)S^{-1}$ is a unitary representation.*

Proof. For any $\xi, \eta \in \mathcal{H}$, we define a bounded continuous function $f_{\xi, \eta}$ on G as follows:

$$f_{\xi, \eta}(g) = (\varphi(g^{-1})\xi \mid \varphi(g^{-1})\eta),$$

where $(\cdot \mid \cdot)$ means the usual inner product on \mathcal{H} . So we can define a new inner product on \mathcal{H} by

$$\langle \xi \mid \eta \rangle = m(f_{\xi, \eta}),$$

where m is an invariant mean on G . Then we have, for any $\xi \in \mathcal{H}$,

$$\frac{1}{M^2}(\xi \mid \xi) \leq \langle \xi \mid \xi \rangle \leq M^2(\xi \mid \xi),$$

where $M = \sup_{g \in G} \|\varphi(g)\|$. Since there exists a positive invertible operator T such that

$$\langle \xi \mid \eta \rangle = (T\xi \mid \eta) \quad \text{for any } \xi, \eta \in \mathcal{H},$$

we put $S = T^{1/2}$. Then we have

$$\begin{aligned} (S\varphi(g)S^{-1}\xi \mid S\varphi(g)S^{-1}\eta) &= \langle \varphi(g)S^{-1}\xi \mid \varphi(g)S^{-1}\eta \rangle \\ &= \langle S^{-1}\xi \mid S^{-1}\eta \rangle = (\xi \mid \eta), \end{aligned}$$

this means that $S\varphi(g)S^{-1}$ becomes unitary for any $g \in G$.

By the assumption, for a unitary $u \in A$, we can find a unitary $U_g \in B(\mathcal{H})$ such that

$$\varphi(g^{-1})\psi(u) = U_g\varphi(g^{-1})$$

for each $g \in G$. Then the fact

$$\begin{aligned} f_{\psi(u)\xi, \psi(u)\eta}(g) &= (\varphi(g^{-1})\psi(u)\xi \mid \varphi(g^{-1})\psi(u)\eta) \\ &= (U_g\varphi(g^{-1})\xi \mid U_g\varphi(g^{-1})\eta) \\ &= f_{\xi, \eta}(g) \end{aligned}$$

implies that

$$S\psi(\mathcal{U}(A))S^{-1} \subset \mathcal{U}(B(\mathcal{H})).$$

Clearly $S\psi(\cdot)S^{-1}$ is a bounded representation of A . It is sufficient to prove that

$$S\psi(x^*)S^{-1} = (S\psi(x)S^{-1})^* \quad \text{for any } x \in A.$$

There exist a non-negative scalar α and unitaries u, v in A such that

$$x = \alpha(u + u^{-1} + iv + iv^{-1}).$$

Then we have

$$\begin{aligned} S\psi(x^*)S^{-1} &= \alpha(S\psi(u^{-1})S^{-1} + S\psi(u)S^{-1} - iS\psi(v^{-1})S^{-1} - iS\psi(v)S^{-1}) \\ &= \alpha(S\psi(u)S^{-1} + S\psi(u^{-1})S^{-1} + iS\psi(v)S^{-1} + iS\psi(v^{-1})S^{-1})^* \\ &= (S\psi(x)S^{-1})^* \end{aligned}$$

Therefore $S\psi(\cdot)S^{-1}$ is a *-representation of A . \square

Remark. In the above proof, we can see that the operator S satisfies

$$\|S\|\|S^{-1}\| \leq \sup_{g \in G} \|\varphi(g)\|^2$$

from its construction.

The fact that a unitary preserving homomorphism is also *-preserving has stated in [9:Lemma 8.6].

As stated in introduction, if unitarizable operators satisfies some relations, then Proposition 1 works very well to determine their simultaneous unitarizability in the sense of the following statement, where we consider the case that each G_i is \mathbb{Z} .

Lemma 3. *Let $G = ((G_1 \rtimes G_2) \rtimes \cdots) \rtimes G_n$ and $\varphi : G \rightarrow B(\mathcal{H})$ a continuous representation of G . If the restriction of φ on G_i is uniformly bounded for each i , then φ is uniformly bounded.*

In particular, if each G_i is amenable, then φ is similar to a continuous unitary representation.

Proof. By the definition of semi-direct product, any element $g \in G$ is uniquely represented by

$$g = g_1 g_2 \cdots g_n,$$

where $g_i \in G_i$ ($i = 1, 2, \dots, n$). This implies

$$\sup_{g \in G} \|\varphi(g)\| \leq \sup_{g_1 \in G_1} \|\varphi(g_1)\| \sup_{g_2 \in G_2} \|\varphi(g_2)\| \cdots \sup_{g_n \in G_n} \|\varphi(g_n)\|,$$

that is, φ is uniformly bounded.

If H, K is amenable groups, then $H \rtimes K$ is also amenable. So the last part follows from Proposition 1. \square

Applying Theorem 2 for unitarizable operators, we have the following result:

Proposition 4. Let a_1, a_2, \dots, a_n be unitarizable elements in $B(\mathcal{H})$ satisfying

$$a_i a_j = a_j a_i \quad (i, j = 1, 2, \dots, n)$$

and let A be the closed subalgebra of $B(\mathcal{H})$ generated by $\{a_i, a_i^{-1} \mid i = 1, 2, \dots, n\}$. If $\varphi : G \rightarrow B(\mathcal{H})$ be a uniformly bounded, continuous representation of a locally compact, amenable group G with

$$\varphi(g)A\varphi(g)^{-1} \subset A \quad \text{for all } g \in G,$$

then we have $\{a_i, \varphi(g) \mid i = 1, 2, \dots, n, g \in G\}$ is simultaneously unitarizable.

Proof. By the amenability of \mathbb{Z} and Lemma 3, there exists an invertible operator $S \in B(\mathcal{H})$ such that $Sa_i S^{-1}$ is unitary for $i = 1, 2, \dots, n$. Thus SAS^{-1} is commutative unital C^* -algebra. For a unitary u in SAS^{-1} , we have

$$S\varphi(g)S^{-1}uS\varphi(g^{-1})S^{-1} \in S\varphi(g)A\varphi(g^{-1})S^{-1} \subset SAS^{-1}.$$

This means that

$$Sp(S\varphi(g)S^{-1}uS\varphi(g^{-1})S^{-1}) = Sp(u) \subset \{z \in \mathbb{C} \mid |z| = 1\}$$

and $S\varphi(g)S^{-1}uS\varphi(g^{-1})S^{-1}$ is normal, that is,

$$S\varphi(g)S^{-1}\mathcal{U}(SAS^{-1})S\varphi(g^{-1})S^{-1} \subset \mathcal{U}(SAS^{-1}) \quad \text{for all } g \in G.$$

Applying Theorem 2 to the canonical embedding of SAS^{-1} and the uniformly bounded representation $S\varphi(\cdot)S^{-1}$ of G , we can get the desired result. \square

As an application of Proposition 4, we can give some examples of simultaneous unitarizable operators.

Examples 1. Let a, b be unitarizable. If the commutator $aba^{-1}b^{-1}$ of a, b is commuting with a, b , then $\{a, b\}$ is simultaneously unitarizable.

In fact, if we put $c = aba^{-1}b^{-1}$, then we have $c^n = ab^n a^{-1} b^{-n}$. Since c is uniformly bounded, $\{a, c\}$ becomes a commuting pair of unitarizable operators and

$$bab^{-1} = c^{-1}a.$$

This implies that a, b, c are simultaneously unitarizable.

Examples 2. Let a, b be unitarizable and let $f : Sp(a) \rightarrow Sp(a)$ be a homeomorphism satisfying the relation

$$bab^{-1} = f(a),$$

where $Sp(a)$ means the spectrum of a . Then we have $\{a, b\}$ is simultaneously unitarizable.

In the case that $Sp(a)$ is infinite and, for any closed subset $F (\neq \emptyset)$ of $Sp(a)$, $f(F) \subset F$ implies $F = Sp(a)$, then it is known that the crossed product C^* -algebra $C(Sp(a)) \rtimes_f \mathbb{Z}$ is simple. It is also known that a closed two sided ideal of a C^* -algebra is automatically $*$ -invariant. So we have that the closed subalgebra A generated by a, a^{-1}, b, b^{-1} is similar to the simple C^* -algebra $C(Sp(a)) \rtimes_f \mathbb{Z}$ and is simple as a Banach algebra. In particular, a, b is unitarizable and

$$aba^{-1}b^{-1} = e^{2\pi i\theta}, \quad \theta \in \mathbb{R} \setminus \mathbb{Q},$$

then the algebra A is similar to the irrational rotation C^* -algebra A_θ , so A is uniquely determined depending on only θ as a Banach algebra.

3. SIMILARITY PROBLEM FOR C^* -ALGEBRAS

Let A be a C^* -algebra and $\varphi : A \rightarrow B(\mathcal{H})$ be a bounded representation of A on \mathcal{H} . If there exists an invertible operator $S \in B(\mathcal{H})$ such that

$$S\varphi(\cdot)S^{-1} \text{ is a } *\text{-representation of } A \text{ on } \mathcal{H},$$

then it is said that φ is similar to a $*$ -representation. If any bounded representation of A is similar to a $*$ -representation, then it is said that the similarity problem can be solved for A (see [9],[10]). J. W. Bunce [1] and E. Christensen [2] show that the similarity problem can be solved for any nuclear C^* -algebra. E. Christensen [3] also shows that every II_1 -factor with property Γ can be solved the similarity problem. U. Haagerup [6] shows that every completely bounded homomorphism φ from A to $B(\mathcal{H})$ is similar to a $*$ -homomorphism and

$$\begin{aligned} \|\varphi\|_{cb} &= \|id \otimes \varphi : B(\ell^2) \otimes A \rightarrow B(\ell^2) \otimes B(\mathcal{H})\| \\ &= \inf\{\|S\|\|S^{-1}\| : S\varphi(\cdot)S^{-1} \text{ is a } *\text{-representation}\}. \end{aligned}$$

In the following theorem, we prove that the similarity problem can be solved for some C^* -crossed product by an amenable group, which is not necessarily nuclear.

Theorem 5. *Let A be a unital C^* -algebra and G an amenable discrete group. If the similarity problem can be solved for A , then the similarity problem can be solved for the C^* -crossed product $A \rtimes_{\alpha} G$ of A by G .*

Moreover, for every homomorphism φ of $A \rtimes_{\alpha} G$, we have

$$\|\varphi\|_{cb} \leq \max\{\|\varphi|_{C_{red}^*(G)}\|_{cb}, \|\varphi|_A\|_{cb}\}^5,$$

where we canonically regard $C_{red}^*(G)$ and A as subalgebras of $A \rtimes_{\alpha} G$.

Proof. Let φ be a non-degenerate bounded representation of $A \rtimes_{\alpha} G$ on \mathcal{H} . We can regard $A \rtimes_{\alpha} G$ as the C^* -algebra generated by $\{x, u_g \mid x \in A, g \in G\}$ satisfying the relation $u_g^* = u_g^{-1}$, $u_g u_h = u_{gh}$, $u_e = 1$ and

$$u_g x u_g^{-1} = \alpha_g(x),$$

where α is the action of G on A . For every $\varepsilon > 0$ there exists an invertible operator S_1 such that $S_1\varphi(\cdot)S_1^{-1}$ is a $*$ -representation of A and

$$\|S_1\|\|S_1^{-1}\| < \|\varphi|_A\|_{cb} + \varepsilon.$$

We set $\varphi_1(\cdot) = S_1\varphi(\cdot)S_1^{-1}$. For every unitary element u of A , we have

$$\varphi_1(u_g)\varphi_1(u)\varphi_1(u_g)^{-1} = \varphi_1(u_g u u_g^{-1}) = \varphi_1(\alpha_g(u)) \in \mathcal{U}(B(\mathcal{H})).$$

Using Theorem 2, there exists an invertible operator $S_2 \in B(\mathcal{H})$ such that $S_2\varphi_1(\cdot)S_2^{-1}$ is a $*$ -representation,

$$\|S_2\|\|S_2^{-1}\| \leq \sup_{g \in G} \|\varphi_1(g)\|^2$$

and

$$g \in G \rightarrow S_2\varphi_1(u_g)S_2^{-1} \in B(\mathcal{H})$$

is a unitary representation.

If we set $\varphi_2(\cdot) = S_2\varphi_1(\cdot)S_2^{-1}$, then $\varphi_2(a)$ and $\varphi_2(u_g)$ ($a \in A$, $g \in G$) clearly satisfy the covariant relation. So we have φ is similar to this *-representation φ_2 of $A \rtimes_\alpha G$ on \mathcal{H} .

The rest part is as follows:

$$\begin{aligned} \|\varphi\|_{cb} &\leq \|S_1\| \|S_1^{-1}\| \|S_2\| \|S_2^{-1}\| \\ &\leq \|S_1\| \|S_1^{-1}\| \sup_{g \in G} \|\varphi_1(g)\|^2 \\ &\leq \sup_{g \in G} \|\varphi(g)\|^2 \|S_1\|^3 \|S_1^{-1}\|^3 \\ &\leq \|\varphi|_{C_{red}^*(G)}\|_{cb}^2 (\|\varphi|_A\|_{cb} + \varepsilon)^3. \end{aligned}$$

Since ε is arbitrary, we have

$$\|\varphi\|_{cb} \leq \max\{\|\varphi|_{C_{red}^*(G)}\|_{cb}, \|\varphi|_A\|_{cb}\}^5.$$

□

G. Pisier [11] introduced a notion of length for a C*-algebra generated by two subalgebras. Let A be a unital C*-algebra and let A_1, A_2 be two unital subalgebras of A , which algebraically generate a dense *-subalgebra \mathcal{A} of A . We say that (A_1, A_2) generate A with length $\leq d$ if there exists a constant K such that for any $\varepsilon > 0$ and any $x \in \mathcal{K} \otimes_{min} A$ there are d -tuples $(i_1, i_2, \dots, i_d) \in \{1, 2\}^d$ and $(k_1, k_2, \dots, k_d) \in \{1, 2\}^d$ and elements $x_j \in \mathcal{K} \otimes_{min} A_{i_j}$ and $y_j \in \mathcal{K} \otimes_{min} A_{k_j}$ such that

$$\|x - x_1 x_2 \dots x_d - y_1 y_2 \dots y_d\|_{\mathcal{K} \otimes_{min} A} < \varepsilon$$

and

$$\prod_{j=1}^d \|x_j\|_{\mathcal{K} \otimes_{min} A} + \prod_{j=1}^d \|y_j\|_{\mathcal{K} \otimes_{min} A} \leq K \|x\|_{\mathcal{K} \otimes_{min} A},$$

where \mathcal{K} is the set of all compact operators and \otimes_{min} means the spatial tensor product.

In [11: Theorem 6], he get the following result: For any homomorphism φ of \mathcal{A} which is algebraically generated by subalgebras A_1 and A_2 , if the completely boundedness of $\varphi|_{A_1}$ and $\varphi|_{A_2}$ implies that of φ , then there is an integer d such that (A_1, A_2) generates A with length d . Moreover, the smallest possible d coincides with the smallest real number $\alpha \geq 1$ for which there is a constant K such that any homomorphism $\varphi : A \rightarrow B(\mathcal{H})$ satisfies

$$\|\varphi\|_{cb} \leq K (\max\{\|\varphi|_{A_1}\|_{cb}, \|\varphi|_{A_2}\|_{cb}\})^\alpha.$$

In the proof of Theorem 5, we construct a *-representation of $A \rtimes_\alpha G$ from *-representations of A and $C_{red}^*(G)$ using the covariant relation of A and G . So, by these fact, we can immediately get the following:

Corollary 6. *Let the similarity problem can be solved for a unital C*-algebra A and let α be an action of discrete amenable group G on A . Then $(A, C_{red}^*(G))$ generate $A \rtimes_\alpha G$ with length ≤ 5 .*

Added in Proof. Professor G. Pisier remarks us that we do not use the assumption,

the similarity problem can be solved for a unital C*-algebra A ,

in the argument of Corollary 6. So we can omit this assumption in Corollary 6.

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