Hyperidentities for Varieties of Star Bands

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Abstract. Hyperidentities have been studied both in general and for specific varieties of algebras. In the latter case, attention has focussed on varieties of type (2), varieties of semigroups, and recently on inverse semigroup varieties of type $(2,1)$. In this paper we investigate hyperidentities for another family of type $(2,1)$ varieties, the varieties of star-bands. Using Petrich's equational description of the lattice of all star-band varieties, we present separating hyperidentities satisfied by these varieties.

1 Introduction Just as identities are used to classify algebras into varieties, hyperidentities are used to classify varieties into collections called hypervarieties. A hyperidentity is formally the same as an identity, built up of operation symbols and variables. Let V be any variety, and let $u \approx v$ be any identity, of arbitrary type. For each operation symbol in the identity, choose a term of V of the same arity. Substitution of these terms into $u \approx v$ leads to an identity in the variables of $u \approx v$ which may or may not hold in V. If every choice of terms of V of appropriate arity leads to an identity which holds in V, we say that V hypersatisfies $u \approx v$, and that $u \approx v$ is a hyperidentity for V. (This notion of substitution of terms of V into an identity may be made more precise using the concept of hypersubstitution; details may be found in [G-S].)

Hyperidentities have been much studied, both in general and for specific varieties of algebras. In the general situation, the concept of solidity and the generalization to pre-hyperidentities and other M-hyperidentities have been significant. (See for example [T], [G-S], [D] and [D-R].) For specific varieties, attention has focussed on algebras of type (2) such as semigroups, (see [D-W], [P]) and recently on inverse semigroups as algebras of type $(2,1)$, as in [C-W], and [C-W1]. In this paper we consider another specific context, that of the star-band algebras of type $(2,1)$. In Section 2 we describe the unary hyperidentities and binary iterative hyperidentities satisfied by the star-band varieties, and in Section 3 we produce for each star-band variety a hyperidentity satisfied by it but not by any larger varieties.

A star-regular band, or star-band for short, is an algebra of type $(2,1)$ with a binary multiplication indicated by juxtaposition and a unary operation *, which satisfies the following identities:

$$x(yz) \approx (xy)z,\ x^{**} \approx x,\ (xy)^* \approx y^*x^*,\ xx^*x \approx x, \text{ and } xx \approx x.$$

Let B^* be the variety of all star-bands. The lattice of all subvarieties of B^* was first described by Adair in [Ad]; we use here a later description by Petrich ([Pet]). This lattice consists of four special varieties, then a countably infinite chain of varieties V_n, with each variety being defined within B^* by one additional identity. For $u \approx v$ an identity of type $(2,1)$, we will use the notation $V(u \approx v)$ for the subvariety of B^* determined by $u \approx v$. For

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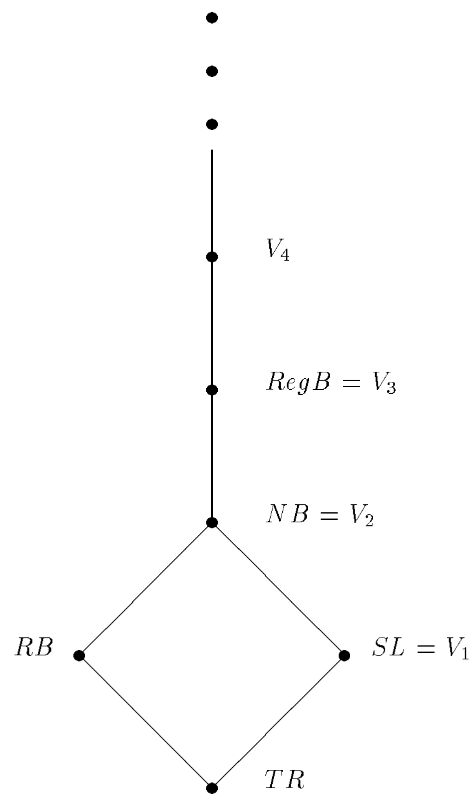

Figure 1: The lattice of varieties of star-bands

any term u of type $(2,1)$, we use the notation $\overline{u}$ for the left-to-right dual of u, so that for instance $\overline{x_1 x_2^* x_2} = x_2 x_2^* x_1$. The lattice of all varieties of star-bands is shown in Figure 1. We list here the varieties which appear in this diagram:

$TR = V(x \approx y)$, the trivial variety,
$V_1 = SL = V(x^* \approx x) = V(x \approx xx^*) = V(xy \approx yx)$, the variety of semilattice star-bands,
$RB = V(xyx \approx x)$, the variety of rectangular star-bands,
$V_2 = NB = V(xy \approx xy^*xy) = V(xyzw \approx xzyw)$, the variety of normal star-bands,
$V_3 = RegB = V(xy \approx xx^*yxy) = V(xyzx \approx xyxzx)$, the variety of regular star-bands.

Above $n = 2$, the varieties V_n are defined inductively as follows ([Pet]):

$$V_{2n-4} = V(P_n \approx \overline{P}_n) = V(\overline{P}_n P_n \approx R_n), \text{ for } n \geq 3,$$
$$\text{and } V_{2n-3} = V(Q_n \approx \overline{Q}_n) = V(\overline{Q}_n Q_n \approx R_n), \text{ for } n \geq 3.$$

Here the words P_n, Q_n and R_n are inductively defined, as follows:

$P_2 = x_2, \qquad P_n = R_{n-1}x_n\overline{P}_{n-1};$
$Q_2 = x_2x_1, \qquad Q_n = R_{n-1}x_n\overline{Q}_{n-1};$
$R_2 = x_2x_1x_2, \qquad R_n = R_{n-1}x_nR_{n-1}.$

For any word w, the content $c(w)$ of w is the set of letter variables (ignoring stars) which occur in w. Note that $c(xx^*) = \{x\} = c(x^*)$. (The content is also sometimes denoted by $Var(w)$, the set of all variables which occur in the term w.) Many of our calculations for identities will use the following important fact about contents.

Lemma 1.1 *([Ad]) For any words u, v and w, if $c(v) \subseteq c(u) = c(w)$, then the identity $uvw \approx uw$ holds in every variety of star-bands.*

2 Some Hyperidentities for Star-Band Varieties In this section we give some examples of hyperidentities satisfied by some of the varieties of star-bands, looking at unary hyperidentities and binary iterative hyperidentities.

Example 2.1 A unary hyperidentity of the form

$$F_1(F_2(\cdots F_k(x)\cdots) \approx G_1(G_2(\cdots G_l(x)\cdots),$$

for some unary operation symbols $F_1, \ldots, F_k$, $G_1, \ldots, G_l$, not necessarily all distinct, is either satisfied by B^* or holds only in the varieties SL and TR. This is because there are only four unary terms in B^*, namely x, x^*, xx^* and x^*x, and substitution of any of these into the hyperidentity leads to an identity of the form $u \approx v$, where u and v are again among the four terms. This means that $u \approx v$ is trivial, or is one of the six identities $x \approx x^*$, $x \approx xx^*$, $x \approx x^*x$, $x^* \approx xx^*$, $x^* \approx x^*x$ or $xx^* \approx x^*x$. It is well known (see [Ad]) that each of these six identities defines the variety SL. Thus either all the identities $u \approx v$ are trivial, and the hyperidentity holds in B^*, or we obtain one or more SL identities, and the hyperidentity holds only in SL (and its one subvariety TR).

Example 2.2 Non-trivial unary iterative hyperidentities have the form $F^a(x) \approx F^b(x)$ for some $b > a \geq 1$. For star-bands these hyperidentities reduce to two cases:

a) When a and b have opposite parity, i.e. when one is even and the other odd, using x^* for F forces $x \approx x^*$, so the hyperidentity $F^a(x) \approx F^b(x)$ holds only in SL and TR.

b) When a and b have the same parity, so that either both are odd or both are even, substitution of any of the four unary terms x, x^*, xx^* or x^*x leads to a trivial identity which holds in all of B^*.

Example 2.3 A unary hyperidentity

$$F_1(F_2(\cdots F_k(x)\cdots) \approx G_1(G_2(\cdots G_l(x)\cdots),$$

in which the two rightmost operation symbols F_k and G_l are different holds only in SL and TR. This is because use of xx^* for F_k and x^*x for G_l forces the identity $xx^* \approx x^*x$, which defines SL.

We next consider binary iterative hyperidentities for varieties of star-bands. We define $F^2(x,y) = F(F(x,y),y)$ and inductively, $F^{k+1}(x,y) = F(F^k(x,y),y)$, for $k \geq 2$. A binary iterative hyperidentity is a hyperidentity of the form $F^a(x,y) \approx F^b(x,y)$, for some $b > a \geq 1$.

Theorem 2.4 *(i) If $b > a \geq 1$ and a and b have opposite parity, then the hyperidentity $F^a(x,y) \approx F^b(x,y)$ holds only in SL and TR.*

(ii) If $b \geq 3$ with b odd, then $F(x,y) \approx F^b(x,y)$ holds only in SL and TR.

(iii) If $b \geq a \geq 2$ have the same parity, then the hyperidentity $F^a(x,y) \approx F^b(x,y)$ is satisfied in every variety of star-bands.

Proof. (i) Using x^* for F forces $x^* \approx x$, and hence SL. It is clear that the hyperidentity does hold in TR and SL.

(ii) It is easy to verify inductively that for b odd, the result of using x^*y^* for F in $F^b(x,y)$ is the word yx^*y^*. Thus using x^*y^* in the hyperidentity gives $x^*y^* \approx yx^*y^*$, which defines the variety SL.

(iii) Let t be any binary term. We will let t_n denote the result of substituting t for F in the term $F^n(x,y)$, for $n \geq 1$. We must show that when $b > a \geq 2$ have the same parity, the identity $t_a \approx t_b$ holds in every variety of star-bands.

We can show directly that if t is any of the terms x, y, x^*, y^*, x^*x, xx^*, xy, yx, xyx or yxy, then in fact $t_a \approx t_b$ is a trivial identity. So we may assume that t contains both letters x and y in its content, and has at least one occurrence of the star operator.

We introduce some notation from Adair ([Ad]). For any word p, we define the initial part of p to be the word obtained from p by keeping only the first occurrence of each letter in p, in the order in which they first occur in p; dually, the final part of p is the word obtained from p by keeping only the last occurrence of each letter in p, in the order in which they make their last occurrence. We also define $\gamma(p)$ to be the longest left cut of p with one variable, and dually, $\delta(p)$ to be the longest right cut of p with one variable. Then it follows from [Ad] that it will suffice to prove that the words t_a and t_b always have the same initial and final parts, and the same γ and δ values.

This means that we will be interested in the "starts" of words like t_a: the longest one-letter prefix of the word, and the order in which the variables make their first appearance; and dually for the "ends". In order to investigate starts and ends, we classify the original term t, as follows. There are eight possible one-letter prefixes, x, x^*, xx^* and x^*x for x and similarly for y, and for each of these there are two choices for the next symbol. This gives rise to 16 possible starts for t, and dually we get sixteen possible ends for t.

We will use the notation $t \approx swe$ to indicate that t starts with the word s, ends with the word e, and has some (possibly empty) word w in between. Note that because of idempotence, we may assume that the start s and the end e do not overlap.

The start and end of the word t_n, for $n \geq 2$, depends on the start and end of $t = t_1$, in an inductive manner determined according to the following key observation. If $F^n(x,y)$ has produced a word t_n, then we can obtain the word t_{n+1} from it by going through t_n and replacing every occurrence of x or x^* by t_n or t_n^* respectively, while leaving each y or y^* alone. This means that occurrences of y in t play a much less significant role than occurrences of x, and suggests the following convenient notation. For α and β equal to any of the words y, y^*, yy^* or y^*y, we will write $t = [\alpha]w$ (or $t = w[\beta]$) to indicate that t has the form either w or αw (respectively w or $w\beta$). This notation allows us to group our two hundred and fifty-six possible start-end combinations into the following four cases.

Case 1: $t = t_1 = [\alpha]xw_1x[\beta]$, for some (possibly empty) word w_1.
Note that this means that $[\alpha]t_1 \approx t_1$ and $t_1[\beta] \approx t_1$. Let us inductively define w_{n+1} as the result of using t_n for the x-input and y as the y-input in the subword w_n. Then we have $t_2 \approx [\alpha]t_1w_2t_1[\beta] \approx t_1w_2t_1$, and for each $k \geq 2$, $t_{k+1} \approx [\alpha]t_kw_{k+1}t_k[\beta]$. Thus by induction, for all $n \geq 2$ we have t_n starting with $[\alpha]t_1 \approx t_1$ and ending with $t_1[\beta] \approx t_1$. In particular, this means that t_a and t_b have the same initial, final, γ and δ values, since they are all equal to the corresponding values for t_1.

Case 2: $t = t_1 = [\alpha]xw_1x^*[\beta]$, for some (possibly empty) word w_1.
Then $t_2 \approx [\alpha]t_1w_2t_1^*[\beta]$, and for each $k \geq 2$, $t_{k+1} \approx [\alpha]t_kw_{k+1}t_k^*[\beta]$. By induction, we get that for $n \geq 2$, the word t_n starts with $[\alpha]t_1 \approx t_1$, and so has the same initial and γ value as t_1. This also means that for $n \geq 2$, t_n ends with $t_1^*[\beta]$, and so has the same final and δ values as $t_1^*[\beta]$. In particular, t_a and t_b have the same initial, final, γ and δ values.

Case 3: $t = t_1 = [\alpha]x^*w_1x[\beta]$, for some (possibly empty) word w_1.
This case is dual to Case 2, and is proved in a similar way.

Case 4: $t = t_1 = [\alpha]x^*w_1x^*[\beta]$, for some (possibly empty) word w_1.
Then $t_1^* \approx [\beta^*]xw_1^*x[\alpha^*]$, so that $[\beta^*]t_1^* \approx t_1^* \approx t_1^*[\alpha^*]$. We have $t_2 \approx [\alpha]t_1^*w_1t_1^*[\beta]$, and for any $k \geq 1$, $t_{k+1} \approx [\alpha]t_k^*w_{k+1}t_k^*[\beta]$, and
$t_{k+2} \approx [\alpha]t_{k+1}^*w_{k+2}t_{k+1}^*[\beta] \approx [\alpha][\beta^*]t_kw_{k+1}^*t_k[\alpha^*]w_{k+2}[\beta^*]t_kw_{k+1}^*t_k[\alpha^*][\beta]$.

Therefore by induction, for $n \geq 3$ and odd, the word t_n starts with $[\alpha][\beta^*]t_1$ and ends with $t_1[\alpha^*][\beta]$. Thus if $b \geq a$ are both odd, t_a and t_b have the same initial, final, γ and δ values. Similarly, for $n \geq 4$ and even, we have t_n starting with $[\alpha][\beta^*][\alpha]t_1^*$ and ending with $t_1^*[\beta][\alpha^*][\beta]$, and $t_a \approx t_b$ holds. Finally, for the special case $a = 2$ and $b \geq 2$ is even, we note that t_2 also starts with $[\alpha][\beta^*][\alpha]t_1^*$, since $[\alpha]t_1^* \approx [\alpha][\beta^*]t_1^* \approx [\alpha][\beta^*][\alpha][\beta^*]t_1^* \approx [\alpha][\beta^*][\alpha]t_1^*$; and similarly for the ends.

■

3 Separating Varieties of Star-Bands by Hyperidentities

So far all the hyperidentities considered for varieties of star-bands either hold in all of B^*, or hold only in SL and TR. This raises the question of whether given any two distinct varieties of star-bands, we can find a hyperidentity satisfied by one but not the other. In particular, for each variety in the lattice, can we find a hyperidentity satisfied by it but not by the next higher variety? In this section we produce such separating hyperidentities.

Lemma 3.1 *(i) The following hyperidentity holds in RB, but not in SL or NB:*

$$F(F(F(a, F(F(a,a),a)),a),a) \approx F(F(F(a, F(F(a,x),a)),a),a).$$

(ii) The following hyperidentity holds in NB but not in $RegB$:

$$F(F(F(a, F(F(a,x), F(y,a))),a),a) \approx F(F(F(a, F(F(a,y), F(x,a))),a),a).$$

Proof. (i) This hyperidentity fails in SL because using xy for F gives $a \approx axa$, which defines RB. We can verify directly, by substitution of all 16 terms, that it does hold in RB. (ii) Using xy for F yields the identity $axya \approx ayxa$, which is known to define NB. Thus the hyperidentity cannot hold in $RegB$. It does hold in $NB = SL \bigvee RB$, by direct verification of terms. ■

To produce a hyperidentity for each V_n, $n \geq 3$, which is not satisfied by the next variety in the chain, we consider Petrich's identities from Section 1. For $n \geq 3$, we have $V_{2n-3} = V(Q_n \approx \overline{Q}_n) = V(\overline{Q}_nQ_n \approx R_n)$. It is easily verified that these two defining identities are also equivalent to the identity $\overline{Q}_n \approx R_n$, with an analogous identity for V_{2n-4}. We modify this identity by doubling each variable, then left associating, to form a potential hyperidentity $A_n \approx B_n$ for the variety V_{2n-3}. For instance, for $n = 3$ and the variety V_3, we obtain the identity $x_2x_1 \quad x_3x_2x_1x_2 \approx x_2x_1x_2x_3x_2x_1x_2$ and the potential hyperidentity $A_3 \approx B_3$:

$$F(x_2, F(x_2, F(x_1, F(x_1, \quad F(x_3, F(x_3, F(x_2, F(x_2, F(x_1, F(x_1, F(x_2, x_2)\ldots) \approx$$
$$F(x_2, F(x_2, F(x_1, F(x_1, F(x_2, F(x_2, F(x_3, F(x_3, F(x_2, F(x_2, F(x_1, F(x_1, F(x_2, x_2)\ldots).$$

It is clear that this identity $A_n \approx B_n$ is not a hyperidentity in the next highest variety V_{2n-2}. Our goal now is to show that is does hold as a hyperidentity in the variety V_{2n-3}. A similar process and proof may be carried out for the even varieties V_{2n-4}, using analogous identities.

To prove that this identity $A_n \approx B_n$ is a hyperidentity for V_{2n-3}, we must show that every identity obtained from it by replacing F by a term t still holds in V_n. We shall verify this in a number of lemmas, for which we first introduce some necessary notation. We shall denote the identity resulting from use of a term t for F in the hyperidentity by $A_n^t \approx B_n^t$. It is clear that to evaluate such an identity, we always begin at the right hand end of each term with $t(x_2, x_2)$, and continue evaluating subwords of the form $F(l, F(l, w))$, where l represents a variable letter and w is any interim word. We note that the right hand side of the hyperidentity is almost the same as the left hand side, except that it has an additional occurrence of "$F(x_2, F(x_2,$" at one stage, and also that all the variables in the hyperidentity occur at least once to the right of this extra occurrence. Of particular interest therefore is the interim word $d = d^t$ which is obtained by evaluation up to the point where the two sides of the hyperidentity differ. We then denote by $e = e^t$ the right hand side interim word obtained by evaluating $F(x_2, F(x_2, d))$. We shall use notation such as $u(x, y)$ to mean that u is a word whose content is one of $\{x\}$, $\{y\}$ or $\{x, y\}$.

The first Lemma deals with words whose first and last content variable is y; that is words which begin with either y or y^* and end with either y or y^*.

Lemma 3.2 *Let* $t \approx p(y)u(x, y)q(y)$, *where* p, $q \in \{y, y^*\}$ *and* $u(x, y)$ *is a word with content* $\{x, y\}$. *Then* $A_n^t \approx B_n^t$ *holds in* V_{2n-3}.

Proof. Here we examine the interim words d and e. We note that

$e \approx F(x_2, F(x_2, d))$
$\approx F(x_2, p(d)u(x_2, d)q(d))$
$\approx F(x_2, p(d)q(d))$, since x_2 occurs in d,
$\approx p(p(d)q(d))Uq(p(d)q(d))$ for some word U,
$\approx p(p(d)q(d))q(p(d)q(d))$, since $c(U) \subseteq c(d)$.

There are four possible combinations for p and q. When $p = q = y$ or $p = q = y^*$, it is easy to see that $e = d$, so that all subsequent evaluations in the hyperidentity give the same result on either side. When $p = y$ and $q = y^*$, or vice versa, we find that $e = dd^*$, or analogously $e = d^*d$. But here we claim that d is star-dual, meaning that $d^* = d$, so in both cases this reduces to d as well. To see that d is star-dual when $p = y$ and $q = y^*$, we note that any evaluation $F(l, F(l, w))$ gives a star-dual result in this case. ∎

Thus in the remaining Lemmas we consider terms t which either start or end with x or x^*. We classify such terms according to whether the first y variable encountered on each side carries a star or not. We use λ for the empty word.

Lemma 3.3 *Let* $t \approx r(x)yu(x, y)ym(x)$, *where* $r, m \in \{\lambda, x, x^*, xx^*, x^*x\}$ *are not both empty and* u *is any word. Then* $A_n^t \approx B_n^t$ *holds in* V_{2n-3}.

Proof. For any letter l and any interim word w, we obtain

$F(l, F(l, w)) \approx F(l, r(l)wu(l, w)wm(l))$
$\approx r(l)r(l)wu(l, w)wm(l)Ur(l)wu(l, w)wm(l)m(l)$ (*) for some word U.

Case 1: If neither r nor m is empty, we see that (*) reduces to $r(l)wm(l)$. In this case it is clear that the final identity $A_n^t \approx B_n^t$ is a consequence of the defining identity for V_{2n-3}; the base identity is copied twice, with each variable x being replaced by $r(x)$ in one copy and by $m(x)$ in the other.

Case 2: If r is non-empty but m is empty, so that $t = r(x)yu(x, y)y$, then we have

$e = F(x_2, F(x_2, d))$
$\approx F(x_2, r(x_2)du(x_2, d)d)$
$\approx F(x_2, r(x_2)d)$, since x_2 occurs in d,

$\approx r(x_2)r(x_2)dUr(x_2)d$ for some word U,
$\approx r(x_2)d$, by content.

Moreover, for any subsequent evaluation $F(l, F(l, w))$ the fact that all variables have now been encountered means that we obtain $r(l)w$. This means that our final identity $A_n^t \approx B_n^t$ has the form $ar(x_2)b \approx ab$, for some words a and b. But both a and b contain $r(x_2)$, so that by content this identity holds in V_{2n-3}.

Case 3: If r is empty but m is not, we obtain a situation analogous to Case 2. We get $e \approx dm(x_2)$, and subsequent evaluations with letter l merely attach $m(l)$ to the right hand side of the previous word. Thus we obtain an identity of the form $am(x_2)b \approx ab$, which does hold in V_{2n-3}. ■

Lemma 3.4 *Let $t \approx r(x)y^*u(x,y)ym(x)$, where $r, m \in \{\lambda, x, x^*, xx^*, x^*x\}$ are not both empty and u is any word. Then $A_n^t \approx B_n^t$ holds in V_{2n-3}.*

Proof. Since r and m are not both empty, we consider three cases.

Case 1: When neither r nor m is empty, it is easy to verify that in the first stages of evaluation we have $F(x_1, F(x_1, F(x_2, x_2))) = r(x_1)m(x_1)^*m(x_2)^*m(x_2)m(x_1)$. Then for further evaluations we see that

$F(l, F(l, w)) \approx F(l, r(l)w^*u(l, w)wm(l))$
$\approx F(l, r(l)w^*wm(l))$, by content,
$\approx r(l)m(l)^*w^*wr(l)^*Ur(l)w^*wm(l)m(l)$, for some word U,
$\approx r(l)m(l)^*w^*wm(l)$, by content.

It follows that the final identity $A_n^t \approx B_n^t$ is again a consequence of the defining identity for V_{2n-3}, with the identity copied twice and each variable x replaced by $m(x)^*$ in one copy and by $m(x)$ in the other (with a final $r(x_2)$ on the left end).

Case 2: Let r be non-empty but m empty, so $t = r(x)y^*u(x,y)y$. Then we have $e = F(x_2, F(x_2, d)) \approx F(x_2, r(x_2)d^*u(x_2, d)d) \approx r(x_2)d^*d$, by content, since x_2 occurs in d. Then in the subsequent evaluations, we obtain $F(l, F(l, e)) \approx r(l)e^*e \approx r(l)d^*dr(x_2)^*r(x_2)d^*d \approx r(l)d^*d$ on one side and $F(l, F(l, d)) \approx r(l)d^*d$ on the other. Thus the two sides give exactly the same results in this case.

Case 3: Let r be empty but m non-empty, so $t = y^*u(x,y)ym(x)$. Then for any evaluation we have

$F(l, F(l, w)) \approx F(l, w^*u(l, w)wm(l))$
$\approx m(l)^*w^*u(m, l)^*wUw^*u(l, w)wm(l)m(l)$, for some word U,
$\approx m(l)^*w^*wm(l)$,

which is star-dual, and so reduces to $m(l)^*wm(l)$. Thus the final identity $A_n^t \approx B_n^t$ has the form of a two-copy instance of the original defining identity, with each variable x replaced by $m(x)^*$ in one copy and dually in the other. ■

Lemma 3.5 *Let $t \approx r(x)y^*u(x,y)y^*m(x)$, where $r, m \in \{\lambda, x, x^*, xx^*, x^*x\}$ are not both empty and u is any word. Then $A_n^t \approx B_n^t$ holds in V_{2n-3}.*

Proof. The first evaluation on either side gives $F(x_2, x_2) = r(x_2)x_2^*u(x_2, x_2)x_2^*m(x_2)$, which by content reduces to $r(x_2)x_2^*m(x_2)$. Subsequent evaluations then give

$F(l, F(l, w)) \approx F(l, r(l)w^*u(l, w)w^*m(l))$
$\approx r(l)m(l)^*wu(l, w)^*wr(l)^*Um(l)^*wu(l, w)^*wr(l)^*m(l)$, for some word U,
$\approx r(l)m(l)^*wr(l)^*m(l)$, by content.

¿From this we see that the final identity $A_n^t \approx B_n^t$ is again a doubled instance of the defining identity for V_{2n-3}; here each variable x is replaced by $r(x)m(x)^*$ in one copy and by $m(x)r(x)^*$ in the other. ■

We have now tested all the terms necessary, establishing the following theorem.

Theorem 3.6 *For any $n \geq 3$, the identity $A_n \approx B_n$ is a hyperidentity satisfied by the variety V_{2n-3}, but not by the variety V_{2n-2}.*

This has given us hyperidentities for each of the varieties V_n in the chain for which n is odd. We can carry out a similar process for n even, using a modified version of the Petrich identities. In this case too we obtain hyperidentities; the proof is very similar, so details are omitted.

Corollary 3.7 *For every star-band variety V_n, $n \geq 3$, there is a hyperidentity separating it from the next variety in the chain.*

We conclude this investigation of hyperidentities for varieties of star-bands with a remark about solid varieties. A variety is said to be solid if each of its identities also holds as a hyperidentity. Solid varieties have been much investigated, both in general and for specific varieties, especially varieties of semigroups (see [P]). However, it is easy to see that there are no solid varieties of star-bands. This is because the defining identity $(xy)^* \approx y^*x^*$ does not hold as a hyperidentity in any non-trivial variety of star-bands.

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