Scientiae Mathematicae **Vol.3,** No. 3(2000), 385–398

QUASI–IDEALS AND EXTENSIONS OF BCK–ALGEBRAS

S. F. Farag and A. N. Babiker

Received December 3, 1999

Abstract. In this paper we introduce the notion of a quasi-ideal K of a BCK-algebra X which is a special kind of subalgebras. The purpose of this paper is to investigate the extension of X by the quasi-ideal K over a BCK-algebra Γ.

Introduction

BCK-algebras were introduced as an algebraic formulation of a propositional calculus by K. Iseki and E. Y. Imai in 1966 [7]. A lot of literature dealing with algebraic theory using first order properties (see. e.g.[4,5,11,16]) and ideal theory of BCK-algebras (see. e.g. [1,2,6,9,15]) is available. Let X and Y be two BCK-algebras and $X \cap Y = \{0\}$, K. Iseki and E. Y. Imai, define the union $X \cup Y$ to be a new BCK-algebra (see [7]). In this union X and Y are subalgebras. H. Yutani [18] and H. Jaing [12] studied the extensions of BCK-algebras. We introduce a new notion, called quasi-ideal and use it to get an extension of any BCK-algebra X by another BCK-algebra Γ in which X is an ideal and Γ is a subalgebra. More precisely, let X be any BCKalgebra, a non-empty subset K of X is called a quasi-ideal of X if $K * X = K$. Then K is a subalgebra of X and any ideal of X is a quasi-ideal. J. G. Varlet, in [17], introduced the notion of Varlet ideals. In 1977, K. Iseki also introduced the notion of additive ideals (see [10]). We show, see Theorem 2.2, that any Varlet or additive ideal is a quasi-ideal.

The quasi-ideal K is then used to construct the extension $E_\Phi(X, K : \Gamma)$ of X over Γ. This extension satisfies the exact sequence $\{0\} \to X \to E_\Phi(X, K : \Gamma) \to \Gamma \to \{0\}$. This means that $E_\Phi(X, K : \Gamma)/X$ is isomorphic to Γ.

1. Preliminaries

A BCK-algebra is a system $(X, *, 0, \leq)$ satisfying the following axioms for all $x, y, z \in X$:
(1) $(x * y) * (x * z) \leq (z * y)$
(2) $x * (x * y) \leq y$
(3) $x \leq x$
(4) $0 \leq x$
(5) $x \leq y, y \leq x$ imply $x = y$
(6) $x \leq y$ if and only if $x * y = 0$.

If X contains an element 1 such that $x \leq 1$, for all $x \in X$, then X is said to be bounded. X is called commutative if $x \wedge y = y \wedge x$ for all $x, y \in X$ where $x \wedge y = y * (y * x)$. A bounded commutative BCK-algebra X is a distributive lattice with respect to $\wedge$ and $\vee$, where $x \vee y = N(Nx \wedge Ny)$ for all $x, y \in X$, and $Nx = 1 * x$ (see [4], [11],[16]). X is called implicative if $x * (y * x) = x$ for all $x, y \in X$. It is well-known that every implicative BCK-algebra is commutative but the converse is not true in general [11]. A non-empty subset J of a BCK-algebra X is called an ideal if $0 \in J$ and $x, y * x \in J$ imply $y \in J$. It follows that if J is an ideal, $x \in J$ and $y \leq x$ then $y \in J$. An ideal of a commutative BCK-algebra is closed with respect to $\wedge$ and $\vee$ in the sense that for any $x, y \in J$, we have $x \wedge y, x \vee y \in J$.

Indeed $x \wedge y \leq x, x \in J$ implies that $x \wedge y \in J$. To prove that $x \vee y \in J$, we recall (see Hoo [6, Theorem 6 (iii)]) that $(x \vee y) * y = x * y \leq x$ implies that $(x \vee y) * y \in J$. Again $y \in J$ and J being an ideal implies that $(x \vee y) \in J$. A non-empty subset Y of a BCK-algebra X is said to be a subalgebra of X if for any $x, y \in Y, x * y \in Y$, i.e., Y is called under the binary operation $*$ of X. It follows that if Y is a subalgebra, then $0 \in Y$. It is easy to see that $\{0\}$ and X are subalgebras in any BCK-algebra X. Moreover $\{0, x\}$ is a subalgebra for any $x \in X$. For some further properties of BCK-algebras and undefined terminology and notions used here, we refer to [9,11,16].

Definition 1.1. Let X be any BCK-algebra, for any subsets $E, F \subseteq X$, define
(i) $E * F = \{x * y : x \in E, y \in F\}$
(ii) $A(E) = \{x \in X : x \leq e \text{ for some } e \in E\}$

Lemma 1.2. Let x be any BCK-algebra. If $0 \in F \subseteq X$, then for each $E \subseteq X$ we have $E \subseteq E * F$.

Proof. Obvious. □

Lemma 1.3. Let X be any BCK-algebra. If E is a subset of X, then
(i) $E * X$ is a subset of $A(E)$.
(ii) If X is commutative, then $E * X = A(E)$

Proof. (i) $x \in E * X$, then $x = e * y \leq e$ for some e in E and y in X. Therefore x is in $A(E)$ and hence $E * X \subseteq A(E)$. (ii) It is enough to prove that $A(E) \subseteq E * X$. Let $x \in A(E)$, then $x \leq e$ for some e in E. Hence $x * e = 0$. Therefore $x = x * (x * e) = e * (e * x) \in E * X$. So $A(E) \subseteq E * X$.□

Lemma 1.4. Let $A, B, E \subseteq X$, then we have
(i) $A \subseteq B$ implies that $A * E \subseteq B * E$ and $E * A \subseteq E * B$.
(ii) $(A \cap B) * E \subseteq (A * E) \cap (B * E)$
(iii) $E * (A \cap B) \subseteq (E * A) \cap (E * B)$
(iv) $(A \cup B) * E = (A * E) \cup (B * E)$
(v) $E * (A \cup B) = (E * A) \cup (E * B)$

Proof. (i) Since $x \in A * E \Rightarrow x = a * e : a \in A$ and $e \in E \Rightarrow x = a * e : a \in B$ and $e \in E \Rightarrow x \in B * E$. Therefore $A * E \subseteq B * E$. the other assertion follows similarly.
(ii) Since $A \cap B \subseteq A, B \Rightarrow (A \cap B) * E \subseteq A * E, B * E$. Hence $(A \cap B) * E \subseteq (A * E) \cap (B * E)$.
(iii) The proof follows similarly as in (ii)
(iv) Since $A, B \subseteq A \cup B$ implies that $A * E, B * E \subseteq (A \cup B) * E$. Therefore $(A * E) \cup (B * E) \subseteq (A \cup B) * E$. Now let $x \in (A \cup B) * E \Rightarrow x = y * e : y \in A \cup B, e \in E$. This implies that $x = y * e : (y \in A$ or $y \in B)$ and $e \in E$. Therefore $x = y * e : (y \in A, e \in E)$ or $(y \in B, e \in E)$. Hence $x = y * e : y * e \in (A * E)$ or $y * e \in (B * E)$. Thus $x \in (A * E) \cup (B * E)$. So $E * (A \cup B) \subseteq (E * A) \cup (E * B)$. Therefore $E * (A \cup B) = (E * A) \cup E * B)$.
(v) follows like (iv).□

2. Quasi-ideals and Ideals in BCK-Algebras

In this section we define the quasi-ideals and then we use the above results to prove several properties of them.

Definition 2.1. Let X be a BCK-algebra. A non-empty subset K of X is said to be a quasi-ideal in X if and only if $K * X = K$. We shall denote this by $K <S X$.

Theorem 2.2. Let X be any BCK-algebra. Then we have
(1) Any quasi-ideal in X is a subalgebra of X.
(2) Any ideal of X is a quasi-ideal in X.

(3) If J is a Varlet ideal (or an additive ideal) of X, then J is a quasi-ideal in X.

Proof. (1) Let K be a quasi-ideal. Since $\emptyset \neq K = K * X$, hence there exists $x \in K$ such that $0 = x * x \in K * X = K$. Moreover for any x and y in $K, x * y \in K * K \subseteq K * X = K$. Hence K is a subalgebra.
(2) Let I be an ideal in X. Then $0 \in I$, and hence $I \neq \emptyset$. For any x in I and y in $X, x*y \in I$, since $x * y \leq x$. Therefore $I * X \subseteq I$. Moreover $I = I * 0 \subseteq I * X$. Hence $I * X = I$; i.e., I is a quasi ideal.
(3) Obvious.□

Proposition 2.3. Let X be a BCK-algebra and H, K be quasi-ideals in X. Then
(1) $H \cap H$ is a quasi-ideal in X.
(2) $H \cup K$ is a quasi-ideal in X.

Proof. (1) Let $L = H \cap K$, Then $L \subseteq L * X = (H \cap K) * X \subseteq (H * X) \cap (K * X) = H \cap K = L$. Hence $H \cap K$ is a quasi-ideal in X.
(2) As above, let $L = H \cup K$, then $L \subseteq L * X = (H \cup K) * X = (H * X) \cup (K * X) = H \cup K = L$. Therefore $H \cup K$ is a quasi-ideal in X.

In general, it is known that, the union of two ideals is not an ideal. The above two propositions lead to the following result.

Corollary 2.4. The union of two ideals of a BCK-algebra is a quasi-ideal.□

Proposition 2.5. Let X, Y be two BCK-algebras. If H is a quasi-ideal in X and K is a quasi-ideal in Y, then $H \times K$ is a quasi-ideal in $X \times Y$.

Proof. $(H \times K) * (X \times Y) = (H * X) \times (K * Y) = H \times K$.

Counterexample 2.6. " The union of a quasi-ideal K and a subalgebra B, of a BCK-algebra X, is not, in general, a quasi-ideal."

Proof. Let $X = \{0, 1, 2, 3, 4\}, n*m = \max.\{n-m, 0\}, K = \{0, 1, 2\}$ and $B = \{0, 4\}$. It is easy to check that K is a quasi-ideal in X and B is a subalgebra of X. However $K \cup B = \{0, 1, 2, 4\}$ is neither a quasi-ideal nor a subalgebra. This is clear since $4 * 1 = 3 \notin K \cup B$.

Remark. " The above example shows also that B is a subalgebra which is not a quasi-ideal, and K is a quasi-ideal which is not an ideal."

Proposition 2.7. Let X be a BCK-algebra. If K is a quasi-ideal in X and B is a subalgebra of X, then $K \cap B$ is a quasi-ideal in B.

Proof. We have:

$$\begin{aligned} (K \cap B) * B &\subseteq (K * B) \cap (B * B) \\ &\subseteq (K * X) \cap B \qquad \text{“Since } B < X\text{”} \\ &= K \cap B \qquad \text{“Since } K <SX\text{”} \\ &\subseteq (K \cap B) * B. \qquad \text{“ Since } 0 \in B\text{”} \end{aligned}$$

This implies that $(K \cap B) * B = K \cap B$. Therefore $K \cap B <SB$.

Proposition 2.8. Let X be a BCK-algebra. For any $x \in X, A(x)$ is a quasi-ideal in X.

Proof. Since $0 \in X$, then $A(x) \subseteq A(x) * X$. Let $y \in X$, then $x * y \leq x$. Hence $x * y \in A(x)$. Therefore $A(x) = A(x) * X$, i.e., $A(x)$ is a quasi-ideal in X.

Corollary 2.9. If E is a non-empty subset of a BCK-algebra X, then $A(E)$ is a quasi-ideal in X.

Proof. See Proposition 2.3 and Proposition 2.8.

Lemma 2.10. Let K be a quasi-ideal in a BCK-algebra X. If $0 \in E \subseteq X$, then $K * E = K$.

Proof. Since $K \subseteq K * E \subseteq K * X = K$, result follow.□

Lemma 2.11. For any subset E of a BCK-algebra $X, X * E$ is a quasi ideal in X.

Proof. Let $(x * e) * y \in (X * E) * X$. Since $(x * e) * y = (x * y) * e \in X * E$, therefore $(X * E) * X \subseteq X * E$. Obviously $X * E \subseteq (X * E) * X$. Hence $(X * E) * X = X * E$. Thus $X * E$ is a quasi-ideal of X.□

Corollary 2.12. Every ideal I of a BCK-algebra X contains a quasi-ideal.

Proof. For any set $E \neq \emptyset$ in X, we have $(I * E) * X = (I * X) * E = I * E$. Therefore $I * E$ is a quasi-ideal in X. On the other hand, for each $k \in I * E$, there exists $h \in I$ and $e \in E$ such that $k = h * e$. Thus $k \leq h$. This implies that $k \in I$. Hence $I * E \subseteq I$ This completes the proof.□

We shall need the following well-known result, given by Isaki in [10], to state and prove our next lemma.

Proposition 2.13. If X is a bounded BCK-algebra, then we have the following:
(1) The set of all involutions $J = \{k \in X : NNk = k\}$ is a subalgebra of X.
(2) $NNNx = Nx$, for each x in X.
(3) $(Nx) * y = (Ny) * x$, for all $x, y \in X$.□

Remark. It is clear that $Nx \in J$, for all $x \in X$.

Lemma 2.14. Let X be a bounded BCK-algebra. Then the set of all involutions J, is a quasi-ideal of X.

Proof. Let $k \in J$ and $x \in X$. Then $NN(k * x) = NN((NNk) * x) = NN((Nx) * (Nk) = (Nx) * (Nk) = (NNk) * x = k * x$. Therefore $k * x \in J$. This means that J is a quasi-ideal in X.□

Lemma 2.15. If X is a bounded BCK-algebra and K is a quasi-ideal in X such that $1 \in K$, then $K \supseteq J = \{x \in X : NNx = x\}$.

Proof. Since $Nx = 1 * x$ and $1 \in K$, therefore $Nx \in K$, for each $x \in X$. This implies that $(1 * x) * x \in K$, for each $x \in X$. So $x \in K$ whenever $NNx = x$. This means that $K \supseteq J$.□

Using the above two lemmas, we get the following

Theorem 2.16. In any bounded BCK-algebra X, the set of all involutions, J, is the smallest quasi-ideal in X, that contains 1.□

Lemma 2.17. If X a bounded BCK-algebra such that $NNx = x$, for all $x \in X$ and K is a quasi-ideal in X, then $1 \in K$ if and only if $K = X$.

Proof. Suppose that K is a quasi-ideal in X and $1 \in K$, therefore $1 * x \in K$, for all $x \in X$. Similarly $1 * (1 * x) \in K$, for all $x \in X$. Since $1 * (1 * x) = x$, for all $x \in X$, so $x \in K$, for all $x \in X$. This proves that $K = X$. The other implication is obvious.□

Theorem 2.18. In any bounded commutative BCK-algebra X, there is no proper quasi-ideal in X that contains 1.□

Theorem 2.19. Let $f : X \to Y$ be an onto BCK-homomorphism.
(1) If K is a quasi-ideal in X, then $f(K)$ is a quasi-ideal in Y.
(2) If H is a quasi-ideal in Y, then $f^{-1}(H)$ is a quasi-ideal in X.

Proof. (1) Let $f(k) \in f(K)$, for some $k \in K$. Then for any $y \in Y$, there exists $x \in X$ such that $f(k) * y = f(k) * f(x) = f(k * x) \in f(K)$. Therefore $f(K)$ is a quasi-ideal in Y.
(2) If $k \in f^{-1}(H)$ and $x \in X$ are arbitrary, then $f(k) \in H$ and $f(x) \in Y$. Since H is a quasi-ideal in Y, thus $f(k * x) = f(k) * f(x) \in H$. This implies that $k * x \in f^{-1}(H)$. Hence $f^{-1}(H)$ is a quasi ideal in X.□

Theorem 2.20. If J is an ideal of a BCK-algebra X and $x \notin J$, then the set

$$\Omega_{x,J} = \{y \in X : y * x \in J\}$$

is a quasi-ideal of X, which contains x and J.

Proof. If $u \in \Omega_{x,J}$ and $z \in X$, then $(u * z) * x = (u * x) * z \leq u * x \in J$. Since J is an ideal, then $(u * z) * x \in J$. This implies that $u * z \in \Omega_{x,J}$. Therefore $\Omega_{x,J}$ is a quasi ideal.□
Note: The following example shows that $\Omega_{x,J}$ is not the smallest quasi-ideal containing x and J.

Counterexample 2.21. Let $X = \{0, 1, 2, 3, 4\}$ in which the operation $*$ is given by the following table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	3	3	0	0
4	4	4	4	4	0

Then $(X, *, 0)$ is a BCK-algebra. It is easy to verify that $J = \{0, 1, 2\}$ is an ideal of X. The quasi-ideal $\Omega_{4,J} = \{y \in X : y * 4 \in \{0, 1, 2\}\} = \{0, 1, 2, 3, 4\} = X$. This implies that $\Omega_{4,J}$ is not the minimal quasi-ideal containing 4 and J, since $K = \{0, 1, 2, 4\}$ is a quasi ideal of X.

3. Extensions of BCK-algebras

Let $(X, *_x, .0_x)$ and $(\Gamma, *_\Gamma, 0_\Gamma)$ be BCK-algebras. If K is a quasi-ideal of X, then for each $\gamma \in \Gamma - \{0_\Gamma\}$, let $\varphi_\gamma(K)$ denotes a set of new distinct elements equal in number to the elements of K. Let $\varphi_\Gamma(K)$ be the union of all $\varphi_\gamma(K), \gamma \neq 0$.

Definition 3.1. Let $E_\Phi(X, K : \Gamma) = \varphi_0(x) \cup \varphi_\Gamma(K)$ such that:
(E-1) $\varphi_0(x) = x$, for each x in X.
(E-2) If $\alpha \neq \beta$, then $\varphi_\alpha(x) \neq \varphi_\beta(y)$, for all α, β in Γ and all x, y in X.

Theorem 3.2. If the operation $*$ and the relation $\leq$ are defined in $E_\Phi(X, K : \Gamma)$ by
(E-3) $\varphi_\alpha(x) * \varphi_\beta(y) = \varphi_{\alpha*\beta}(x * y)$, for all α, β in Γ and all x, y in X and
(E-4) $\varphi_\alpha(x) \leq \varphi_\beta(y)$ if and only if $\alpha \leq \beta$ and $x \leq y$, then
(1) $E_\Phi(X, K : \Gamma)$ is a BCK-algebra and
(2) X is an ideal of $E_\Phi(X, K : \Gamma)$.

Proof. From now on we shall write $*$ instead of $*_x$ or $*_\Gamma$ and 0 instead of 0_x or 0_Γ and $E(X, K : \Gamma)$ instead of $E_\Phi(X, K : \Gamma)$ whenever there is no confusion. First we show that $E(X, K : \Gamma)$ is a BCK-algebra. Hence let $x, y, z \in X$ and $\alpha, \beta, \gamma \in \Gamma$. We observe that $(x * y) * (x * z) \leq z * x$ and $(\alpha * \beta) * (\alpha * \gamma) \leq \gamma * \beta$. By axioms (E-3) and (E-4), therefore

1.
$$\begin{aligned}[\varphi_\alpha(x) * \varphi_\beta(y)] * [\varphi_\alpha(x) * \varphi_\gamma(z)] &= [\varphi_{\alpha*\beta}(x * y)] * [\varphi_{\alpha*\gamma}(x * z)] \\ &= \varphi_{[(\alpha*\beta)*(\alpha*\gamma)]}[(x * y) * (x * z)] \\ &\leq \varphi_{\gamma*\beta}(z * y) \\ &= \varphi_\gamma(z) * \varphi_\beta(y).\end{aligned}$$

2. Since $\alpha * (\alpha * \beta) \leq \beta$ and $x * (x * y) \leq y$, therefore by (E-3) and (E-4) we have

$$\varphi_\alpha(x) * [\varphi_\alpha(x) * \varphi_\beta(y)] = \varphi_{\alpha*(\alpha*\beta)}(x * (x * y)) \leq \varphi_\beta(y).$$

3. Since $\alpha \leq \alpha$ and $x \leq x$, then (E-4) gives $\varphi_\alpha(x) \leq \varphi_\alpha(x)$.
4. By (E-1) and (E-4): $0 = \varphi_0(0) \leq \varphi_\alpha(x)$.
5. By (E-3) and (E-4); $\varphi_\alpha(x) \leq \varphi_\beta(y)$ if and only if $\alpha \leq \beta, x \leq y$ if and only if $\alpha * \beta = 0, x * y = 0$ if and only if $\varphi_{\alpha*\beta}(x * y) = 0$ if and only if $\varphi_\alpha(x) * \varphi_\beta(y) = 0$.
6. Using 5. above, (E-3) and (E-4) we have $\varphi_\alpha(x) \leq \varphi_\beta(y)$ and $\varphi_\beta(y) \leq \varphi_\alpha(x)$ imply that $\varphi_{\alpha*\beta}(x * y) = 0 = \varphi_{\beta*\alpha}(y * x)$. Hence, by (E-2); $\alpha * \beta = 0 = \beta * \alpha$ and $x * y = 0 = y * x$. So we have $\alpha = \beta$ and $x = y$. Therefore $\varphi_\alpha(x) = \varphi_\beta(y)$. This implies that $E(X, K : \Gamma)$ is a BCK-algebra

Secondly to prove that X is an ideal, we observe that $\varphi_\alpha(x)$ is in X if and only if $\alpha = 0$. Hence $\varphi_0(x) = x$. Now let $\varphi_\alpha(x)$ and $\varphi_\beta(y) * \varphi_\alpha(x)$ be in X. Hence $\alpha = 0$ and $\beta * \alpha = 0$. Therefore $\beta = 0$. So $\varphi_\beta(y) = y \in X$. This proves that X is an ideal in $E(X, K : \Gamma)$.□

Definition 3.3. $E(X, K : \Gamma)$ is called the extension of X by K over Γ.

Theorem 3.4. If X and Γ are commutative, positive implicative or implicative, then so is $E(X, K : \Gamma)$.

proof. Let X and Γ be commutative, then: $\varphi_\alpha(x) * [\varphi_\alpha(x) * \varphi_\beta(y)] = \varphi_{\alpha*(\alpha*\beta)}(x * (x * y) = \varphi_{\beta*((\beta*\alpha)}(y * (y * x)) = \varphi_\beta(y)[\varphi_\beta(y) * \varphi_\alpha(x)]$. Hence $E(X, K : \Gamma)$ is commutative. The rest of the proof is similar.□

The following lemmas give various situations in which $E(X, K : \Gamma)$ is not bounded though X, K or Γ may be bounded.

Lemma 3.5. Let X and Γ be bounded BCK-algebras. If K is a quasi-ideal of X such that $1 \notin K$, then $E(X, K : \Gamma)$ is not bounded.

Proof. Suppose $\varphi_\beta(h)$ is an upper bound for some $\beta \in \Gamma$ and $h \in X$. Hence $\varphi_\alpha(k) * \varphi_\beta(h) = 0.$ for all $\alpha \in \Gamma$ and $k \in X$, i.e., $\varphi_{\alpha*\beta}(k * h) = 0$. This implies that $\alpha * \beta = 0$ and $k * h = 0$. Therefore $\alpha \leq \beta$ and $k \leq h$. Hence $\beta = 1$ and $h = 1$. Since $1 \notin K$, therefore $\varphi_1(1) \notin E(X, K : \Gamma)$ and thus we have a contradiction.□

Corollary 3.6. If $1 \in K$, then $\varphi_1(1)$ is an upper bound in $E(X, K : \Gamma)$.□

We observe that K may be a bounded quasi-ideal with an upper bound different from 1, i.e., $A(x)$ for some $x \in X$. Thus we have:

Corollary 3.7. If K is bounded and $1 \notin K$, then $E(X, K : \Gamma)$ is not bounded.□

Corollary 3.8. If $E(X, K : \Gamma)$ is bounded, then X, K and Γ are bounded and $1 \in K$.

Proof. See Lemma 3.5 and Corollary 3.7.□

Lemma 3.9. Let $E(X, K : \Gamma)$ be the extension of X by K over Γ, then for all $\varphi_\alpha(x)$ and $\varphi_\beta(y)$ in $E(X, K : \Gamma)$ we have:
(1) $\varphi_\alpha(x) \wedge \varphi_\beta(y) = \varphi_{\alpha\wedge\beta}(x \wedge y)$
(2) If $E(X, K : \Gamma)$ is bounded, then

$$\text{(i)} \quad N(\varphi_\alpha(x)) = \varphi_{N\alpha}(Nx) \qquad \text{(ii)} \quad \varphi_\alpha(x) \vee \varphi_\beta(y) = \varphi_{\alpha\vee\beta}(x \vee y)$$

Proof.

$$\begin{aligned} (1) \quad \varphi_\alpha(x) \wedge \varphi_\beta(y) &= \varphi_\beta(y) * [\varphi_\beta(y) * \varphi_\alpha(x)] \\ &= \varphi_\beta(y) * [\varphi_{\beta*\alpha}(y * x)] \\ &= \varphi_{\beta*(\beta*\alpha)}(y * (y * x)) = \varphi_{\alpha\wedge\beta}(x \wedge y) \end{aligned}$$

(2) (i) $N\varphi_\alpha(x) = \varphi_1(1) * \varphi_\alpha(x) = \varphi_{1*\alpha}(1 * x) = \varphi_{N\alpha}(Nx)$.

(ii)
$$\begin{aligned}\varphi_\alpha(x) \vee \varphi_\beta(y) &= N[N\varphi_\alpha(x) \wedge N\varphi_\beta(y)] = N[\varphi_{N\alpha}(Nx) \wedge \varphi_{N\beta}(Ny)] \\ &= N[\varphi_{N\alpha\wedge N\beta}(Nx \wedge Ny)] = \varphi_{N(N\alpha\wedge N\beta)}(N(Nx \wedge Ny)) \\ &= \varphi_{\alpha\vee\beta}(x \vee y). \square\end{aligned}$$

Corollary 3.10. In $E(X, K : \Gamma), \varphi_\alpha(x)$ is an involution if and only if x and α are involutions in X and Γ respectively.

Proof. Clear by 2(i) above. $\square$

Lemma 3.11. If $E(X, K : \Gamma)$ is bounded, then $NN\varphi_\alpha(x) = x \vee \varphi_\alpha(0)$, for any x in X and α in Γ.

Proof. For any x in X and α in Γ, we have

$$\begin{aligned}x \vee \varphi_\alpha(0) &= N[Nx \wedge N\varphi_\alpha(0)] \\ &= N[N\varphi_0(x) \wedge N\varphi_\alpha(0)] \\ &= N[\varphi_{N0}(Nx) \wedge \varphi_{N\alpha}(N0)] \\ &= N[\varphi_1(Nx) \wedge \varphi_{N\alpha}(1)] \\ &= N[\varphi_{1\wedge N\alpha}(Nx \wedge 1) \\ &= N[\varphi_{N\alpha}(Nx)] \\ &= NN\varphi_\alpha(x). \square\end{aligned}$$

Corollary 3.12. If $E(X, K : \Gamma)$ is bounded such that $NNx = x$ and $NN\alpha = \alpha$, for each x in X and α in Γ, then $\varphi_\alpha(x) = x \vee \varphi_\alpha(0)$. $\square$

Corollary 3.13. If $E(X, K : \Gamma)$ is bounded and commutative, then $\varphi_\alpha(x) = x \vee \varphi_\alpha(0)$. $\square$

Notation (1) If $\Lambda \subseteq \Gamma$ and $Y \subseteq X$, then we write $\varphi_\Lambda(Y) = \{\varphi_\beta(y) : \beta \in \Lambda \text{ and } y \in Y\}$.
(2) If $x \wedge y = 0$, we call $z = x \vee y$ the direct join of x and y.

Notice. In a bounded involutive or commutative BCK-Extension $E(X, K : \Gamma), \varphi_\alpha(x)$ is the direct join of x and $\varphi_\alpha(0)$.

Lemma 3.14. In $E(X, K : \Gamma)$, we have: $A(\varphi_\alpha(x)) = \varphi_{A(\alpha)}(A(x))$.

Proof. $\varphi_\beta(y) \in A(\varphi_\alpha(x))$ if and only if $\varphi_\beta(y) \leq \varphi_\alpha(x)$ if and only if $\beta \leq \alpha, y \leq x$ if and only if $\beta \in A(\alpha), y \in A(x)$ if and only if $\beta \in A(\alpha), \varphi_\beta(y) \in \varphi_\beta(A(x))$ if and only if $\varphi_\beta(y) \in \varphi_{A(\alpha)}(A(x))$. $\square$

Lemma 3.15. $E(X, K : \Gamma)$ is not, in general, a chain eventhough X, K and Γ are chains.

Proof. Let $\varphi_\alpha(x), \varphi_\beta(y) \in E(X, K, \Gamma)$ such that $\alpha \leq \beta$ and $y \leq x; y \neq x$. Then $\varphi_\alpha(x) * \varphi_\beta(y) \neq 0 \neq \varphi_\beta(y) * \varphi_\alpha(x)$. $\square$.

4. Subalgebras, Quasi-ideals and Ideals of BCK-Extensions

Lemma 4.1. Let X and Γ be BCK-algebras. If Y is a subalgebra of X, K is a quasi-ideal in X and $\gamma \in \Gamma$ then $Z = Y \cup \varphi_\gamma(K \cap Y)$ is a subalgebra of $E_\Phi(X, K : \Gamma)$.

Proof. It is clear $0 \in Z$. To prove that $x * y \in Z$, for any $x, y \in Z$, consider the following cases:
(1) If $x, y \in Y$, then $x * y \in Y \subseteq Z$.

(2) If $x, y \in \varphi_\gamma(K \cap Y)$, then there exist $h, k \in K \cap Y$ such that $x = \varphi_\gamma(h)$ and $y = \varphi_\gamma(k)$. This implies that $x * y = \varphi_\gamma(h) * \varphi_\gamma(k) = h * k \in Y \subseteq Z$.
(3) I $x \in Y$ and $y \in \varphi_\gamma(K \cap Y)$, then $x * y = x * \varphi_\gamma(k)$, for some $k \in K \cap Y$. Therefore $x * y = x * k$. Hence $x * y \in Z$. On the other hand $y * x = \varphi_\gamma(k) * x = \varphi_\gamma(k * x) \in \varphi_\gamma(K \cap Y)$. This implies that $y * x \in Z$.□

Note. In 4.1 $Z = Y \cup \varphi_\gamma(K \cap Y)$ can be written as $E_\Phi(Y, K \cap Y : \{0, \gamma\})$.

Corollary 4.2. If X, Y, K and Γ are as above and $B = \{0, \gamma\}$, for some $\gamma \in \Gamma$, then $E_\Phi(X, K : B)$ and $E_\Phi(K, K : B)$ are subalgebras of $E_\Phi(X, K : \Gamma)$.□

Corollary 4.3. Let X, Γ be BCK-algebras and K be a quasi-ideal of X. Then we have the following:
(1) If Y is subalgebra of X, then Y is a subalgebra of $E_\Phi(X, K : \Gamma)$.
(2) If Y is a subalgebra of K, then $Y \cup \varphi_\gamma(Y)$ is a subalgebra of $E_\Phi(X, K : \Gamma)$.□

Lemma 4.4. Let X, Γ be BCK-algebras and K be a quasi-ideal of X. If B and Y are subalgebras of Γ and X respectively; and $H = K \cap Y$, then $E_\Phi(Y, H : B)$ is a subalgebra of $E_\Phi(X, K : \Gamma)$.

Proof. It is clear that $0 \in E_\Phi(Y, H : B)$. To prove that $x * y \in E_\Phi(Y, H : B)$, for any $x, y \in E_\Phi(Y, H : B)$, consider the following cases:
(1) If $x, y \in Y$, then $x * y \in Y$.
(2) If $x \in Y$ and $y = \varphi_t(h)$, where $t \in B, h \in H$, then $x * y = x * \varphi_t(h) = x * h \in Y$ (since $H \subseteq Y$). Since H is a quasi-ideal of Y, then $h * y \in H$. Therefore $\varphi_t(h) * x \in E_\Phi(Y, H : B)$. This implies that $y * x = \varphi_t(h) * x = \varphi_t(h * x) \in E_\Phi(Y, H : B)$.
(3) If $x = \varphi_r(h)$ and $y = \varphi_t(k)$ are in $E_\Phi(Y, H : B)$, then $x * y = \varphi_r(h) * \varphi_t(k) = \varphi_{r*t}(h * k)$ is in $E_\Phi(Y, H : B)$, since $r * t \in B$ and $h * k \in H$.□

Corollary 4.5. Let X and Γ be BCK-algebras. If H is a quasi-ideal of X, then H is a quasi-ideal of $E_\Phi(X, K : \Gamma)$.

Proof. By corollary 4.3 H is a subalgebra. Let $h \in H, x \in X$ and $k \in K$. Then $h * x \in H$. Moreover $h * \varphi_\gamma(k) = \varphi_0(h) * \varphi_\gamma(k) = \varphi_{0*\gamma}(h * k) = \varphi_0(h * k) = h * k \in H$.□

Theorem 4.6. Let Y and K be quasi-ideals of a BCK-algebra X. If B is quasi-ideal of a BCK-algebra Γ and $H = Y \cap K$, then $E_\Phi(Y, H : B)$ is a quasi-ideal of $E_\Phi(X, K : \Gamma)$.

Proof. Let $\varepsilon \in E_\Phi(Y, H : B)$ and $e \in E_\Phi(X, K : \Gamma)$ be arbitrary elements, then we need to prove that $\varepsilon * e$ is in $E_\Phi(Y, H : B)$. Consider the following four cases: (1) If $\varepsilon = y \in Y$ and $e = x \in X$, then $\varepsilon * e = y * x \in Y$; since Y is a quasi-ideal of X. Therefore $\varepsilon * e$ is in $E_\Phi(Y, H : B)$. (2) If $\varepsilon = y \in Y$ and $e = \varphi_\gamma(k)$, for some $\gamma \in \Gamma$ and $k \in K$. In this case $\varepsilon * e = y * \varphi_\gamma(k) = y * k \in Y$, as above. (3) If $e = x \in X$ and $\varepsilon = \varphi_t(h)$, for some $t \in B$ and $h \in H$. Then $\varepsilon * e = \varphi_t(h) * x = \varphi_t(h * x)$. since H is a quasi-ideal of $X, h * x \in H$. This implies that $\varepsilon * e \in E_\Phi(Y, H : B)$. (4) Finally, if $\varepsilon = \varphi_t(h)$ and $e = \varphi_\gamma(k)$ for some $t \in B, h \in H, \gamma \in \Gamma$ and $k \in K$, then $\varepsilon * e = \varphi_t(h) * \varphi_\gamma(k) = \varphi_{t*\gamma}(h * k)$. Since B and H are quasi-ideals in Γ and X, respectively, we get $t * \gamma \in B$ and $h * k \in H$. Therefore $\varepsilon * e = \varphi_{t*\gamma}(h * k)$ is in $E_\Phi(X, K : \Gamma)$.□

For any two subsets $B \subseteq \Gamma$ and $Z \subseteq X$, let $\varphi_B(Z) = \{\varphi_t(z) : t \in B \text{ and } z \in Z\}$. Using this notation we get the following.

Theorem 4.7. If K is a quasi-ideal of a BCK-Algebra X and Γ is another BCK-algebra, then $F = K \cup \varphi_{\Gamma'}(K)$ is a quasi-ideal of $E_\Phi(X, K : \Gamma)$, where $\Gamma' = \Gamma - \{0\}$.

Proof. In fact $F = E_\Phi(K, K : \Gamma)$. Therefore, by putting $Y = K$ result follows by Theorem 4.6.□

Theorem 4.8. Let X and Γ be BCK-algebras. If J and E are ideals of X and Γ, respectively; and K is any quasi-ideal of X, then , we have $E_\Phi(J, J\cap K : E)$ is an ideal of $E_\Phi(X, K : \Gamma)$.

Proof. If ε and $e * \varepsilon$ are in $E_\Phi(J, J \cap K : E)$, for some e in $E_\Phi(X, K : \Gamma)$, then we need to prove that e is in $E_\Phi(J.J \cap K : E)$. Consider the following four cases: (1) If $e = x \in X$ and $\varepsilon = k \in J$, then $x * k = e * \varepsilon \in E_\Phi(J, J \cap K : E)$. This implies that $x * k \in J$. Therefore $x \in J$. Hence $e = x$ is in $E_\Phi(J, J \cap K : E)$. (2) If $e = x \in X$ and $\varepsilon = \varphi_\beta(t)$, for some $\beta \in E$ and $t \in J \cap K$, then $e * \varepsilon = x * \varphi_\beta(t) = x * t \in J$ (as above). Therefore $x \in J$. Hence $e = x$ is in $E_\Phi(J, J \cap K : E)$. (3) If $e = \varphi_\gamma(k)$, for some $\gamma \in \Gamma$ and $k \in K$ and, $\varepsilon = t \in J$, then $e * \varepsilon = \varphi_\gamma(k) * t = \varphi_\gamma(k * t)$. This implies that $\gamma \in E$ and $k * t \in J \cap K$. Therefore $\gamma \in E$ and $k \in J$. Hence $\gamma \in E$ and $k \in J \cap K$. Thus $e = \varphi_\gamma(k)$ is in $E_\Phi(J, J \cap K : E)$. (4) Let $e = \varphi_\gamma(k)$ and $\varepsilon = \varphi_\beta(t)$, for some $\gamma \in \Gamma, k \in K, \beta \in E$ and $t \in J \cap K$. Then $e * \varepsilon = \varphi_\gamma(k) * \varphi_\beta(t) = \varphi_{\gamma*\beta}(k * t) \in E_\Phi(J, J \cap K : E)$. This means that $\gamma * \beta \in E$ and $k * t \in J \cap K$. Since E is an ideal of Γ, we get $\gamma \in E$. Meanwhile $k * t \in J$ implies that $k \in J$. Therefore $\gamma \in E$ and $k \in H \cap K$. Thus $e = \varphi_\gamma(k)$ is in $E_\Phi(J, J \cap K : E)$.□

Using this theorem we get the following results.

Corollary 4.9. Let X and Γ be BCK-algebras such that J and E are ideals of X and Γ, respectively. If $K \supseteq K_1$ and they are quasi-ideals of X and J, respectively, then $E_\Phi(J, K_1 : E)$ is an ideal of $E_\Phi(X, K : \Gamma)$.

Proof. The proof is similar to that of Theorem 4.8, since $K_1 \subseteq J \cap K$.□

Corollary 4.10. Let X and Γ be BCK-algebras such that J and E are ideals of X and Γ, respectively. IF K is any quasi-ideal of X then, $J \cup E$ is an ideal of $E_\Phi(X, K : \Gamma)$.

Proof. In Corollary 4.9, let $K_1 = \{0\}$. Since $E_\Phi(J, \{0\} : E) = J \cup E$, result follows. We observe that E represents the set $\{\varphi_\gamma(0) : \gamma \in E\}$ for obvious reasons.□

Theorem 4.11. If $E_\Phi(X, K : \Gamma)$ and $E_\Psi(X, H : \Gamma)$ are the extensions of X by K and H respectively over Γ, then $E_\Psi(E_\Phi(X, K : \Gamma), H : \Gamma) = E_\Phi(E_\Psi(X, H : \Gamma), K : \Gamma)$.

Proof. Let $Y = E_\Phi(X, K : \Gamma)$ and $Z = E_\Psi(X, H : \Gamma)$. Since H is a quasi-ideal of Y, see Corollary 4.5, then $Y \cup \Psi_\Gamma(H)$ can be considered as the extension $E_\Psi(Y, H : \Gamma)$. Similarly $Z \cup \Phi_\Gamma(K)$ can be considered as the extension $E_\Phi(Z, K : \Gamma)$. Then setwise and by definition, we have

$$\begin{aligned}
E_\Psi(E_\Phi(X, K : \Gamma), H : \Gamma) &= E_\Psi(Y, H : \Gamma) \\
&= Y \cup \Psi_\Gamma(H) \\
&= E_\Phi(X, K : \Gamma) \cup \Psi_\Gamma(H) \\
&= [X \cup \Phi_\Gamma(K)] \cup \Psi_\Gamma(H) \\
&= [X \cup \Psi_\Gamma(H)] \cup \Phi_\Gamma(K) \\
&= E_\Psi(X, H : \Gamma) \cup \Phi_\Gamma(K) \\
&= Z \cup \Phi_\Gamma(K) \\
&= E_\Phi(Z, K : \Gamma) \\
&= E_\Phi(E_\Psi(X, H : \Gamma), K : \Gamma).
\end{aligned}$$

Moreover, for each $\varphi_\alpha(k)$ and $\psi_\gamma(h)$ in $E_\Psi(Y, H : \Gamma) = E_\Phi(Z, K : \Gamma)$, we have

$$\begin{aligned}
\psi_\gamma(h) * \varphi_\alpha(k) &= \psi_\gamma(h * \varphi_\alpha(k)) \\
&= \psi_\gamma(h * k)
\end{aligned}$$

is in $\Psi_\Gamma(H)$, for $H \subseteq X \subseteq Y$ and $H * K = H$. Similarly

$$\begin{aligned}\varphi_\alpha(k) * \psi_\gamma(h) &= \varphi_\alpha(k * \psi_\gamma(h))\\ &= \varphi_\alpha(k * h)\end{aligned}$$

is in $\Phi_\Gamma(K)$, for $K \subseteq X \subseteq Z$ and $K * H = K$. Similarly one can prove that $a * b$ is well defined by considering $a, b \in E_\Phi(E_\Psi(X, H : \Gamma), K : \Gamma = E_\Psi(E_\Phi(X, K : \Gamma), H : \Gamma)$. Hence

$$E_\Phi(E_\Psi(X, H : \Gamma), K : \Gamma) = E_\Psi(E_\Phi(X, K : \Gamma), H : \Gamma)$$

as BCK-algebras.□

Corollary 4.12. Let X, Y, Z, Γ, K and H be given as above. Then $Y = E_\Phi(X, K : \Gamma)$ and $Z = E_\Psi(X, H : \Gamma)$ are ideals in $E_\Psi(E_\Phi(X, K : \Gamma), H : \Gamma)$.

Proof. See Theorems 3.2,4.11.□

5. Isomorphism Theorems

Theorem 5.1 (First Isomorphism Theorem Part 1) Let X and Γ be BCK-algebras. If K is a quasi-ideal of X, then $E_\Phi(X, K : \Gamma)/X$ is isomorphic to Γ.

Proof. From Theorem 3.2, we have X is an ideal in $E_\Phi(X, K : \Gamma)$. Next we show that for any α in $\Gamma, \varphi_\alpha(K)$ is an equivalence class, that is an element of $E_\Phi(X, K : \Gamma)/X$. Let $\varphi_\alpha(x)$ and $\varphi_\beta(y)$ be in $E_\Phi(X, K : \Gamma)$; and $\sim$ be the equivalence relation: $\varphi_\alpha(x) \sim \varphi_\beta(y)$ if and only if $\varphi_\alpha(x) * \varphi_\beta(y)$ and $\varphi_\beta(y) * \varphi_\alpha(x)$ are in X. Therefore $\varphi_{\alpha*\beta}(x * y)$ and $\varphi_{\beta*\alpha}(y * x)$ are in X. This implies that $\alpha * \beta = 0 = \beta * \alpha$, whence $\alpha = \beta$. Therefore, if $\alpha \neq 0, [\varphi_\alpha(h)] = [\varphi_\beta(k)]$ if and only if $\alpha = \beta$. So $[\varphi_\alpha(h)] = \varphi_\alpha(K)$ is an equivalence class for each $\alpha \neq 0$. If $\alpha = 0$, then $\varphi_0(x) \sim \varphi_0(y)$ for each x and each y in X. Therefore $[\varphi_0(x)] = X$ is an equivalence class. It should be noted that $\varphi_\alpha(K) * \varphi_\beta(K) = \varphi_{\alpha*\beta}(K), \varphi_\alpha(K) * X = \varphi_\alpha(K * X) = \varphi_\alpha(K)$ and $X * \varphi_\alpha(K) = X * K = X$, whenever $\alpha \neq 0 \neq \beta$. Now let the function $f : E(X, K : \Gamma)/X \to \Gamma$ be defined as: $f(X) = f([\varphi_0(x)]) = 0$ and $f(\varphi_\alpha(K)) = f([\varphi_\alpha(k)]) = \alpha$. This implies that:

$$f(\varphi_\alpha(K) * \varphi_\beta(K)) = \alpha * \beta = f(\varphi_\alpha(K)) * f(\varphi_\beta(K)),$$

$$f(\varphi_\alpha(K) * X) = f(\varphi_\alpha(K)) = \alpha = \alpha * 0 = f(\varphi_\alpha(K)) * f(X) \quad \text{and}$$

$$f(X * \varphi_\alpha(K)) = f(X) = 0 = 0 * \alpha = f(X) * f(\varphi_\alpha(K)).$$

Hence f is an epimorphism (homomorphism and onto). To prove that f is a one to one function (injective), let $f(\varphi_\alpha(K)) = f(\varphi_\beta(K))$. Then by definition of $f, \alpha = \beta$. Thus $\varphi_\alpha(K) = \varphi_\beta(K)$. Therefore result follows.□

Definition 5.2. Let K be a quasi-ideal of a BCK-algebra X and Γ be any BCK-algebra. If $E_\Phi(X, K : \Gamma)$ is the extension of X by K over Γ, Z is any subset of Γ and $x \in X$, then $\Phi_Z(x)$ is defined to be $\{\varphi_\gamma(x) : \gamma \in Z\}$.

Lemma 5.3. If B is a subalgebra, an ideal or a quasi-ideal of Γ, then $\Phi_B(0)$ is a subalgebra, an ideal or a quasi-ideal of $E_\Phi(X, K : \Gamma)$ respectively. Moreover B is isomorphic to $\Phi_B(0)$.

Proof. It is easy to see that $\Phi_B(0) = E_\Phi(\{0\}, \{0\} : B)$. Then the first part of lemma follows directly by Lemma 4.4, Theorem 4.6 and Theorem 4.8. To prove the second part, let the function $\zeta : B \to \Phi_B(0)$ be given by $\zeta(\beta) = \varphi_\beta(0)$. It is clear that this map is one-to-one. If α and β are in B, then $\zeta(\alpha * \beta) = \varphi_{\alpha*\beta}(0) = \varphi_{\alpha*\beta}(0 * 0) = \varphi_\alpha(0) * \varphi_\beta(0) = \zeta(\alpha) * \zeta(\beta)$. This means that ζ is a homomorphism and result follows.□

Theorem 5.4 (First Isomorphism Theorem Part 2). Let X and Γ be BCK-algebras. If K is a quasi-ideal of X, then Γ can be embedded as an ideal in $E_\Phi(X,K:\Gamma)$ in which case $E_\Phi(X,K:\Gamma)/\Gamma$ is isomorphic to X.

Proof. From Lemma 5.3, Γ is isomorphic to $\Phi_\Gamma(0)$. Hence Γ can be considered as an ideal in $E_\Phi(X,K:\Gamma)$ for $\Phi_\Gamma(0)$ is an ideal in $E_\Phi(X,K:\Gamma)$, again by Lemma 5.3. So we can define $E_\Phi(X,K:\Gamma)/\Gamma$ which in fact is $E_\Phi(X,K:\Gamma)/\Phi_\Gamma(0)$. We note that the elements of $E_\Phi(X,K:\Gamma)/\Gamma$ are of the form $\Phi_\Gamma(x)$ and the operation is given as $\Phi_\Gamma(x)*\Phi_\Gamma(y)=\Phi_\Gamma(x*y)$, where x and y are in X. Let the map $\zeta: X\to E_\Phi(X,K:\Gamma)/\Gamma$ be defined as $\zeta(x)=\Phi_\Gamma(x)$. Let $\Phi_\Gamma(x)=\Phi_\Gamma(y)$. This implies that $\Phi_\Gamma(x)*\Phi_\Gamma(y)=\Phi_\Gamma(0)$. Therefore $\Phi_\Gamma(x*y)=\Phi_\Gamma(y*x)=\Phi_\Gamma(0)$. Hence $x*y=y*x=0$. This means that $x=y$. Therefore ζ is one to one. It is clear that ζ is onto. Hence ζ is a one-to-one correspondence. To prove that ζ is a homomorphism, let x and y be in X. Then $\zeta(x*y)=\Phi_\Gamma(x*y)=\Phi_\Gamma(x)*\Phi_\Gamma(y)=\zeta(x)*\zeta(y)$. This implies that ζ is a homorphism and theorem is proved.□

Theorem 5.5 (Second Isomorphism Theorem). If X and Γ are BCK-algebras and K is a quasi-ideal in X, then $E_\Phi(X,K:\Gamma)/\Gamma$ is an ideal in $E_\Psi(E_\Phi(X,K:\Gamma),X:\Gamma)/\Gamma$ and $[E_\Psi(E_\Phi(X,K:\Gamma),X:\Gamma)/\Gamma]/[E_\Phi(X,K:\Gamma)/\Gamma]$ is isomorphic to $E_\Psi(E_\Phi(X,K:\Gamma),X:\Gamma)/E_\Phi(X,K:\Gamma)$.

Proof. Using Theorem 5.1 and Theorem 5.4, we get $E_\Psi(E_\Phi(X,K:\Gamma),X:\Gamma)/\Gamma$ is isomorphic to $E_\Phi(X,K:\Gamma)$ and $E_\Phi(X,K:\Gamma)/\Gamma$ is isomorphic to X. Since X is an ideal in $E_\Phi(X,K:\Gamma)$, thus $E_\Phi(X,K:\Gamma)/\Gamma$ is an ideal in $E_\Psi(E_\Phi(X,K:\Gamma),X:\Gamma)/\Gamma$. Now

$$[E_\Psi(E_\Phi(X,K:\Gamma),X:\Gamma)/\Gamma]/[E_\Phi(X,K:\Gamma)/\Gamma]\cong E_\Phi(X,K:\Gamma)/X\cong\Gamma. \tag{1}$$

Moreover

$$E_\Psi(E_\Phi(X,K:\Gamma),X:\Gamma)/E_\Phi(X,K:\Gamma)\cong\Gamma. \tag{2}$$

From (1) and (2) above result follows.□

Lemma 5.6. If X and Γ are BCK-algebras; then $E_\Phi(X,\{0\}:\Gamma)$ is isomorphic to $X\cup\Gamma$.

Proof. We have $E_\Phi(X,K:\Gamma)=X\cup\Phi_\Gamma(K)$. Therefore $E_\Phi(X,\{0\}:\Gamma)=X\cup\Phi_\Gamma(0)$. This implies that $E_\Phi(X,\{0\}:\Gamma)$ is isomorphic to $X\cup\Gamma$, for $\Phi_\Gamma(0)$ is isomorphic to Γ, see Lemma 5.3□

Using this lemma we deduce Iseki's and Imai's theorem (see[7]) in the following corollary.

Corollary 5.7. Let X and Y be BCK-algebras such that $X\cap Y=\{0\}$, then $X\cup Y$ is a BCK-algebra.
Proof. $X\cup Y=E_\Phi(X,\{0\}:Y)$.□

Theorem 5.8. Let X and Γ be BCK-algebras. If K is an ideal of X, then K is an ideal of $E_\Phi(X,K:\Gamma)$. Moreover $E_\Phi(X,K:\Gamma)/K$ is isomorphic to $E_\Phi(X/K,[0]:\Gamma)$ which is isomorphic to $(X/K)\cup\Gamma$.

Proof. By Theorem 3.2, we have X is an ideal of $E_\Phi(X,K:\Gamma)$. Since K is an ideal of X, then K is an ideal of $E_\Phi(X,K:\Gamma)$. Let $\varphi_\alpha(k)$ and $\varphi_\beta(h)$ be in $E_\Phi(X,K:\Gamma)$; and $\sim$ be the equivalence relation: $\varphi_\alpha(k)\sim\varphi_\beta(h)$ if and only if $\varphi_\alpha(k)*\varphi_\beta(h)$ and $\varphi_\beta(h)*\varphi_\alpha(k)$ are in K. Then it is clear that the equivalence classes are the elements of $E_\Phi(X,K:\Gamma)/K$. Let us compute these classes; $\varphi_\alpha(k)*\varphi_\beta(h)$ and $\varphi_\beta(h)*\varphi_\alpha(k)$ are in K; if and only if $\varphi_{\alpha*\beta}(k*h)$ and $\varphi_{\beta*\alpha}(h*k)$ are in K; if and only if $\alpha*\beta=0=\beta*\alpha$ and $k*h,h*k\in K$; if and only if $\alpha=\beta$ and $k*h,h*k\in K$. This implies that whenever $\alpha\neq0$, we get $h,k\in K$ and therefore the equivalence class $[\varphi_\alpha(k)]=\Phi_\alpha(K)$, for any $\alpha\neq0$ and $k\in K$. Moreover $[\varphi_\alpha(k)]\cap X=\Phi_\alpha(K)\cap X=\emptyset$. Next we prove that any class in $E_\Phi(X,K:\Gamma)/K$ other than

$\Phi_\alpha(K), \alpha \neq 0$; is in X/K. This means that these classes are of the form $[\varphi_0(x)] = [x] \in X/K$. Let $\varphi_\alpha(k) \in [\varphi_0(x)]$, for some $\alpha \in \Gamma, k \in K$ and $x \in X$. Therefore $\varphi_\alpha(k) * \varphi_0(x)$ and $\varphi_0(x) * \varphi_\alpha(k)$ are in K. Hence $\varphi_{\alpha*0}(k*x)$ and $\varphi_{0*\alpha}(x*k)$ are in K. This implies that $\alpha = 0$ and $k*x, x*k \in K$. Therefore $k \in [x]$ as desired. Hence any class $[x]$ in X/K is a class in $E_\Phi(X, K : \Gamma)/K$. So we get $E_\Phi(X, K : \Gamma)/K = (X/K) \cup \{\Phi_\alpha(K) : \alpha \neq 0\}$ Since $K = [0]$, we have $(X/K) \cup \{\Phi_\alpha(K) : \alpha \neq 0\} = E_\Phi(X/K, [0] : \Gamma)$. This proves that $E_\Phi(X, K : \Gamma)/K = E_\Phi(X/K, [0] : \Gamma)$.□

Remark. It should be noted that

$$\begin{aligned}\Phi_\alpha(K) * \Phi_\beta(K) &= \{\varphi_\alpha(k) * \varphi_\beta(h) : k, h \in K\}\\ &= \{\varphi_{\alpha*\beta}(k*h) : k, h \in K\}\\ &= \{\varphi_{\alpha*\beta}(k') : k' \in K\}\\ &= \Phi_{\alpha*\beta}(K).\end{aligned}$$

And the product, in $E_\Phi(X, K : \Gamma)/K$, of $[x]$ and $[y]$ is the same as in X/K.

Theorem 5.9. For any BCK-algebras X and Γ, if H and K are ideals of X such that $H \subseteq K$, then H is an ideal in $E_\Phi(X, K : \Gamma)$ and $E_\Phi(X, K : \Gamma)/H$ is isomorphic to $E_\Phi(X/H, K/H : \Gamma)$.

Proof. Clear by the above theorem. □

Theorem 5.10. If X and Γ are BCK-algebras and K is a quasi-ideal in X, then $K \times \Gamma$ is embedded in $E_\Phi(X, K : \Gamma)$ as a quasi-ideal.

Proof. By Theorem 4.10, we have $F = K \cup \Phi_\Gamma(K) = E_\Phi(K, K : \Gamma)$ is a quasi-ideal in $E_\Phi(X, K : \Gamma)$. Now let the map $\zeta : K \times \Gamma \to F$ be given by $\zeta(k, \gamma) = \varphi_\gamma(k)$. Then it is clear that ζ is one-to-one and onto. To prove that ζ is a homomorphism, let (h, β) and $k, \gamma)$ be in $K \times \Gamma$, then $\zeta((h, \beta)*(k, \Gamma)) = \zeta(h*k, \beta*\gamma) = \varphi_{\beta*\gamma}(h*k) = \varphi_\beta(h)*\varphi_\gamma(k) = \zeta(h, \beta)*\zeta(k, \gamma)$. Therefore $K \times \Gamma$ is isomorphic to a quasi-ideal in $E_\Phi(X, K : \Gamma)$ and result follows.□

Theorem 5.11. If H and K are quasi-ideals of a BCK-algebra X and B is a quasi-ideal of a BCK-algebra Γ, then $H \cup B$ is embedded in $E_\Phi(X, K : \Gamma)$ as a quasi-ideal.

Proof. The proof is similar to that of Lemma 5.6.□

References

1. J. Ahsan and A. B. Thaheem, *On ideals in BCK-algebras*, Math. Seminar Notes, 5 (1977), 167-172.
2. J. Ahsan, E. Y. Deeba and A. B. Thaheem, *On prime ideals of BCK-algebras*, Math. Japonica 36(1991), 875-882.
3. M. Aslam and A. B. Thaheem, *On certain ideals in BCK-algebras*, Math. Japonica 36(1991), 895-906.
4. W. H. Cornich, *On Iseki' BCK-algebras*, Algebraic structures and applications Proc. of the First Western Australian Conference on Algebra, Marcel Dekker, Inc. New York, 101-122.
5. .
6. C. S. Hoo and P. V. Ramana Murty, *The ideals of bounded commutative BCK-algebras*, Math. Japonica 32 (1987), 723-233.
7. K. Iseki and E. Y. Imai, *On axiom system of propositional calculi*, Proc. Japan Academy 42(1966), 19-22.
8. K. Iseki, *On some ideals in BCK-algebras*, Seminar Notes 3(1975), 65-70.
9. K. Iseki and S. Tanaka, *Ideal Theory of BCK-algebras*, Math. Japonica 21(1976), 351-366.
10. K. Iseki, *On implicative BCK-algebras with condition (S)*, Math. Seminar Notes 5(1977), 249-253.
11. K. Iseki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japonica 23(1978), 1-26.
12. H. Jiang, *Computational methods in the stydy of finite BCK-algebras with low orders*, Kobe J. Math. 7(1990), 33-46.

13. M. Palasinski, *Ideals in BCK-algebra which are lower semilattice*, Math. Japonica 26(1981), 245-250.
14. M. Palasinski, *Decomposition of ideals in certain BCK-algebras*, Math. Seminar Notes 10 (1982), 467-471.
15. A. B. Thaheem, *Characterizations of certain ideals in BCK-algebras*, Math. Seminar Notes 6(1978), 475-481.
16. T. Traczyk, *On the variety of bounded commutative BCK-algebras*, Math. Japonica 24(1979), 283-292.
17. J. C. Varlet, *Relative annihilators in semilattices*, Bull. Austral. Math. Soc. 9(1973), 169-185.
18. H. Yutani, *Disjoint unions of BCK-algebras*, Matth. Seminar Notes 8(1980), 421-423.