CHARACTERIZATIONS OF USUAL AND CHAOTIC ORDER VIA FURUTA AND KANTOROVICH INEQUALITIES

MASATOSHI FUJII, MASASHI HASHIMOTO, YUKI SEO AND MASAHIRO YANAGIDA

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ABSTRACT. As a continuation of our previous papers, we shall show characterizations of usual order and chaotic order via operator equations and Kantorovich-type inequalities as applications of Furuta inequality.

1. INTRODUCTION

An operator means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and also T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. The following Theorem F is an extension of the celebrated Löwner-Heinz theorem: $A \geq B \geq 0$ *ensures* $A^\alpha \geq B^\alpha$ *for any* $\alpha \in [0, 1]$.

Theorem F (Furuta inequality [12]).

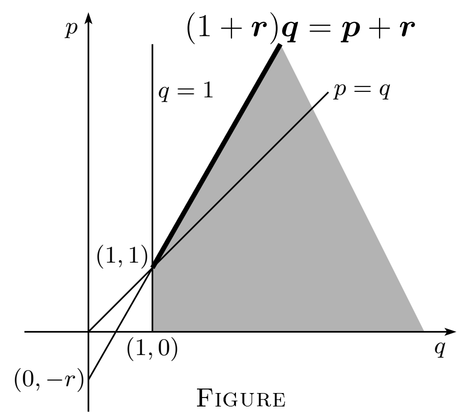
If $A \geq B \geq 0$*, then for each* $r \geq 0$*,*

(i) $$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ *and* $q \geq 1$ *with* $(1 + r)q \geq p + r$.



FIGURE

We remark that Theorem F yields Löwner-Heinz theorem when we put $r = 0$ in (i) or (ii) stated above. Alternative proofs of Theorem F are given in [6][29] and also an elementary one-page proof in [13]. It is shown in [34] that the domain of the parameters p, q and r drawn in Figure is the best possible for Theorem F.

The following Theorem G was established in [17] as an extention of Theorem F.

Theorem G ([17]). *If* $A \geq B \geq 0$ *with* $A > 0$*, then for each* $t \in [0, 1]$ *and* $p \geq 1$*,*

$$F_{p,t}(A, B, r, s) = A^{\frac{-r}{2}} \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} A^{\frac{-r}{2}}$$

is decreasing for $r \geq t$ *and* $s \geq 1$*, and* $F_{p,t}(A, A, r, s) \geq F_{p,t}(A, B, r, s)$*, that is, for each* $t \in [0, 1]$ *and* $p \geq 1$*,*

$$A^{1-t+r} \geq \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{1.1}$$

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holds for any $r \geq t$ *and* $s \geq 1$.

Ando-Hiai [2] established log majorization results and showed the following inequality which is equivalent to the main log majorization theorem: *If* $A \geq B \geq 0$ *with* $A > 0$, *then* $A^r \geq \{A^{\frac{r}{2}}(A^{\frac{-1}{2}}B^pA^{\frac{-1}{2}})^rA^{\frac{r}{2}}\}^{\frac{1}{p}}$ *holds for any* $p \geq 1$ *and* $r \geq 1$. Theorem G interpolates the inequality stated above by Ando-Hiai and Theorem F itself, and also extends results of [7][14] and [15]. A mean theoretic proof of Theorem G is shown in [9] and a one-page proof of (1.1) is shown in [21]. The best possibility of the outside exponents on both sides of (1.1) is shown in [11][35] and [37]. In [25], Furuta, Ito and one of the authors showed equivalence relation among the inequality (1.1), monotonicity of the function $F_{p,t}(A,B,r,s)$ in Theorem G, and related results.

Related to Löwner-Heinz theorem, the following proposition is also well known: $A \geq B \geq 0$ *does not always assure* $A^\alpha \geq B^\alpha$ *for any* $\alpha > 1$. Associated with this result, Furuta [20] showed the following result.

Theorem A.1 ([20]). *If* $A \geq B \geq 0$ *and* $MI \geq A \geq mI > 0$, *then*

$$\left(\frac{M}{m}\right)^{p-1} A^p \geq K_+(m,M,p)A^p \geq B^p \quad \text{for } p > 1,$$

where

$$K_+(m,M,p) = \frac{(p-1)^{p-1}}{p^p}\frac{(M^p-m^p)^p}{(M-m)(mM^p-m^pM)^{p-1}}. \tag{1.2}$$

We remark that Theorem A.1 is related to both Hölder-McCarthy inequality [31] and Kantorovich inequality: *If* $MI \geq A \geq mI > 0$, *then* $(A^2x,x) \leq \frac{(m+M)^2}{4mM}(Ax,x)^2$ *holds for every unit vector* x *in* H. The number $\frac{(m+M)^2}{4mM}$ is called Kantorovich constant and $K_+(m,M,2) = \frac{(m+M)^2}{4mM}$ where $K_+(m,M,p)$ is defined in (1.2), so that $K_+(m,M,p)$ is a generalization of Kantorovich constant. $K_+(m,M,p)$ is called Ky Fan-Furuta constant.

On the other hand, for positive and invertible operators A and B, the order defined by $\log A \geq \log B$ is called chaotic order. We remark that chaotic order is weaker than usual order $A \geq B$ since $\log t$ is an operator monotone function. In [38], as applications of the results in [20], Yamazaki and one of the authors showed the following characterizations of chaotic order associated with Kantorovich inequality.

Theorem B.1 ([38]). *Let* A *and* B *be positive and invertible operators satisfying* $MI \geq A \geq mI > 0$. *Then* $\log A \geq \log B$ *if and only if*

$$\frac{(m^p+M^p)^2}{4m^pM^p}A^p \geq B^p \quad \text{for all } p \geq 0. \tag{1.3}$$

Theorem B.2 ([38]). *Let* A *and* B *be positive and invertible operators satisfying* $MI \geq A \geq mI > 0$. *Then* $\log A \geq \log B$ *if and only if*

$$M_h(p)A^p \geq B^p \quad \text{for all } p > 0,$$

where $h = \frac{M}{m} > 1$ *and*

$$M_h(p) = \frac{h^{\frac{p}{h^p-1}}}{e\log h^{\frac{p}{h^p-1}}}. \tag{1.4}$$

We remark that $M_h(1) = \frac{(h-1)h^{\frac{1}{h-1}}}{e\log h}$ is called Specht's ratio [33], which is the best upper bound of the ratio of the arithmetic mean $A = \frac{1}{n}\sum_{i=1}^n x_i$ to the geometric mean $G = (\prod_{i=1}^n x_i)^{\frac{1}{n}}$ for positive real numbers x_i satisfying $M \geq x_i \geq m > 0$ for $i = 1, 2, \cdots, n$, that is, $\frac{(h-1)h^{\frac{1}{h-1}}}{e\log h}G \geq A \geq G$ holds where $h = \frac{M}{m} > 1$. In [3], J.I.Fujii, Furuta, Yamazaki and one of the authors showed a simple proof of Theorem B.2 by applying the results in [4] and [5].

In order to compare Theorem B.1 and Theorem B.2, we cite the following result.

Proposition B.3 ([38]). *Let $K_+(m,M,p)$ and $M_h(p)$ be defined in* (1.2) *and* (1.4), *respectively. Then $F(p,r,m,M)=K_+\left(m^r,M^r,\frac{p+r}{r}\right)$ is an increasing function of p, r and M, and also a decreasing function of m for $p>0$, $r>0$ and $M>m>0$. Moreover,*

$$\lim_{r\searrow 0} K_+\left(m^r,M^r,\frac{p+r}{r}\right)=M_h(p) \tag{1.5}$$

and

$$h^p=\left(\frac{M}{m}\right)^p\geq K_+\left(m^r,M^r,\frac{p+r}{r}\right)\geq M_h(p)>1 \tag{1.6}$$

hold for $p>0$, $r>0$ and $M>m>0$, where $h=\frac{M}{m}>1$.

It turns out by (1.6) of Proposition B.3 that Theorem B.2 gives a more precise sufficient condition for chaotic order than Theorem B.1 since

$$\frac{(m^p+M^p)^2}{4m^pM^p}=K_+(m^p,M^p,2)=K_+\left(m^p,M^p,\frac{p+p}{p}\right)\geq M_h(p)$$

holds for any $p>0$.

In [26], as an application of Theorem G, Furuta and one of the authors showed the following characterization of chaotic order via operator equations, which is an extension of [16, Theorem 2.1].

Theorem C.1 ([26]). *Let A and B be positive and invertible operators. Then the following assertions are mutually equivalent:*

(i) $\log A\geq\log B$.

(ii) *For each $\alpha\in[0,1]$, $p\geq 0$, $u\geq 0$ and $s\geq 1$ such that $(p+\alpha u)s\geq(1-\alpha)u$, there exists the unique invertible positive contraction T satisfying*

$$TA^{(p+\alpha u)s}T=\left(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}}\right)^s.$$

(iii) *For each $p\geq 0$, there exists the unique invertible positive contraction T satisfying*

$$TA^pT=B^p.$$

By applying Theorem C.1, they also showed the following result which includes Theorem B.1.

Theorem C.2 ([26]). *Let A and B be positive and invertible operators satisfying $MI\geq A\geq mI>0$. Then the following assertions are mutually equivalent:*

(i) $\log A\geq\log B$.

(ii) *For each $\alpha\in[0,1]$, $p\geq 0$ and $u\geq 0$,*

$$\frac{\left(M^{(p+\alpha u)s}+m^{(p+\alpha u)s}\right)^2}{4m^{(p+\alpha u)s}M^{(p+\alpha u)s}}A^{(p+\alpha u)s}\geq\left(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}}\right)^s$$

holds for all $s\geq 1$ and $(p+\alpha u)s\geq(1-\alpha)u$.

(iii) $\dfrac{(m^p+M^p)^2}{4m^pM^p}A^p\geq B^p$ *holds for all $p\geq 0$.*

In [27], by using Theorem G more generally, Yamazaki and one of the authors extended Theorem C.1 and Theorem C.2 as follows.

Theorem D.1 ([27]). *Let A and B be positive and invertible operators. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each natural number n, $\alpha \in [0,1]$, $p \geq 0$, $u \geq 0$, $s \geq 1$ and $r \geq 1-\alpha$ such that $\{nr+(n+1)\alpha\}u \geq (p+\alpha u)s$, there exists the unique invertible positive contraction T satisfying*

$$T(A^{\frac{(p+\alpha u)s+ru}{n+1}}T)^n = A^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}\left(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}}\right)^sA^{\frac{-(p+\alpha u)s+nru}{2(n+1)}}.$$

(iii) *For each natural number n, $\alpha \in [0,1]$, $p \geq 0$, $u \geq 0$ and $s \geq 1$ such that $(p+\alpha u)s \geq n(1-\alpha)u$, there exists the unique invertible positive contraction T satisfying*

$$T(A^{\frac{(p+\alpha u)s}{n}}T)^n = \left(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}}\right)^s.$$

(iv) *For each natural number n and $p \geq 0$, there exists the unique invertible positive contraction T satisfying*

$$T(A^{\frac{p}{n}}T)^n = B^p.$$

Theorem D.2 ([27]). *Let A and B be positive and invertible operators satisfying $MI \geq A \geq mI > 0$, and let $K_+(m,M,p)$ and $M_h(p)$ be defined in (1.2) and (1.4), respectively. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each natural number n, $\alpha \in [0,1]$, $p \geq 0$ and $u \geq 0$,*

$$K_+\left(m^{\frac{(p+\alpha u)s+ru}{n+1}}, M^{\frac{(p+\alpha u)s+ru}{n+1}}, n+1\right)A^{(p+\alpha u)s} \geq \left(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}}\right)^s$$

holds for all $s \geq 1$ and $r \geq 1-\alpha$ such that $\{nr+(n+1)\alpha\}u \geq (p+\alpha u)s$.

(iii) *For each natural number n, $\alpha \in [0,1]$, $p \geq 0$ and $u \geq 0$,*

$$K_+\left(m^{\frac{(p+\alpha u)s-\alpha u}{n}}, M^{\frac{(p+\alpha u)s-\alpha u}{n}}, n+1\right)A^{(p+\alpha u)s} \geq \left(A^{\frac{\alpha u}{2}}B^pA^{\frac{\alpha u}{2}}\right)^s$$

holds for all $s \geq 1$ and $(p+\alpha u)s \geq (n+\alpha)u$.

(iv) *For each natural number n and $p \geq nu \geq 0$,*

$$K_+\left(m^{\frac{p+ru}{n+1}}, M^{\frac{p+ru}{n+1}}, n+1\right)A^p \geq B^p$$

holds for all real number r such that $nru \geq p$.

(v) *$M_h(p)A^p \geq B^p$ holds for all $p > 0$, where $h = \frac{M}{m} > 1$.*

We remark that Theorem D.2 includes Theorem B.2 which is more precise than Theorem B.1 as mentioned above.

On the other hand, in [32], one of the authors showed the following results on usual order which are parallel to Theorem C.1 and Theorem C.2 on chaotic order.

Theorem E.1 ([32]). *Let A and B be positive and invertible operators. Then the following assertions are mutually equivalent:*

(i) $A \geq B$.

(ii) *For each $t \in [0,1]$, $p \geq 1$ and $s \geq 1$ such that $(p-t)s \geq t$, there exists the unique invertible positive contraction T satisfying*

$$TA^{(p-t)s}T = (A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^s.$$

(iii) *For each $p \geq 2$, there exists the unique invertible positive contraction T satisfying*

$$TA^{p-1}T = A^{\frac{-1}{2}}B^pA^{\frac{-1}{2}}.$$

Theorem E.2 ([32]). *Let A and B be positive and invertible operators satisfying $MI \geq A \geq mI > 0$. Then the following assertions are mutually equivalent:*

(i) $A \geq B$.

(ii) *For each $t \in [0,1]$,*

$$\frac{(m^{(p-t)s} + M^{(p-t)s})^2}{4m^{(p-t)s}M^{(p-t)s}} A^{(p-t)s} \geq (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s$$

holds for any $p \geq 1$, $s \geq 1$ and $(p-t)s \geq t$.

(iii) $$\left(\frac{(m^{(p-1)s} + M^{(p-1)s})^2}{4m^{(p-1)s}M^{(p-1)s}}\right)^{\frac{1}{s}} A^p \geq B^p$$

holds for any $s \geq 1$ and $p \geq \frac{1}{s} + 1$.

(iv) $\left(\frac{M}{m}\right)^{p-1} A^p \geq B^p$ *holds for all $p \geq 1$.*

He also showed the following result which is an immediate corollary of Theorem E.2.

Corollary E.3 ([32]). *If $A \geq B \geq 0$ and $MI \geq A \geq mI > 0$, then*

$$\frac{(m^{p-1} + M^{p-1})^2}{4m^{p-1}M^{p-1}} A^p \geq B^p \tag{1.7}$$

holds for all $p \geq 2$.

We remark that several related results are also shown in [10] and [30].

In this paper, by applying Theorem G and operator inequalities which are parallel to Theorem G, we shall show characterizations of usual order and chaotic order via operator equations and Kantorovich-type operator inequalities as extensions of our previous results.

2. Characterizations of usual and chaotic order

Firstly, we show the following Theorem 1 and Theorem 2 which are characterizations of usual and chaotic order via operator equations. We remark that Theorem 1 and Theorem 2 are extensions of Theorem E.1 and Theorem D.1, respectively.

Theorem 1. *Let A and B be positive and invertible operators, and $q > 1$. Then the following assertions are mutually equivalent:*

(i) $A \geq B$.

(ii-1) *For each $p \geq 0$, $t \geq 0$, $s \geq 0$ and $r \geq 0$ with $(p+t+r)q \geq (p+t)s + r$ and $(1+t+r)q \geq (p+t)s + r$, there exists the unique invertible positive contraction $T = T(p,t,s,r,q)$ satisfying*

$$T^{\frac{1}{2}}(T^{\frac{1}{2}} A^{\frac{(p+t)s+r}{q}} T^{\frac{1}{2}})^{q-1} T^{\frac{1}{2}} = A^{\frac{-(p+t)s+(q-1)r}{2q}} (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^s A^{\frac{-(p+t)s+(q-1)r}{2q}}. \tag{2.1}$$

(ii-2) *For each $p \geq 1$, $t \in [0,1]$, $s \geq 1$ and $r \geq t$ with $(1-t+r)q \geq (p-t)s + r$, there exists the unique invertible positive contraction $T = T(p,t,s,r,q)$ satisfying*

$$T^{\frac{1}{2}}(T^{\frac{1}{2}} A^{\frac{(p-t)s+r}{q}} T^{\frac{1}{2}})^{q-1} T^{\frac{1}{2}} = A^{\frac{-(p-t)s+(q-1)r}{2q}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{-(p-t)s+(q-1)r}{2q}}. \tag{2.2}$$

(iii-1) *For each $p \geq 0$, $t \geq 0$ and $s \geq 0$ with $(p+t)s \geq q$ and $p+t \geq 1$, there exists the unique invertible positive contraction $T = T(p,t,s,q)$ satisfying*

$$T^{\frac{1}{2}}(T^{\frac{1}{2}} A^{\frac{(p+t)s-1}{q-1}} T^{\frac{1}{2}})^{q-1} T^{\frac{1}{2}} = A^{\frac{-1}{2}} (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^s A^{\frac{-1}{2}}. \tag{2.3}$$

(iii-2) *For each $p \geq 1$, $t \in [0,1]$ and $s \geq 1$ with $(p-t)s \geq (q-1)t$, there exists the unique invertible positive contraction $T = T(p,t,s,q)$ satisfying*

$$T^{\frac{1}{2}}(T^{\frac{1}{2}}A^{\frac{(p-t)s}{q-1}}T^{\frac{1}{2}})^{q-1}T^{\frac{1}{2}} = (A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^s. \tag{2.4}$$

(iv) *For each $p \geq q$, there exists the unique invertible positive contraction $T = T(p,q)$ satisfying*

$$T^{\frac{1}{2}}(T^{\frac{1}{2}}A^{\frac{p-1}{q-1}}T^{\frac{1}{2}})^{q-1}T^{\frac{1}{2}} = A^{\frac{-1}{2}}B^pA^{\frac{-1}{2}}.$$

Theorem 2. *Let A and B be positive and invertible operators, and $q > 1$. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) *For each $p \geq 0$, $t \geq 0$, $s \geq 0$ and $r \geq 0$ with $(t+r)q \geq (p+t)s+r$, there exists the unique invertible positive contraction $T = T(p,t,s,r,q)$ satisfying*

$$T^{\frac{1}{2}}(T^{\frac{1}{2}}A^{\frac{(p+t)s+r}{q}}T^{\frac{1}{2}})^{q-1}T^{\frac{1}{2}} = A^{\frac{-(p+t)s+(q-1)r}{2q}}(A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^sA^{\frac{-(p+t)s+(q-1)r}{2q}}. \tag{2.5}$$

(iii) *For each $p \geq 0$, $t \geq 0$ and $s \geq 0$ there exists the unique invertible positive contraction $T = T(p,t,s,q)$ satisfying*

$$T^{\frac{1}{2}}(T^{\frac{1}{2}}A^{\frac{(p+t)s}{q-1}}T^{\frac{1}{2}})^{q-1}T^{\frac{1}{2}} = (A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^s. \tag{2.6}$$

(iv) *For each $p \geq 0$, there exists the unique invertible positive contraction $T = T(p,q)$ satisfying*

$$T^{\frac{1}{2}}(T^{\frac{1}{2}}A^{\frac{p}{q-1}}T^{\frac{1}{2}})^{q-1}T^{\frac{1}{2}} = B^p.$$

Secondly, we show the following Theorem 3 and Theorem 4 which are characterizations of usual and chaotic order via Kantorovich-type operator inequalities. We remark that Theorem 3 and Theorem 4 are extensions of Theorem E.2 and Theorem D.2, respectively.

Theorem 3. *Let A and B be positive and invertible operators satisfying $MI \geq A \geq mI > 0$, and let $K_+(m,M,p)$ be defined in (1.2). Then the following assertions are mutually equivalent:*

(i) $A \geq B$.

(ii-1)
$$K_+\left(m^{\frac{(p+t)s+r}{q}}, M^{\frac{(p+t)s+r}{q}}, q\right)A^{(p+t)s} \geq (A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^s$$

holds for all $p \geq 0$, $t \geq 0$, $s \geq 0$, $r \geq 0$ and $q > 1$ with $(p+t+r)q \geq (p+t)s+r$ and $(1+t+r)q \geq (p+t)s+r$.

(ii-2)
$$K_+\left(m^{\frac{(p-t)s+r}{q}}, M^{\frac{(p-t)s+r}{q}}, q\right)A^{(p-t)s} \geq (A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^s$$

holds for all $p \geq 1$, $t \in [0,1]$, $s \geq 1$, $r \geq t$ and $q > 1$ with $(1-t+r)q \geq (p-t)s+r$.

(iii-1)
$$K_+\left(m^{1+t}, M^{1+t}, \frac{(p+t)s}{1+t}\right)A^{(p+t)s} \geq (A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^s$$

holds for all $p \geq 1$, $t \geq 0$ and $s \geq 0$ with $(p+t)s > 1+t$.

(iii-2)
$$K_+(m, M, (p-t)s+t)A^{(p-t)s} \geq (A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^s$$

holds for all $p \geq 1$, $t \in [0,1]$ and $s \geq 1$ with $(p-t)s > 1-t$.

(iv-2)
$$\frac{(m^{(p-t)s-(1-t)} + M^{(p-t)s-(1-t)})^2}{4m^{(p-t)s-(1-t)}M^{(p-t)s-(1-t)}}A^{(p-t)s} \geq (A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^s$$

holds for all $p \geq 1$, $t \in [0,1]$ and $s \geq 1$ with $(p-t)s \geq 2-t$.

(v) *$K_+(m,M,p)A^p \geq B^p$ holds for all $p > 1$.*

(vi) $\left(\frac{M}{m}\right)^{p-1} A^p \geq B^p$ *holds for all* $p \geq 1$.

Theorem 4. *Let A and B be positive and invertible operators satisfying $MI \geq A \geq mI > 0$, and let $K_+(m, M, p)$ and $M_h(p)$ be defined in (1.2) and (1.4), respectively. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $$K_+\left(m^{\frac{(p+t)s+r}{q}}, M^{\frac{(p+t)s+r}{q}}, q\right) A^{(p+t)s} \geq (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^s$$

holds for all $p \geq 0$, $t \geq 0$, $s \geq 0$, $r \geq 0$ and $q > 1$ with $(t+r)q \geq (p+t)s + r$.

(iii) $$K_+\left(m^t, M^t, \frac{(p+t)s}{t}\right) A^{(p+t)s} \geq (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^s$$

holds for all $p \geq 0$, $t > 0$ and $s \geq 0$ with $(p+t)s > t$.

(iv) $$\frac{(m^{(p+t)s-t} + M^{(p+t)s-t})^2}{4m^{(p+t)s-t} M^{(p+t)s-t}} A^{(p+t)s} \geq (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^s$$

holds for all $p \geq 0$, $t \geq 0$ and $s \geq 0$ with $(p+t)s \geq 2t$.

(v) $M_h(p)A^p \geq B^p$ *holds for all* $p > 0$, *where* $h = \frac{M}{m} > 1$.

(vi) $\frac{(m^p + M^p)^2}{4m^p M^p} A^p \geq B^p$ *holds for all* $p \geq 0$.

Lastly, we show the following Corollary 5 which is an immediate corollary of Theorem 3. We remark that Corollary 5 is an extension of Corollary E.3.

Corollary 5. *If $A \geq B \geq 0$ with $MI \geq A \geq mI > 0$, then*

$$\frac{(m^{(p+t)s-(1+t)} + M^{(p+t)s-(1+t)})^2}{4m^{(p+t)s-(1+t)} M^{(p+t)s-(1+t)}} A^{(p+t)s} \geq (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^s$$

holds for all $p \geq 1$, $t \geq 0$ and $s \geq 0$ with $(p+t)s \geq 2(1+t)$.

3. Parallel inequalities to Theorem G under usual and chaotic order

First, we show the following result under usual order which is parallel to Theorem G.

Proposition 6. *If $A \geq B \geq 0$, then*

$$A^{\frac{(p+t)s+r}{q}} \geq \{A^{\frac{r}{2}} (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1}{q}} \tag{3.1}$$

holds for any $p \geq 0$, $t \geq 0$, $s \geq 0$, $r \geq 0$ and $q \geq 1$ with $(p+t+r)q \geq (p+t)s + r$ and $(1+t+r)q \geq (p+t)s + r$.

We remark that the case $p \geq 1$ and $s \geq \frac{1+t}{p+t}$ of Proposition 6 is shown in [19, Corollary 2.4]. We also remark that Proposition 6 yields Theorem F by putting $t = 0$ and $s = 1$.

Proof. The case $p = 0$ is trivial, so that we have only to prove the case $p > 0$. By (ii) of Theorem F, we have

$$A^{\frac{p+t}{q_1}} \geq (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^{\frac{1}{q_1}} \tag{3.2}$$

holds for any $p > 0$, $t \geq 0$ and $q_1 = \max\{1, \frac{p+t}{1+t}\}$. Put $A_1 = A^{\frac{p+t}{q_1}}$ and $B_1 = (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^{\frac{1}{q_1}}$, then $A_1 \geq B_1 \geq 0$ by (3.2). By again applying (ii) of Theorem F to A_1 and B_1, we have

$$A_1^{\frac{p_1+r_1}{q}} \geq (A_1^{\frac{r_1}{2}} B_1^{p_1} A_1^{\frac{r_1}{2}})^{\frac{1}{q}},$$

that is,

$$A^{\frac{(p+t)(p_1+r_1)}{qq_1}} \geq \{A^{\frac{(p+t)r_1}{2q_1}}(A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^{\frac{p_1}{q_1}}A^{\frac{(p+t)r_1}{2q_1}}\}^{\frac{1}{q}} \tag{3.3}$$

holds for any $p_1 \geq 0$, $r_1 \geq 0$ and $q \geq 1$ with $(1+r_1)q \geq p_1+r_1$. Put $p_1 = sq_1 \geq 0$ and $r_1 = \frac{rq_1}{p+t} \geq 0$, then (3.3) turns to be (3.1), and $(1+r_1)q \geq p_1+r_1$ is equivalent to $(p+t+r)q \geq (p+t)s+r$ and $(1+t+r)q \geq (p+t)s+r$. Hence the proof is complete. □

Next, we show the following result under chaotic order which is parallel to Proposition 6.

Proposition 7. *Let A and B be positive and invertible operators. If $\log A \geq \log B$, then*

$$A^{\frac{(p+t)s+r}{q}} \geq \{A^{\frac{r}{2}}(A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1}{q}} \tag{3.1}$$

holds for any $p \geq 0$, $t \geq 0$, $s \geq 0$, $r \geq 0$ and $q \geq 1$ with $(t+r)q \geq (p+t)s+r$.

We remark that the case $s \geq \frac{t}{p+t}$ of Proposition 7 is an immediate corollary of [28, Theorem 1], which is a function version of Proposition 7. We also remark that Proposition 7 yields (i) ⇒ (iii) of the following Theorem H by putting $t = 0$ and $s = 1$.

Theorem H ([7][8][15])**.** *Let A and B be positive and invertible operators. Then the following assertions are mutually equivalent:*

(i) $\log A \geq \log B$.
(ii) $A^p \geq (A^{\frac{p}{2}}B^pA^{\frac{p}{2}})^{\frac{1}{2}}$ *holds for all* $p \geq 0$.
(iii) $A^r \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}}$ *holds for all* $p \geq 0$ *and* $r \geq 0$.

(i)⇔(ii) of Theorem H is shown in [1], and a simple proof of (ii)⇒(i) is shown in [26]. Recently, Uchiyama [36] gave a simple and excellent proof of (i)⇒(iii) directly by Theorem F. Here we give a direct proof of Proposition 7 via Proposition 6 by using his technique and the idea in [23].

Proof of Proposition 7. Put $A_n = 1+\frac{1}{n}\log A$ and $B_n = 1+\frac{1}{n}\log B$ for each natural number n, then

$$A_n^n \to A \quad \text{and} \quad B_n^n \to B \quad \text{as } n \to \infty \tag{3.4}$$

since $(1+\frac{1}{n}X)^n \to e^X$ holds for any operator X. Let n be sufficiently large. Then $A_n \geq B_n \geq 0$, so that by applying Proposition 6 to A_n and B_n, we have

$$A_n^{\frac{(p_1+t_1)s+r_1}{q}} \geq \{A_n^{\frac{r_1}{2}}(A_n^{\frac{t_1}{2}}B_n^{p_1}A_n^{\frac{t_1}{2}})^sA_n^{\frac{r_1}{2}}\}^{\frac{1}{q}} \tag{3.5}$$

holds for any $p_1 \geq 0$, $t_1 \geq 0$, $s \geq 0$, $r_1 \geq 0$ and $q \geq 1$ with $(t_1+r_1)q \geq (p_1+t_1)s+r_1$. Put $p_1 = np \geq 0$, $t_1 = nt \geq 0$ and $r_1 = nr \geq 0$ in (3.5), then we have

$$A_n^{\frac{n\{(p+t)s+r\}}{q}} \geq \{A_n^{\frac{nr}{2}}(A_n^{\frac{nt}{2}}B_n^{np}A_n^{\frac{nt}{2}})^sA_n^{\frac{nr}{2}}\}^{\frac{1}{q}} \tag{3.6}$$

holds for any $p \geq 0$, $t \geq 0$, $s \geq 0$, $r \geq 0$ and $q \geq 1$ with $(t+r)q \geq (p+t)s+r$. By tending $n \to \infty$, (3.6) turns to be (3.1) by (3.4). Hence the proof is complete. □

4. Proofs of the results in Section 2

We use the following lemma in order to give proofs of the results.

Lemma F ([17])**.** *Let A be a positive and invertible operator and B be an invertible operator. Then*

$$(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

holds for any real number λ.

Proof of Theorem 1.
(i) $\Rightarrow$ (ii-1): By Proposition 6, (i) ensures

$$A^{\frac{(p+t)s+r}{q}} \geq \{A^{\frac{r}{2}}(A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1}{q}} \tag{3.1}$$

for $p \geq 0$, $t \geq 0$, $s \geq 0$, $r \geq 0$ and $q > 1$ with $(p+t+r)q \geq (p+t)s+r$ and $(1+t+r)q \geq (p+t)s+r$. Define $T = T(p,t,s,r,q)$ as follows:

$$T = A^{\frac{-\{(p+t)s+r\}}{2q}}\{A^{\frac{r}{2}}(A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1}{q}}A^{\frac{-\{(p+t)s+r\}}{2q}}.$$

Then T is an invertible positive contraction by (3.1), and

$$(A^{\frac{(p+t)s+r}{2q}}TA^{\frac{(p+t)s+r}{2q}})^q = A^{\frac{r}{2}}(A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^sA^{\frac{r}{2}} \tag{4.1}$$

holds. By applying Lemma F to the left hand side of (4.1), we have

$$(A^{\frac{(p+t)s+r}{2q}}TA^{\frac{(p+t)s+r}{2q}})^q = A^{\frac{(p+t)s+r}{2q}}T^{\frac{1}{2}}(T^{\frac{1}{2}}A^{\frac{(p+t)s+r}{q}}T^{\frac{1}{2}})^{q-1}T^{\frac{1}{2}}A^{\frac{(p+t)s+r}{2q}}, \tag{4.2}$$

so that (4.1) is equivalent to (2.1).

Uniqueness of T can be shown as follows: Assume that for each $p \geq 0$, $t \geq 0$, $s \geq 0$, $r \geq 0$ and $q > 1$ with $(p+t+r)q \geq (p+t)s+r$ and $(1+t+r)q \geq (p+t)s+r$, there exists an invertible positive contraction $S = S(p,t,s,r,q)$ satisfying

$$S^{\frac{1}{2}}(S^{\frac{1}{2}}A^{\frac{(p+t)s+r}{q}}S^{\frac{1}{2}})^{q-1}S^{\frac{1}{2}} = A^{\frac{-(p+t)s+(q-1)r}{2q}}(A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^sA^{\frac{-(p+t)s+(q-1)r}{2q}}. \tag{4.3}$$

By (2.1) and (4.3), we have

$$S^{\frac{1}{2}}(S^{\frac{1}{2}}A^{\frac{(p+t)s+r}{q}}S^{\frac{1}{2}})^{q-1}S^{\frac{1}{2}} = T^{\frac{1}{2}}(T^{\frac{1}{2}}A^{\frac{(p+t)s+r}{q}}T^{\frac{1}{2}})^{q-1}T^{\frac{1}{2}}. \tag{4.4}$$

By applying Lemma F to both sides of (4.4), (4.4) is equivalent to

$$(A^{\frac{(p+t)s+r}{2q}}SA^{\frac{(p+t)s+r}{2q}})^q = (A^{\frac{(p+t)s+r}{2q}}TA^{\frac{(p+t)s+r}{2q}})^q,$$

that is, $S = T$.
(i) $\Rightarrow$ (ii-2) can be proved in the same way as (i) $\Rightarrow$ (ii-1) by applying Theorem G instead of Proposition 6.
(ii-1) $\Rightarrow$ (iii-1): Put $r = \frac{(p+t)s-q}{q-1}$ in (ii-1), then $\frac{(p+t)s+r}{q} = \frac{(p+t)s-1}{q-1}$ and $\frac{-(p+t)s+(q-1)r}{2q} = \frac{-1}{2}$, so that (2.1) turns to be (2.3). And one condition $(1+t+r)q \geq (p+t)s+r$ holds and the other conditions $r \geq 0$ and $(p+t+r)q \geq (p+t)s+r$ are equivalent to $(p+t)s \geq q$ and $p+t \geq 1$, respectively. Hence we have (iii-1).
(ii-2) $\Rightarrow$ (iii-2): Put $r = \frac{(p-t)s}{q-1}$ in (ii-2), then $\frac{(p-t)s+r}{q} = \frac{(p-t)s}{q-1}$ and $\frac{-(p-t)s+(q-1)r}{2q} = 0$, so that (2.2) turns to be (2.4). And one condition $(1-t+r)q \geq (p-t)s+r$ holds and the other condition $r \geq t$ is equivalent to $(p-t)s \geq (q-1)t$. Hence we have (iii-2).
(iii-1) $\Rightarrow$ (iv): We have only to put $t = 0$ and $s = 1$ in (iii-1).
(iii-2) $\Rightarrow$ (iv): We have only to put $t = 1$ and $s = 1$ in (iii-2).
(iv) $\Rightarrow$ (i): Assume (iv). Then we have

$$\begin{aligned}(A^{\frac{p-1}{2(q-1)}}TA^{\frac{p-1}{2(q-1)}})^q &= A^{\frac{p-1}{2(q-1)}}T^{\frac{1}{2}}(T^{\frac{1}{2}}A^{\frac{p-1}{q-1}}T^{\frac{1}{2}})^{q-1}T^{\frac{1}{2}}A^{\frac{p-1}{2(q-1)}} && \text{by Lemma F}\\ &= A^{\frac{p-q}{2(q-1)}}B^pA^{\frac{p-q}{2(q-1)}} && \text{by (iv).}\end{aligned} \tag{4.5}$$

By taking the $\frac{1}{q}$-th power of both sides of (4.5), we have

$$A^{\frac{p-1}{q-1}} \geq A^{\frac{p-1}{2(q-1)}}TA^{\frac{p-1}{2(q-1)}} = (A^{\frac{p-q}{2(q-1)}}B^pA^{\frac{p-q}{2(q-1)}})^{\frac{1}{q}} \quad \text{for any } p \geq 0, \tag{4.6}$$

and the first inequality holds since $I \geq T > 0$. Put $p = q$ in (4.6), then we have $A \geq B$.

Consequently, the proof is complete. $\square$

Proof of Theorem 2.
(i) $\Rightarrow$ (ii) can be proved in the same way as (i) $\Rightarrow$ (ii-1) of Theorem 1 by applying Proposition 7 instead of Proposition 6.
(ii) $\Rightarrow$ (iii): Put $r = \frac{(p+t)s}{q-1}$ in (ii), then $\frac{(p+t)s+r}{q} = \frac{(p+t)s}{q-1}$ and $\frac{-(p+t)s+(q-1)r}{2q} = 0$, so that (2.5) turns to be (2.6), and the condition $(t+r)q \geq (p+t)s+r$ holds. Hence we have (iii).
(iii) $\Rightarrow$ (iv): We have only to put $t = 0$ and $s = 1$ in (iv).
(iv) $\Rightarrow$ (i): Assume (iv). Then we have

$$(4.7) \qquad \begin{aligned} (A^{\frac{p}{2(q-1)}} T A^{\frac{p}{2(q-1)}})^q &= A^{\frac{p}{2(q-1)}} T^{\frac{1}{2}} (T^{\frac{1}{2}} A^{\frac{p}{q-1}} T^{\frac{1}{2}})^{q-1} T^{\frac{1}{2}} A^{\frac{p}{2(q-1)}} && \text{by Lemma F} \\ &= A^{\frac{p}{2(q-1)}} B^p A^{\frac{p}{2(q-1)}} && \text{by (iv).} \end{aligned}$$

By taking the $\frac{1}{q}$-th power of both sides of (4.7), we have

$$(4.8) \qquad A^{\frac{p}{q-1}} \geq A^{\frac{p}{2(q-1)}} T A^{\frac{p}{2(q-1)}} = (A^{\frac{p}{2(q-1)}} B^p A^{\frac{p}{2(q-1)}})^{\frac{1}{q}} \quad \text{for any } p \geq 0,$$

and the first inequality holds since $I \geq T > 0$. By applying Löwner-Heinz theorem to (4.8), we have

$$(4.9) \qquad A^{\frac{pq}{n(q-1)}} \geq (A^{\frac{p}{2(q-1)}} B^p A^{\frac{p}{2(q-1)}})^{\frac{1}{n}}$$

holds for a natural number n such that $n \geq q$. Put $X = (A^{\frac{p}{2(q-1)}} B^p A^{\frac{p}{2(q-1)}})^{\frac{1}{n}}$, then we have

$$(4.10) \qquad \begin{aligned} \frac{A^{\frac{pq}{n(q-1)}} - I}{p} &\geq \frac{(A^{\frac{p}{2(q-1)}} B^p A^{\frac{p}{2(q-1)}})^{\frac{1}{n}} - I}{p} \qquad \text{by (4.9)} \\ &= \frac{X - I}{p} = \frac{(X^n - I)(X^{n-1} + \cdots + X + I)^{-1}}{p} \\ &= \left(\frac{A^{\frac{p}{2(q-1)}} (B^p - I) A^{\frac{p}{2(q-1)}}}{p} + \frac{A^{\frac{p}{q-1}} - I}{p} \right) (X^{n-1} + \cdots + X + I)^{-1}. \end{aligned}$$

Tending $p \searrow 0$ in (4.10), we have

$$\frac{q}{n(q-1)} \log A \geq \frac{1}{n} \left(\log B + \frac{1}{q-1} \log A \right)$$

since $X \to I$ as $p \searrow 0$, so that $\log A \geq \log B$.

Consequently, the proof is complete. □

We remark that the idea of factorization which we use in the above proof of (iv) $\Rightarrow$ (i) is due to Furuta [24][22].

Proof of Theorem 3.
(i) $\Rightarrow$ (ii-1): By Proposition 6, (i) ensures

$$(3.1) \qquad A^{\frac{(p+t)s+r}{q}} \geq \{A^{\frac{r}{2}} (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1}{q}}$$

for $p \geq 0$, $t \geq 0$, $s \geq 0$, $r \geq 0$ and $q > 1$ with $(p+t+r)q \geq (p+t)s+r$ and $(1+t+r)q \geq (p+t)s+r$. Put $A_1 = A^{\frac{(p+t)s+r}{q}}$ and $B_1 = \{A^{\frac{r}{2}} (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1}{q}}$, then $A_1 \geq B_1 > 0$ by (3.1) and $MI \geq A \geq mI > 0$ assures $M^{\frac{(p+t)s+r}{q}} I \geq A_1 \geq m^{\frac{(p+t)s+r}{q}} I > 0$. By applying Theorem A.1 to A_1 and B_1, we have

$$K_+ \left(m^{\frac{(p+t)s+r}{q}}, M^{\frac{(p+t)s+r}{q}}, q \right) A_1^q \geq B_1^q,$$

so that we have (ii-1).

(i) $\Rightarrow$ (ii-2) can be proved in the same way as (i) $\Rightarrow$ (ii-1) by applying the following inequality which is easily obtained by Theorem G instead of Proposition 6:

$$A^{\frac{(p-t)s+r}{q}} \geq \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1}{q}}$$

for $p \geq 1$, $t \in [0,1]$, $s \geq 1$, $r \geq t$ and $q > 1$ with $(1-t+r)q \geq (p-t)s+r$.
(ii-1) $\Rightarrow$ (iii-1): We have only to put $r = 0$ and $q = \frac{(p+t)s}{1+t}$ in (ii-1).
(ii-2) $\Rightarrow$ (iii-2): We have only to put $r = t$ and $q = (p-t)s+t$ in (ii-2).
(ii-2) $\Rightarrow$ (iv-2): We have only to put $r = (p-t)s-2(1-t)$ and $q = 2$ in (ii-2).
(iii-1) $\Rightarrow$ (v): We have only to put $t = 0$ and $s = 1$ in (iii-1).
(iii-2) $\Rightarrow$ (v): We have only to put $t = 0$ and $s = 1$ in (iii-2).
(iv-2) $\Rightarrow$ (vi): In (iv-2), put $t = 1$ and apply Löwner-Heinz theorem for $\frac{1}{s} \in [0,1]$, then we have

$$\left(\frac{(m^{(p-1)s}+M^{(p-1)s})^2}{4m^{(p-1)s}M^{(p-1)s}}\right)^{\frac{1}{s}} A^p \geq B^p$$

holds for any $s \geq 1$ and $p \geq \frac{1}{s}+1$, which implies (vi) by Theorem E.2.
(v) $\Rightarrow$ (i): Let $p \searrow 1$ in (v), then we have (i) since

$$\left(\frac{M}{m}\right)^{p-1} \geq K_+(m,M,p) > 1$$

holds by Theorem A.1.
(vi) $\Rightarrow$ (i): We have only to put $p = 1$ in (vi).

Consequently, the proof is complete. □

Proof of Theorem 4.
(i) $\Rightarrow$ (ii) can be proved in the same way as (i) $\Rightarrow$ (ii-1) of Theorem 3 by applying Proposition 7 instead of Proposition 6.
(ii) $\Rightarrow$ (iii): We have only to put $r = 0$ and $q = \frac{(p+t)s}{t}$ in (ii).
(ii) $\Rightarrow$ (iv): We have only to put $r = (p+t)s-2t$ and $q = 2$ in (ii).
(iii) $\Rightarrow$ (v): Put $s = 1$ and let $t \searrow 0$ in (iii), then we have (v) since

$$K_+\left(m^t, M^t, \frac{p+t}{t}\right) \to M_h(p)$$

holds for each $p > 1$ by (1.5) of Proposition B.3.
(iv) $\Rightarrow$ (vi): We have only to put $t = 0$ and $s = 1$ in (ii).
(v) $\Rightarrow$ (i) and (vi) $\Rightarrow$ (i) have been already shown in Theorem B.2 and Theorem B.1, respectively, so that the proof is complete. □

Proof of Corollary 5. We have only to put $r = (p+t)s-2(1+t)$ and $q = 2$ in (ii-1) of Theorem 3. □

5. Concluding remarks

Remark 1. Practically, Theorem D.1 and Theorem D.2 were obtained in [27] by applying the following result which was shown in [18, Theorem 2.2] as an application of Theorem G.

Proposition I ([18])**.** *Let A and B be positive and invertible operators. If $\log A \geq \log B$, then*

$$A^{u(\alpha+r)} \geq \{A^{\frac{ur}{2}}(A^{\frac{u\alpha}{2}}B^pA^{\frac{u\alpha}{2}})^sA^{\frac{ur}{2}}\}^{\frac{u(\alpha+r)}{(p+u\alpha)s+ur}}$$

holds for any $u \geq 0$, $p \geq 0$, $\alpha \in [0,1]$, $s \geq 1$ and $r \geq 1-\alpha$.

By rewriting αu and ru with t and r, respectively, and considering the conditions, it turns out that Proposition I can be rewritten as follows:

Proposition I'. *Let A and B be positive and invertible operators. If $\log A \geq \log B$, then*

$$A^{t+r} \geq \{A^{\frac{r}{2}}(A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{t+r}{(p+t)s+r}}$$

holds for any $p \geq 0$, $t \geq 0$, $s \geq 1$ and $r \geq 0$.

We remark that Proposition 7, which is applied in the proofs of Theorem 3 and Theorem 4, includes Proposition I' as the case $s \geq 1$.

Remark 2. Theorem B.1 and Corollary E.3 can be considered as parallel results under usual order and chaotic order, respectively. By comparing them, the scalar $\frac{(m^{p-1}+M^{p-1})^2}{4m^{p-1}M^{p-1}}$ of (1.7) can be obtained by replacing p with $p-1$ in the scalar $\frac{(m^p+M^p)^2}{4m^pM^p}$ of (1.3). Hence one might expect that the following result holds under usual order as a parallel result to Theorem B.2 under chaotic order.

Conjecture. *Let A and B be positive operators satisfying $MI \geq A \geq mI > 0$. Then $A \geq B$ implies*

$$M_h(p-1)A^p \geq B^p \quad \text{for all } p > 1,$$

where $h = \frac{M}{m} > 1$ and $M_h(p)$ is defined in (1.4).

But we have a counterexample to this conjecture. Put

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

then $A \geq B \geq 0$. And $m = \frac{3-\sqrt{5}}{2}$ and $M = \frac{3+\sqrt{5}}{2}$, so that $h = \frac{M}{m} = \frac{3+\sqrt{5}}{3-\sqrt{5}}$. Then we have $M_h(1) = 1.55441$ and $M_h(2) = 4.77748$. On the other hand, $\alpha A^2 \geq B^2$ holds if and only if $\alpha \geq 2$, and $\beta A^3 \geq B^3$ holds if and only if $\beta \geq 5$. Therefore $M_h(1)A^2 \not\geq B^2$ and $M_h(2)A^3 \not\geq B^3$.

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(Masatoshi Fujii) Department of Mathematics, Osaka Kyoiku University, 4-698-1 Asahigaoka, Kashiwara, Osaka 582-8582, Japan

E-mail address: `mfujii@cc.osaka-kyoiku.ac.jp`

(Masashi Hashimoto) Yugawara Highschool, 1576-31 Yoshihama, Yugawara, Ashigarashimogun, Kanagawa 259-0312, Japan

E-mail address: `mhashi@am.kagu.sut.ac.jp`

(Yuki Seo) Tennoji Branch, Senior Highschool, Osaka Kyoiku University, 4-88 Minami-Kawahori, Tennoji, Osaka 543-0054, Japan

E-mail address: `yukis@cc.osaka-kyoiku.ac.jp`

(Masahiro Yanagida) Department of Applied Mathematics, Faculty of Science, Science University of Tokyo, 1-3 Kagurazaka, Shinjuku, Tokyo 162-8601, Japan

E-mail address: `yanagida@am.kagu.sut.ac.jp`