AN APPLICATION OF A CHARACTERIZATION OF OPERATOR ORDER TO p-HYPONORMAL OPERATORS

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Abstract. The Furuta inequality says that it characterizes the operator order: For $A \geq B \geq 0$, $A \geq C \geq B$ if and only if

$$(A^{\frac{r}{2}}C^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (C^{\frac{r}{2}}C^pC^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}C^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

holds for all $p, r \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

From the viepoint of this, we give some characterizations of p-hyponormal operators and simplify the privious characterization by one of the authors. Furthermore we characterize log-hyponormal operators.

1. Introduction. Throughout this note, an operator means a bounded linear operator acting on a Hilbert space H. An operator T on H is said to be positive (in symbol: $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. The order $A \geq B$ among selfadjoint operators are defined by $A - B \geq 0$.

As a generalization of hyponormal operators, an operator T is said to be p-hyponormal for $p > 0$ if $(T^*T)^p \geq (TT^*)^p$. (The 1-hyponormality is the hyponormality.) In [1], Aluthge discussed the growth of p-hyponormality by introducing the Aluthge transform: $T = U|T| \to \tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, where $T = U|T|$ is the polar decomposition of T, cf. [14]. As a matter of fact, he proved that if T is p-hyponormal for $p \in (0, 1)$, then $\tilde{T}$ is $\min\{p+\frac{1}{2}, 1\}$-hyponormal. It is a nice example of applications of the Furuta inequality. For the sake of convenience, we cite the Furuta inequality [10].

The Furuta inequality

If $A \geq B \geq 0$, then for each $r \geq 0$,

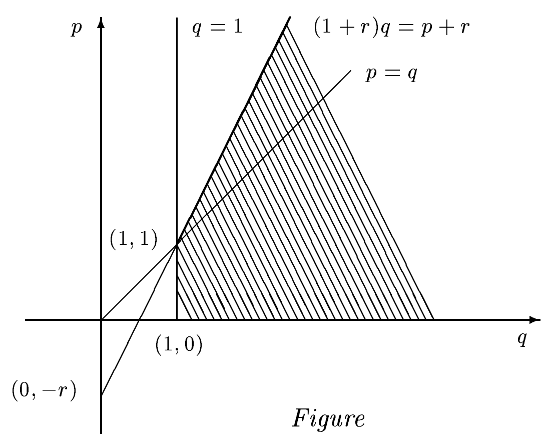
(i) $\left(B^{\frac{r}{2}}A^pB^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(B^{\frac{r}{2}}B^pB^{\frac{r}{2}}\right)^{\frac{1}{q}}$

and

(ii) $\left(A^{\frac{r}{2}}A^pA^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1}{q}}$

hold for $p \geq 0$ and $q \geq 1$ with

$$(1+r)q \geq p+r.$$



Figure

The original proof of the Furuta inequality is in [10], a mean theoretic proofs in [4], [15] and a one-page proof in [11]. The domain drawn for p, q and r in the Figure is the best possible one for (i) and (ii) of it in [17].

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Now the Furuta inequality says that it characterizes the operator order as follows, see also [2]:

Theorem F. *Suppose that $A \geq B \geq 0$. Then $A \geq C \geq B$ if and only if*

$$(C^{\frac{r}{2}} A^p C^{\frac{r}{2}})^{\frac{1}{q}} \geq (C^{\frac{r}{2}} C^p C^{\frac{r}{2}})^{\frac{1}{q}} \geq (C^{\frac{r}{2}} B^p C^{\frac{r}{2}})^{\frac{1}{q}} \tag{1}$$

holds for all $p, r \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

From the viepoint of this, one of the authors gave a characterization of p-hyponormal operators in [16]. In this note, we give some characterizations of p-hyponormal operators and simplify the privious characterization in [16]. Furthermore we characterize log-hyponormal operators from our view point. In addition, we propose an alternative characterization of it in terms of a norm inequality.

2. Characterizations of hyponormal operators. In the below, $T = U|T|$ means the polar decomposition of an operator T. The Furuta inequality implies the following characterization of hyponormal operators:

Lemma 1. *The following statements are mutually equivalent for an operator $R = U|R|$ with $U^*U = UU^*$:*

(1) R is hyponormal.

(2) $U^*|R|^2U \geq |R|^2 \geq U|R|^2U^* = |R^*|^2$.

(3) $U^*(|R^*|^r|R|^{2k}|R^*|^r)^{\frac{1}{q}}U \geq |R|^{\frac{2(k+r)}{q}} \geq (|R|^r|R^*|^{2k}|R|^r)^{\frac{1}{q}}$ for all $k,\ r \geq 0$ and $q \geq 1$ with $(1+r)q \geq k+r$.

(4) $(|R^*|^r|R|^{2k}|R^*|^r)^{\frac{1}{q}} \geq |R^*|^{\frac{2(k+r)}{q}} \geq U(|R|^r|R^*|^{2k}|R|^r)^{\frac{1}{q}}U^*$ for all $k,\ r \geq 0$ and $q \geq 1$ with $(1+r)q \geq k+r$.

Proof. We first note that $R^* = U^*|R^*|$, $|R^*|^s = U|R|^sU^*$ and $(U^*|R|^2U)^s = U^*|R|^{2s}U$ (by $U^*U = UU^*$) for all $s \geq 0$. So it is obvious that (1) implies (2). Next, if (2) holds, then the Furuta inequality ensures that

$$(|R|^r(U^*|R|^2U)^k|R|^r)^{\frac{1}{q}} \geq |R|^{\frac{2(k+r)}{q}} \geq (|R|^r|R^*|^{2k}|R|^r)^{\frac{1}{q}}.$$

Moreover the left hand side of the above is rephrased as

$$(|R|^r(U^*|R|^2U)^k|R|^r)^{\frac{1}{q}} = U^*(U|R|^rU^*|R|^{2k}U|R|^rU^*)^{\frac{1}{q}}U = U^*(|R^*|^r|R|^{2k}|R^*|^r)^{\frac{1}{q}}U,$$

so that (3) is proved. Finally (3) $\Leftrightarrow$ (4) follows from $U^*U = UU^*$.

The following characterization of hyponormal operators follows from Lemma 1.

Theorem 2. *Let $R = UH$ with $U^*U = UU^*$, where $H = |R|$, $\tilde{R} = H^kUH^r$ and $\hat{R} = H^rUH^k$ for given $k, r \geq 0$. Then R is hyponormal if and only if*

$$(\tilde{R}^*\tilde{R})^{\frac{1}{q}} \geq H^{\frac{2(k+r)}{q}} \geq (\hat{R}\hat{R}^*)^{\frac{1}{q}}$$

holds for all $k,\ r \geq 0$ and $q \geq 1$ with $(1+r)q \geq k+r$.

3. Characterizations of p-hyponormal operators. As an application of Theorem 2, we have the following characterizations of p-hyponormal operators.

Theorem 3. *Let $T = U|T|$ with $U^*U = UU^*$. For given $p \in (0,1]$ and $r \geq 0$, put $\tilde{T}_k = |T|^{kp}U|T|^{rp}$ and $\hat{T}_k = |T|^{rp}U|T|^{kp}$ for $k \geq 1$. Then T is p-hyponormal if and only if*

$$(\tilde{T}_k^*\tilde{T}_k)^{\frac{1}{q}} \geq |T|^{\frac{2p(k+r)}{q}} \geq (\hat{T}_k\hat{T}_k^*)^{\frac{1}{q}}$$

holds for all k, $r \geq 0$ and $q \geq 1$ with $(1+r)q \geq k+r$. In this case, both $\tilde{T}_k$ and $\hat{T}_k$ are $\frac{1}{q}$-hyponormal.

Proof. We note that T is p-hyponormal if and only if $R = U|T|^p$ is hyponormal. Applying Theorem 2 to $H = |T|^p$, we have the conclusion. In this case, we have also

$$(\hat{T}_k^*\hat{T}_k)^{\frac{1}{q}} \geq |T|^{\frac{2p(k+r)}{q}} \geq (\tilde{T}_k\tilde{T}_k^*)^{\frac{1}{q}}$$

by changing k and r. Combining them, both $\tilde{T}_k$ and $\hat{T}_k$ are $\frac{1}{q}$-hyponormal.

We have to remark that Theorem 3 is essentially as same as the theorem in [16]. As a matter of fact, Theorem 3 is corresponding to (1) and (2) in the theorem, and what correspnds to (3) and (4) in it is the following:

Theorem 4. *Let $T = U|T|$ with $U^*U = UU^*$. For given $p \in (0,1]$ and $r \geq 0$, put $\tilde{S}_k = |T^*|^{kp}U|T^*|^{rp}$ and $\hat{S}_k = |T^*|^{rp}U|T^*|^{kp}$ for $k \geq 1$. Then T is p-hyponormal if and only if*

$$(\tilde{S}_k^*\tilde{S}_k)^{\frac{1}{q}} \geq |T^*|^{\frac{2p(k+r)}{q}} \geq (\hat{S}_k\hat{S}_k^*)^{\frac{1}{q}}$$

holds for all k, $r \geq 0$ and $q \geq 1$ with $(1+r)q \geq k+r$. In this case, both $\tilde{S}_k$ and $\hat{S}_k$ are $\frac{1}{q}$-hyponormal.

Proof. We use (4) in Lemma 1 instead of (3) with the fact that

$$U(|R|^r|R^*|^{2k}|R|^r)^{\frac{1}{q}}U^* = (|R^*|^rU|R^*|^{2k}U^*|R^*|^r)^{\frac{1}{q}}.$$

Applying Lemma 1 to $R = U|T|^p$, we have the conclusion because $|R^*|^s = U|R|^sU^* = U|T|^{sp} = |T^*|^{sp}$.

In addition, since

$$(\hat{S}_k^*\hat{S}_k)^{\frac{1}{q}} \geq |T^*|^{\frac{2p(k+r)}{q}} \geq (\tilde{S}_k\tilde{S}_k^*)^{\frac{1}{q}}$$

holds for all k, $r \geq 0$ and $q \geq 1$ with $(1+r)q \geq k+r$, both $\tilde{S}_k$ and $\hat{S}_k$ are $\frac{1}{q}$-hyponormal.

4. Monotonicity of operator functions. In this section, we consider the results in the preceding section from the viewpoint of operator functions. For this, we cite the following result by Furuta [12], see also [5].

Theorem A. *If $A \geq B \geq 0$ and $r \geq 0$, then the operator function*

$$F(k) = (B^rA^kB^r)^{\frac{1+2r}{k+2r}}$$

is monotone increasing for $k \geq 1$, and

$$G(k) = (A^rB^kA^r)^{\frac{1+2r}{k+2r}}$$

is monotone decreasing for $k \geq 1$.

Theorem 5. *Let $T = U|T|$ be a p-hyponormal operator for some $p \in (0,1]$ with $U^*U = UU^*$, and for a given $r \geq 1$, $\tilde{T}_k$, $\hat{T}_k$, $\tilde{S}_k$ and $\hat{S}_k$ as in Theorems 3 and 4. Then the following operator functions on $[1,\infty)$ are monotone;*

(1) $\tilde{f}(k) = (\tilde{T}_k^*\tilde{T}_k)^{\frac{1+r}{k+r}}$ is increasing.

(2) $\hat{f}(k) = (\hat{T}_k\hat{T}_k^*)^{\frac{1+r}{k+r}}$ is decreasing.

(3) $\tilde{g}(k) = (\tilde{S}_k^*\tilde{S}_k)^{\frac{1+r}{k+r}}$ is increasing.

(4) $\hat{g}(k) = (\hat{S}_k\hat{S}_k^*)^{\frac{1+r}{k+r}}$ is decreasing.

Proof. We first note that $U^*|T|^{2p}U \geq |T|^{2p} \geq U|T|^{2p}U^* = |T^*|^{2p}$ by Lemma 1. In order to apply Theorem A to each function, we see them:

$$\tilde{f}(k) = (|T|^{rp}U^*|T|^{2kp}U|T|^{rp})^{\frac{1+r}{k+r}} = ((|T|^{2p})^{\frac{r}{2}}(U^*|T|^{2p}U)^k(|T|^{2p})^{\frac{r}{2}})^{\frac{1+r}{k+r}}.$$

Similarly we have

$$\hat{f}(k) = (|T|^{rp}U|T|^{2kp}U^*|T|^{rp})^{\frac{1+r}{k+r}} = (|T|^{rp}|T^*|^{2kp}|T|^{rp})^{\frac{1+r}{k+r}},$$

$$\tilde{g}(k) = (|T^*|^{rp}U^*|T^*|^{2kp}U|T^*|^{rp})^{\frac{1+r}{k+r}} = (|T^*|^{rp}|T|^{2kp}|T^*|^{rp})^{\frac{1+r}{k+r}}$$

and

$$\begin{aligned}\hat{g}(k) &= (|T^*|^{rp}U|T^*|^{2kp}U^*|T^*|^{rp})^{\frac{1+r}{k+r}} = (U|T|^{rp}|T^*|^{2kp}|T|^{rp}U^*)^{\frac{1+r}{k+r}} \\ &= U(|T|^{rp}|T^*|^{2kp}|T|^{rp})^{\frac{1+r}{k+r}}U^* = U\hat{f}(k)U^*.\end{aligned}$$

Hence it ensures the conclusion of Theorem 5.

5. Log-hyponormal operators. In [8], we introduced the chaotic order among positive invertible operators which is weaker than the operator order by $\log A \geq \log B$, in symmbol, $A \gg B$. An invertible operator T is said to be log-hyponormal if $\log T^*T \geq \log TT^*$, [18]. Since $\log x = \lim_{h\to+0} \frac{x^h-1}{h}$, the log-hyponormality is regarded as the 0-hyponomality sometimes. In Section 3, we characterized p-hyponormality. We now show that the log-hyponormality has similar characterizations to those of p-hyponormality. For this, we cite the Furuta inequality for the chaotic order, which has the following similar formulation to the Furuta inequality, [13] and also [5], [3]. We denote by $A > 0$ if A is positive invertible.

Theorem B. *Suppose that $A \gg B$ for $A, B > 0$. Then $A \gg C \gg B$ if and only if*

$$(A^{\frac{r}{2}}C^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (C^{\frac{r}{2}}C^pC^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}C^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

holds for all $p, r \geq 0$ and $q \geq 1$ with $rq \geq p + r$.

Theorem 6. *The following statements are mutually equivalent for an invertible operator $R = U|R|$:*

(1) R is log-hyponormal.

(2) $U^*|R|U \gg |R| \gg |R^*|$, or $U^*|R|^2U \gg |R|^2 \gg |R^*|^2$.

(3) $U^*(|R^*|^r|R|^{2k}|R^*|^r)^{\frac{1}{q}}U \geq |R|^{\frac{2(k+r)}{q}} \geq (|R|^r|R^*|^{2k}|R|^r)^{\frac{1}{q}}$ for all $k,\ r \geq 0$ and $q \geq 1$ with $rq \geq k + r$.

(4) $(|R^*|^r|R|^{2k}|R^*|^r)^{\frac{1}{q}} \geq |R^*|^{\frac{2(k+r)}{q}} \geq U(|R|^r|R^*|^{2k}|R|^r)^{\frac{1}{q}}U^*$ for all $k,\ r \geq 0$ and $q \geq 1$ with $rq \geq k + r$.

Theorem 6 gives us the following characterization of log-hyponormal operators, whose proof is analogous to that of Theorem 3. So we omit the proof.

Theorem 7. *Let $T = U|T|$ be an invertible operator.*

(1) For a given $r \geq 0$, put $\tilde{C}_k = |T|^k U|T|^r$ and $\hat{C}_k = |T|^r U|T|^k$ for $k \geq 1$. Then T is log-hyponormal if and only if

$$(\tilde{C}_k^* \tilde{C}_k)^{\frac{1}{q}} \geq |T|^{\frac{2p(k+r)}{q}} \geq (\hat{C}_k \hat{C}_k^*)^{\frac{1}{q}}$$

holds for all $k,\ r \geq 0$ and $q \geq 1$ with $rq \geq k+r$.

(2) For a given $r \geq 0$, put $\tilde{D}_k = |T^*|^k U|T^*|^r$ and $\hat{D}_k = |T^*|^r U|T^*|^k$ for $k \geq 1$. Then T is log-hyponormal if and only if

$$(\tilde{D}_k^* \tilde{D}_k)^{\frac{1}{q}} \geq |T^*|^{\frac{2p(k+r)}{q}} \geq (\hat{D}_k \hat{D}_k^*)^{\frac{1}{q}}$$

holds for all $k,\ r \geq 0$ and $q \geq 1$ with $rq \geq k+r$.

In these cases, $\tilde{C}_k$, $\hat{C}_k$, $\tilde{D}_k$ and $\hat{D}_k$ are $\frac{1}{q}$-hyponormal.

Next we consider the monotonicity of operator functions on log-hyponormal operators. For this, we prepare the following result proved in Corollaries 8, 9 of [7].

Theorem C. *If $A \gg B$ for $A, B > 0$ and $r \geq 0$, then the operator function*

$$F_c(k) = (B^r A^k B^r)^{\frac{2r}{k+2r}}$$

is monotone increasing for $k \geq 0$, and

$$G_c(k) = (A^r B^k A^r)^{\frac{2r}{k+2r}}$$

is monotone decreasing for $k \geq 0$.

Theorem 8. *Let $T = U|T|$ be a log-hyponormal operator and for a given $r \geq 0$, $\tilde{C}_k$, $\hat{C}_k$, $\tilde{D}_k$ and $\hat{D}_k$ as in Theorem 7. Then the following operator functions on $[0, \infty)$ are monotone;*

(1) $\tilde{f}_c(k) = (\tilde{C}_k^* \tilde{C}_k)^{\frac{r}{k+r}}$ is increasing.

(2) $\hat{f}_c(k) = (\hat{C}_k \hat{C}_k^*)^{\frac{r}{k+r}}$ is decreasing.

(3) $\tilde{g}_c(k) = (\tilde{D}_k^* \tilde{D}_k)^{\frac{r}{k+r}}$ is increasing.

(4) $\hat{g}_c(k) = (\hat{D}_k \hat{D}_k^*)^{\frac{r}{k+r}}$ is decreasing.

Proof. Since U is unitary, we have $U^*|T|^2 U \gg |T|^2 \gg U|T|^2 U^* = |T^*|^2$. Hence it follows from Theorem C that

$$\tilde{f}_c(k) = (|T|^r U^* |T|^{2k} U |T|^r)^{\frac{r}{k+r}} = ((|T|^2)^{\frac{r}{2}} (U^*|T|^2 U)^k (|T|^2)^{\frac{r}{2}})^{\frac{r}{k+r}}$$

is monotone increasing for $k \geq 0$. Similarly (2), (3) and (4) follow from the equations

$$\hat{f}_c(k) = (|T|^r U |T|^{2k} U^* |T|^r)^{\frac{r}{k+r}} = (|T|^r |T^*|^{2k} |T|^r)^{\frac{r}{k+r}},$$

$$\tilde{g}_c(k) = (|T^*|^r U^* |T^*|^{2k} U |T^*|^r)^{\frac{r}{k+r}} = (|T^*|^r |T|^{2k} |T^*|^r)^{\frac{r}{k+r}}$$

and

$$\hat{g}_c(k) = (|T^*|^r U |T^*|^{2k} U^* |T^*|^r)^{\frac{r}{k+r}} = U(|T|^r |T^*|^{2k} |T|^r)^{\frac{r}{k+r}} U^* = U \hat{f}_c(k) U^*.$$

6. Norm inequalities. Finally we give a characterization of p-hyponormal operator. Such an attempt is done in [6] and [9].

Theorem 9. *The following statements are mutually equivalent for an operator $T = U|T|$ on H:*

(1) T is p-hyponormal.

(2) $|(Tx,y)| \leq \||T|^{1-p}x\|\||T|^p y\|$ for all $x,\ y \in H$.

(3) $\||T|^p y\|^2|(x,|T|^{2(1-p)}z)|^2 \leq \||T|^{1-p}x\|^2\||T|^p y\|^2 - |(Tx,y)|^2$ for all $x,\ y \in H$ and $z \in H$ such that $(z, T^*y) = 0$ and $\||T|^{1-p}z\| = 1$.

Proof. If T is p-hyponormal, then

$$|(Tx,y)| = |(|T|^{1-p}x, |T|^p U^* y)| \leq \||T|^{1-p}x\|\||T|^p U^* y\|,$$

so that (2) holds. Next we show (2) implies (3). Suppose that $(T^*y, z) = 0$ and $\||T|^{1-p}z\| = 1$. Put $\alpha = (z, |T|^{2(1-p)}z)$ and $u = x - \alpha z$. Since

$$(|T|^{1-p}u, |T|^{1-p}z) = (u, |T|^{2(1-p)}z) = (x, |T|^{2(1-p)}z) - \alpha(z, |T|^{2(1-p)}z) = 0,$$

we have

$$\||T|^{1-p}x\|^2 = \||T|^{1-p}u + \alpha|T|^{1-p}z\|^2 = \||T|^{1-p}u\|^2 + |\alpha|^2.$$

On the other hand, since $(Tx,y) = (Tu,y) + \alpha(Tz,y) = (Tu,y)$, it follows from (2) that

$$\begin{aligned}
&\||T|^{1-p}x\|^2\||T|^p y\|^2 - |(Tx,y)|^2 \\
&= (\||T|^{1-p}u\|^2 + |\alpha|^2)\||T|^p y\|^2 - |(Tu,y)|^2 \\
&= |\alpha|^2\||T|^p y\|^2 + \||T|^{1-p}u\|^2\||T|^p y\|^2 - |(Tu,y)|^2 \\
&\geq |\alpha|^2\||T|^p y\|^2,
\end{aligned}$$

so that (3) holds under (2).

It is trivial that (3) implies (2). Finally we prove (2) implies (1); (2) says that $|(v, |T|^p U^* y)| \leq \||T|^p y\|$ for all $v \in ran|T|^{1-p}$, where $ranX$ is the range of X. Hence $\||T|^p U^* y\| \leq \||T|^p y\|$ and so $|T^*|^{2p} \leq |T|^{2p}$.

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