Iterative Solution of Nonlinear Equations Involving K-Accretive Operator Equations

CHIKA MOORE*
Department of Mathematics and Computer Science
Nnamdi Azikiwe University
P. M. B. 5025, Awka, Anambra State, Nigeria

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ABSTRACT. Let E be an arbitrary real Banach space and let $A : D(A) \subseteq E \mapsto E$ be a Lipschitz strongly K-accretive operator. It is proved that modified iteration processes of the Mann and Ishikawa types converge strongly to the unique solutions of the operator equations $Ax = f$ and $Kx + Ax = f$ where $f \in E$ is an arbitrary but fixed vector.

1 Introduction Let E be a normed linear space. A mapping U with domain D(U) and range R(U) in E is said to be *accretive* if $\forall x, y \in D(U)$ and all $r \geq 0$ the following inequality holds:

$$\|x - y\| \leq \|x - y + r(Ux - Uy)\| \tag{1}$$

and is said to be *strictly accretive* if the inequality in (**??**) is strict whenever $x \neq y$. U is said to be *strongly accretive* if $\exists\, k > 0$ such that the operator $U - kI$ is accretive. The operator U is said to be *m-accretive* (or, sometimes, *hyper-accretive*) if $\forall\, \lambda > 0$, $I + \lambda U$ is surjective, where I denotes the identity operator on E. A consequence of the above definition is that the operator equation $x + Ux = f$ has a unique solution $x^* \in D(U)$ whenever U is m-accretive. The accretive operators are of interest mainly because they usually occur in several mathematical models for physically significant problems. For instance, it is known that evolution systems of the type

$$\frac{dx}{dt} + Ux - f = 0; \quad x(0) = x_0 \tag{2}$$

are used typically in models involving heat, wave and Shrödinger equations where U is an operator of the accretive type in an appropriate Banach space. Thus several authors have studied mappings of the accretive type with regard to the existence, uniqueness and iterative construction of solutions to operator equations of the forms $Tx = f$ and $x + Tx = f$ (see e.g., [1-7, 9-12, 14-16, 21-23]).

Closely related to the accretive operators is the class of dissipative operators where an operator T is said to be *dissipative* if and only if the operator $U = (-T)$ is accretive. The concepts of *strict, strong* and *m-* (or, *hyper-*) dissipativity are similarly defined. The importance of the dissipative operators derive from their stated connection with the important class of accretive operators and also from their occurrence in several models for physically

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relevant problems. Consequently, several authors have also studied this important class of mappings (see e.g., [2-7, 9-12, 14-16, 21-23]).

Let $1 < p < \infty$ and let $J_p : E \mapsto 2^{E^*}$ denote the *duality mapping* (with gauge function $f(t) = t^p$) defined by

$$J_p x := \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^p;\ \|f^*\| = \|x\|^{p-1}\}$$

where $\langle .,. \rangle$ denotes the generalized duality pairing between elements of E and E^*. If $p = 2$, then $J = J_2$ is called the *normalized duality map* and $\forall\, x \in E,\ J_p x := \|x\|^{p-2} Jx$. Using the duality map and a result of Kato [11], we obtain equivalent formulations of the accretive condition thus: U is said to be accretive if $\forall\, x, y \in D(U)\ \exists\, j_p(x-y) \in J_p(x-y)$ such that

$$\langle Ux - Uy, j_p(x-y) \rangle \geq 0 \tag{3}$$

The equivalent formulations of the other accretive type mappings can easily be obtained now. We remark immediately that the usual definitions of the accretive type mappings using the duality mapping are generally given with the normalized duality map J_2.

Recall that a map T is said to be *closed* if whenever $\{x_n\} \subseteq D(T)$ and $\{Tx_n\} \subseteq R(T)$ are such that $x_n \to x^*,\ Tx_n \to y^*$ as $n \to \infty$ then $x^* \in D(T)$ and $Tx^* = y^*$. A map T is said to be *closeable* if it is closed at the origin $x = o$. Let E,F be normed linear spaces and let T be a map with domain $D(T) \subseteq E$ and range $R(T) \subseteq F$. Then the graph $G(T)$ of T is defined by

$$G(T) := \{(x, y) : x \in D(T) \text{ and } Tx = y\} \subseteq E\text{x}F$$

Then the map T is closed if and only if its graph is a closed subset of $E\text{x}F$.

Let D be a dense subset of a Hilbert space H. A *linear* operator K with domain $D(K) \supset D$ is said to be *continuously D-invertible* if the range $R(K|_D) = K(D)$ of the restriction of K to D is dense in H and K has a bounded inverse on K(D). Then a *linear* operator A is said to be *K-positive definite* (briefly, *K-p.d*) if there exists a continuously D(A)-invertible closed operator K with $D(A) \subseteq D(K)$ and a constant $\mu > 0$ such that

$$\langle Ax, Kx \rangle \geq \mu \|Kx\|^2 \tag{4}$$

The K-p.d operators were introduced by Petryshyn [18] as a natural generalization of the class of *positive definite* or *linear strongly accretive* operators in Hilbert space. It is easy to see that if $K = I$, the identity operator on H, then A is positive definite. The choice $K = A$ shows that the class of K-p.d operators also contains the class of linear invertible operators as a subclass. Moreover, as indicated by Petryshyn [18-20], several differential operators of odd order also belong to this class of operators.

Chidume and Aneke [6] extended the notion of the K-p.d operator to Banach spaces more general than Hilbert spaces thus: Let E be a *smooth* Banach space (so that the duality map is single-valued) and let K be a linear continuously D(A)-invertible closed operator. Then the linear operator A is said to be K-p.d if there exists a positive constant $\mu > 0$ such that

$$\langle Ax, j(Kx) \rangle \geq \mu \|Kx\|^2 \tag{5}$$

Let K be a nonlinear operator with domain D(K) and range R(K) in E and let D be a dense subset of E. K shall be said to be *continuously D-invertible* if the range $R(K|_D) = K(D)$ of the restriction of K to D is dense in E and K has a Lipschitz continuous inverse in K(D). An operator A, in general nonlinear, is said to be *K-accretive* (briefly, *K-a*) if there exists a continuously D(A)-invertible closed operator K with $D(A) \subseteq D(K)$ such that $\forall\, x, y \in D(A)$ and some $j_p(Kx - Ky) \in J_p(Kx - Ky)$ the following inequality holds:

$$\langle Ax - Ay, j_p(Kx - Ky) \rangle \geq 0 \tag{6}$$

A is called *strongly K-accretive* (briefly, *s.K-a*) if A is K-accretive and there exists a constant $\mu > 0$ such that the following inequality holds:

$$\langle Ax - Ay, j_p(Kx - Ky)\rangle \geq \mu \|Kx - Ky\|^p \tag{7}$$

This concept of K-accretivity which was introduced in [13] is an obvious extension of the K-positive property to nonlinear operators in more general spaces. Moreover, the class of accretive operators is a proper subclass of this class of K-accretive operators as has been shown in [14,15].

This class of operators has been studied in connection with the solvability of the operator equations

$$Ax = f \tag{8}$$

and

$$Kx + Ax = f \tag{9}$$

and results on the existence and iterative approximations of solutions obtained for both the linear and nonlinear cases (see e.g., [6, 8, 13-15, 17-20])

It is our purpose in this paper to continue with the study and extend the results to the more general arbitrary real Banach spaces: the most general results so far announced are confined to the *q-uniformly smooth* Banach spaces even for the linear case (see e.g., [8]). Our theorems, therefore, will generalize extend and unify the most important known results in this connection. We shall also discuss the stability of our iteration processes and obtain explicit convergence rates in some cases.

2 Preliminaries We introduce a new duality pairing and norm in $D(K)$ thus:
For $w, x, y, z \in D(K)$ we have

$$\langle w - z, j_p(x - y)\rangle_K := \langle Kw - Kz, j_p(Kx - Ky)\rangle$$

and thus

$$|x - y|^p = \langle x - y, j_p(x - y)\rangle_K = \langle Kx - Ky, j_p(Kx - Ky)\rangle = \|Kx - Ky\|^p$$

It is routine to check that $(D(K), |.|)$ is a normed linear space where D(K) is regarded as a collection of equivalence classes with the equivalence relation defined by

$$x\mathcal{R}y \iff Kx = Ky$$

Clearly, if K is injective (as we have required) then each equivalence class is precisely a singleton. We remark immediately that the zero element of D(K), under this relation, may well be different from the zero element of E. However, prescribing that $K0 = 0$ ensures the coincidence of both zero elements. We denote the completion of the space $(D(K), |.|)$ by E_0. We observe that the norm $\|.\|$ is equivalent to the new norm $|.|$.

If E is a Hilbert space and A, K are linear operators then the above process reduces to the process of Petryshyn (see e.g., [18]) with $z = 0 = y$. Under such a situation, it is proved (see e.g., [18]) that the K-p.d operator A, considered as an operator from E to E_0 is bounded (that is, Lipschitz continuous). In the sequel, whenever we require that A be Lipschitz, this shall mean that it be Lipschitz as an operator from E onto E_0.

We shall need the following result in the sequel.

LEMMA W [25] *Suppose $\{\Psi_n\}_{n\geq 0}$ is a sequence of nonnegative real numbers such that*

$$\Psi_{n+1} \leq (1 - \delta_n)\Psi_n + \sigma_n; \quad n \geq 0 \tag{10}$$

where $\{\delta_n\}_{n\geq 0} \subset [0, 1]$ and $\{\sigma_n\}_{n\geq 0}$ are real sequences satisfying the following conditions: $\sum_{n\geq 0} \delta_n = \infty$ and $\sigma_n = o(\delta_n)$. Then, $\Psi_n \to 0$ as $n \to \infty$.

3 Main Theorems

LEMMA 1 *Let E be an arbitrary real normed linear space. Then the following are equivalent:*

(i) A is strongly K-accretive with $D(A) = D(K) = E$.

(ii) $\exists\, \mu \in (0,1)$, a constant, such that $\forall\, x, y \in D(A)$ and all $r > 0$ the following inequality holds:

$$\|Kx - Ky\| \leq \|Kx - Ky + r[(AK^{-1} - \mu I)Kx - (AK^{-1} - \mu I)Ky]\| \tag{11}$$

PROOF Observe that K is injective and uniquely invertible. Also $D(K) = E$, $K(E)$ is dense in E and is also of the second category in E. Hence, K is surjective on E. Thus for any pair $w, z \in E$ $\exists\, x, y \in D(K)$ such that $w = Kx$, $z = Ky$ and the operator $G := AK^{-1}$ is well-defined. Since A is strongly K-accretive, $\exists \mu \in (0,1)$ such that

$$\langle Ax - Ay, j_p(Kx - Ky)\rangle \geq \mu\|Kx - Ky\|^p$$

Set $w = Kx$, $z = Ky$ and $G = AK^{-1}$ to obtain

$$\langle Gw - Gz, j_p(w - z)\rangle = \langle Ax - Ay, j_p(Kx - Ky)\rangle \geq \mu\|Kx - Ky\|^p = \mu\|w - z\|^p$$

so that G is strongly accretive. Thus, $\forall\, r > 0$ we have that

$$\|w - z\| \leq \|w - z + r[(G - \mu I)w - (G - \mu I)z]\|$$

which immediately yields (**??**) as required. The converse follows immediately from an application of Kato's lemma ([11], Lemma 1) to the last inequality. This completes the proof.

THEOREM 1 *Let E be an arbitrary real Banach space and let $A : E \mapsto E$ be a Lipschitz strongly K-accretive operator with $D(A) \subseteq D(K)$. Let the real sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1)$ and the error sequences $\{u_n\}$, $\{v_n\}$ satisfy the following conditions:*

$$(i)\ \lim_{n\to\infty} \alpha_n = 0 = \lim_{n\to\infty} \beta_n,\ (ii) \sum_{n\geq 0} \alpha_n = \infty, (iii)\ \lim_{n\to\infty} \|v_n\| = 0,\ (iv)\ \|u_n\| \leq M, \exists M > 0$$

Then the sequence $\{Kx_n\}_{n\geq 0}$ generated from an arbitrary $x_0 \in D(A)$ and some $u_0, v_0 \in E$ by

$$Kx_{n+1} = Kx_n + \alpha_n(f - GKy_n + v_n);\quad n \geq 0 \tag{12}$$

$$Ky_n = Kx_n + \beta_n(f - GKx_n + u_n);\quad n \geq 0 \tag{13}$$

where $G = AK^{-1}$ converges strongly to Kx^* where x^* is the unique solution to the equation $Ax = f$.

PROOF Observe that $G : E \mapsto E$ is strongly accretive. Moreover,

$$\|Gw - Gz\| = \|Ax - Ay\| \leq L\|Kx - Ky\| = L\|w - z\|$$

so that G is also L-Lipschitz. Hence, the equation $Gw = f$ has a unique solution $w^* \in E$. But $w^* = Kx^*$ for some unique $x^* \in E$. Then,

$$f = Gw^* = GKx^* = AK^{-1}Kx^* = Ax^*$$

Thus, the equation $Ax = f$ has the unique solution $x^* \in D(A)$. Observe further that,

$$\begin{aligned} Kx_n - Kx^* &= (1+\alpha_n)(Kx_{n+1} - Kx^*) + \alpha_n[(G-\mu I)Kx_{n+1} - (G-\mu I)Kx^*] \\ &\quad -(1-\mu)\alpha_n(Kx_n - Kx^*) - \alpha_n(GKx_{n+1} - GKy_n) \\ &\quad +\mu\alpha_n^2(GKx^* - GKy_n) - (1+2\alpha_n)\alpha_n v_n \end{aligned}$$

So that,

$$\begin{aligned} \|Kx_n - Kx^*\| &\geq (1+\alpha_n)\|Kx_{n+1} - Kx^* + \frac{\alpha_n}{1+\alpha_n}[(G-\mu I)Kx_{n+1} - (G-\mu I)Kx^*]\| \\ &\quad -(1-\mu)\alpha_n\|Kx_n - Kx^*\| - \alpha_n\|GKx_{n+1} - GKy_n\| \\ &\quad -\mu\alpha_n^2\|GKy_n - GKx^*\| - (1+2\alpha_n)\alpha_n\|v_n\| \\ &\geq (1+\alpha_n)\|Kx_{n+1} - Kx^*\| - (1-\mu)\alpha_n\|Kx_n - Kx^*\| \\ &\quad -\alpha_n\|GKx_{n+1} - GKy_n\| - \mu\alpha_n^2\|GKy_n - GKx^*\| \\ &\quad -(1+2\alpha_n)\alpha_n\|v_n\| \end{aligned}$$

Therefore,

$$\begin{aligned} \|Kx_{n+1} - Kx^*\| &\leq \left[\frac{1+(1-\mu)\alpha_n}{1+\alpha_n}\right]\|Kx_n - Kx^*\| + \alpha_n\|GKx_{n+1} - GKy_n\| \\ &\quad +\mu\alpha_n^2\|GKy_n - GKx^*\| + (1+2\alpha_n)\alpha_n\|v_n\| \\ &\leq \left(1 - \frac{\mu}{2}\alpha_n\right)\|Kx_n - Kx^*\| + \alpha_n\|GKx_{n+1} - GKy_n\| \\ &\quad +\mu\alpha_n^2\|GKy_n - GKx^*\| + (1+2\alpha_n)\alpha_n\|v_n\| \end{aligned}$$

where we have used the fact that $(1+\alpha_n)^{-1} \geq 2^{-1}$ since $\alpha_n \leq 1$. We now have the following estimates:

$$\begin{aligned} \|GKx_{n+1} - GKy_n\| &\leq L\|Kx_{n+1} - Ky_n\| \\ &= L\|\alpha_n(GKx^* - GKy_n + v_n) - \beta_n(GKx^* - GKx_n + u_n)\| \\ &\leq L\alpha_n\left(L\|Kx^* - Ky_n\| + \|v_n\|\right) + L\beta_n\left(\|Kx_n - Kx^*\| + \|u_n\|\right) \\ &= L\alpha_n\left(L\|Kx^* - Kx_n - \beta_n(GKx^* - GKx_n + u_n)\| + \|v_n\|\right) \\ &\quad +L\beta_n\left(L\|Kx_n - Kx^*\| + \|u_n\|\right) \\ &\leq L\alpha_n\left\{L\left[(1+L\beta_n)\|Kx_n - Kx^*\| + \beta_n\|u_n\|\right] + \|v_n\|\right\} \\ &\quad +L\beta_n\left(L\|Kx_n - Kx^*\| + \|u_n\|\right) \\ &= L^2[\alpha_n(1+L\beta_n) + \beta_n]\|Kx_n - Kx^*\| \\ &\quad +L(1+\alpha_n)\beta_n\|u_n\| + L\alpha_n\|v_n\| \\ \|GKy_n - GKx^*\| &\leq L\|Ky_n - Kx^*\| \\ &= L\|Kx_n - Kx^* + \beta_n(GKx_n - GKx^* + u_n)\| \\ &\leq L(1+L\beta_n)\|Kx_n - Kx^*\| + L\beta_n\|u_n\| \end{aligned}$$

We then have,

$$\begin{aligned} \|Kx_{n+1} - Kx^*\| &\leq \left\{1 - \frac{\mu}{2}\alpha_n + T(\alpha_n,\beta_n)\alpha_n\right\}\|Kx_n - Kx^*\| \\ &\quad +L(1+2\alpha_n)\alpha_n\beta_n\|u_n\| + [1+(L+2)\alpha_n]\alpha_n\|v_n\| \end{aligned} \tag{14}$$

where $T(\alpha_n,\beta_n) := L^2[\alpha_n(1+L\beta_n)+\beta_n] + \mu L(1+L\beta_n)\alpha_n$. Observe that $T(\alpha_n,\beta_n) \to 0$ as $n \to \infty$. Hence, since $\mu > 0$, $\exists\, N > 0$, sufficiently large, such that

$$T(\alpha_n,\beta_n) \leq \frac{\mu}{4}, \quad \forall\, n \geq N$$

Now define

$$\begin{aligned}\Psi_n &:= \|Kx_n - Kx^*\|\\ \delta_n &:= \frac{\mu}{4}\alpha_n\\ \sigma_n &:= L(1+2\alpha_n)\alpha_n\beta_n\|u_n\| + [1+(L+2)\alpha_n]\alpha_n\|v_n\|\end{aligned}$$

Observe that $\Psi_n \geq 0,\ 0 \leq \delta_n < 1,\ \sum_{n\geq 0}\delta_n = \infty$ and $\sigma_n = o(\delta_n)$. Thus, (**??**) now becomes

$$\Psi_{n+1} \leq (1-\delta_n)\Psi_n + \sigma_n, \quad \forall\, n \geq N$$

And hence, $\Psi_n \to 0$ as $n \to \infty$. This completes the proof.

COROLLARY 1 *Let E, K and A be as in Theorem 1. Let the real sequence $\{C_n\} \subset [0,1)$ and the error sequence $\{u_n\}$ satisfy the following conditions*

$$(i)\ \lim_{n\to\infty} C_n = 0, \quad (ii)\ \sum_{n\geq 0} C_n = \infty, \quad (iii)\ \lim_{n\to\infty}\|u_n\| = 0$$

Let the sequence $\{Kx_n\}_{n\geq 0}$ be iteratively generated from an arbitrary $x_0 \in E$ and some $u_0 \in E$ by

(15) $$Kx_{n+1} = Kx_n + C_n(f - GKx_n + u_n), \quad n \geq 0$$

Then the conclusions of Theorem 1 still hold.

REMARKS 1

1. Our iterative schemes (**??**), (**??**) and (**??**) are well defined. Observe that $A, K : D(A) = D(K) = E \mapsto E$, $K^{-1} : E \mapsto D(A)$ and so $G = AK^{-1} : E \mapsto E$. Also K^{-1} exists and is a Lipschitz continuous map by definition of the operator A so that the operator G exists and is well defined. Choose $x_0 \in D(A)$ then we easily compute $Kx_0 \in E, GKx_0 \in E, u_0 \in E, v_0 \in E$ and $f \in E$. So $Ky_0 = Kx_0 + \beta_0(f - GKx_0 + u_0) \in E$ and hence $Kx_1 = Kx_0 + \alpha_0(f - GKy_0 + v_0) \in E$ are easy to compute. Then GKx_1, Ky_1 and hence Kx_2 can be computed. Proceeding thus, we generate the iterative sequences used in the theorems.

2. In the case where K is linear (in general, we require that K be linear whenever A is linear but the converse need not hold), and K^{-1} commutes with A then our iteration schemes reduce to

$$\begin{aligned} x_{n+1} &= x_n + \alpha_n(f - Ay_n + v_n), \quad n \geq 0\\ y_n &= x_n + \beta_n(f - Ax_n + u_n), \quad n \geq 0\\ &\qquad\textit{and}\\ x_{n+1} &= x_n + C_n(f - Ax_n + u_n), \quad n \geq 0\end{aligned}$$

which look like the more common steepest descent schemes with "errors".

3. Our theorems indicate that our iteration schemes are stable since the introduction of "small perturbations", the so-called "error" terms, does not affect the asymptotic behaviours and convergence properties of the schemes. This can easily be seen by setting $u_n, v_n \equiv 0$ in our theorems.

4. We can obtain the additional information of the convergence rate if specific choices for the real sequences are made. For instance, choosing $\alpha_n \equiv n^{-1}$ yields that $\Psi_n = 0(n^{-1})$.

5. If K is additionally *weakly coercive*, that is, $\forall\, x, y \in D(K),\ \exists \eta > 0$, a constant, such that

$$\|Kx - Ky\| \geq \eta\|x - y\|$$

then our theorems imply that the iteration scheme defined by

$$x_{n+1} := K^{-1}[Kx_n + \alpha_n(f - GKy_n + v_n)],\ n \geq 0 \tag{16}$$

$$y_n := K^{-1}[Kx_n + \beta_n(f - GKx_n + u_n)],\ n \geq 0 \tag{17}$$

converges strongly to x^*. It is, however, clear that (**??**) and (**??**) are not computationally more economical than (**??**) and (**??**).

THEOREM 2 *Let E, K, G, $\{\alpha_n\}$, $\{\beta_n\}$, $\{u_n\}$, $\{v_n\}$ and f be as in Theorem 1. Let $A : E \mapsto E$ be a nonlinear Lipschitz operator satisfying the following condition: $\forall\, x, y \in E$ and some $\mu \in (0,1)$,*

$$\langle Ax - Ay, j_p(Kx - Ky)\rangle \geq -\mu\|Kx - Ky\|^p$$

Then, the nonlinear equation $Kx + Ax = f$ has a unique solution $x^* \in E\, \forall\, f \in E$. Then the sequence $\{Kx_n\}_{n\geq 0}$ iteratively generated from an arbitrary $x_0 \in E$ and some $u_0, v_0 \in E$ by

$$Kx_{n+1} := Kx_n + \alpha_n(f - Kx_n - GKy_n + v_n);\ n \geq 0 \tag{18}$$

$$y_n := Kx_n + \beta_n(f - Kx_n - GKx_n + u_n);\ n \geq 0 \tag{19}$$

converges strongly to Kx^*.

PROOF Let $\lambda := 1 - \mu \in (0,1)$ and observe that $I + G$ as an operator from E_0 to E is m-accretive and so the equation $u + Gu = f$ has a unique solution $u^* \in E_0$. This easily translates into the equation $Kx + Ax = f$ having the unique solution $x^* \in E$. Moreover, the operator $I + G$ is strongly accretive with constant $\lambda \in (0,1)$ and Lipschitz. Then the operator $S := I + G - \lambda I$ is accretive on E_0.

From (**??**) and (**??**) we see that

$$\begin{aligned} Kx_n - Kx^* &= (1+\alpha_n)(Kx_{n+1} - Kx^*) + \alpha_n(SKx_{n+1} - SKx^*) \\ &\quad -(1-\lambda)\alpha_n(Kx_n - Kx^*) - \alpha_n(GKx_{n+1} - GKy_n) \\ &\quad +(2-\lambda)\alpha_n^2(Kx_n - Kx^*) + (2-\lambda)\alpha_n^2(GKy_n - GKx^*) \\ &\quad +[1 + (2-\lambda)\alpha_n]\alpha_n v_n \end{aligned}$$

Thus, we have that

$$\begin{aligned} \|Kx_n - Kx^*\| &\geq (1+\alpha_n)\|Kx_{n+1} - Kx^* + \frac{\alpha_n}{1+\alpha_n}(SKx_{n+1} - SKx^*)\| \\ &\quad -(1-\lambda)\alpha_n\|Kx_n - Kx^*\| - \alpha_n\|GKx_{n+1} - GKy_n\| \\ &\quad -(2-\lambda)\alpha_n^2(\|Kx_n - Kx^*\| + \|GKy_n - GKx^*\|) \\ &\quad -[1 + (2-\lambda)\alpha_n]\alpha_n\|v_n\| \\ &\geq (1+\alpha_n)\|Kx_{n+1} - Kx^*\| - (1-\lambda)\alpha_n\|Kx_n - Kx^*\| \\ &\quad -(2-\lambda)\alpha_n^2(\|Kx_n - Kx^*\| + \|GKy_n - GKx^*\|) \\ &\quad -\alpha_n\|GKx_{n+1} - GKy_n\| - [1 + (2-\lambda)\alpha_n]\alpha_n\|v_n\| \end{aligned}$$

Hence,

$$\begin{aligned}\|Kx_{n+1} - Kx^*\| &\leq \left[\frac{1+(1-\lambda)\alpha_n}{1+\alpha_n}\right]\|Kx_n - Kx^*\| + (2-\lambda)\alpha_n^2\|Kx_n - Kx^*\| \\ &\quad +(2-\lambda)\alpha_n^2\|GKy_n - GKx^*\| + \alpha_n\|GKx_{n+1} - GKy_n\| \\ &\quad +[1+(2-\lambda)\alpha_n]\alpha_n\|v_n\|\end{aligned}$$

We have the following estimates;

$$\begin{aligned}\|GKy_n - GKx^*\| &\leq L[1+(L+1)\beta_n]\|Kx_n - Kx^*\| + L\beta_n\|u_n\| \\ \|GKx_{n+1} - GKy_n\| &\leq L\{\alpha_n\left(1+L[1+(L+1)\beta_n]\right) + (L+1)\beta_n\}\|Kx_n - Kx^*\| \\ &\quad +L(1+L\alpha_n)\beta_n\|u_n\| + L\alpha_n\|v_n\| \\ \frac{1+(1-\lambda)\alpha_n}{1+\alpha_n} &\leq 1-\lambda\alpha_n + \lambda\alpha_n^2\end{aligned}$$

Define

$$\begin{aligned}\Psi_n &:= \|Kx_n - Kx^*\| \\ \delta_n &:= \frac{\lambda}{2}\alpha_n \\ \sigma_n &:= L[1+(L+2-\lambda)\alpha_n]\alpha_n\beta_n\|u_n\| + [1+(L+2-\lambda)\alpha_n]\alpha_n \\ T(\alpha_n,\beta_n) &:= 2\alpha_n + (2-\lambda)L[1+(L+1)\beta_n]\alpha_n \\ &\quad +L\left(1+L[1+(L+1)\beta_n]\right)\alpha_n + L(L+1)\beta_n\end{aligned}$$

Observe that $T(\alpha_n,\beta_n) \to 0$ as $n \to \infty$. Thus, $\exists N_0 > 0$ such that

$$T(\alpha_n,\beta_n) \leq \frac{\lambda}{2}; \quad \forall\, n \geq N_0$$

Then, $\forall\, n \geq N_0$ we have

$$\begin{aligned}\Psi_{n+1} &\leq [1-\lambda\alpha_n + T(\alpha_n,\beta_n)\alpha_n]\Psi_n + \sigma_n \\ &= \{1-[\lambda - T(\alpha_n,\beta_n)]\alpha_n\}\,\Psi_n + \sigma_n \\ &\leq (1-\delta_n)\Psi_n + \sigma_n\end{aligned}$$

This completes the proof.

COROLLARY 2 *Let E, A, K, G be as in Theorem 2 and let the sequences $\{C_n\}$, $\{u_n\}$ be as in Corollary 1. Let the sequence $\{Kx_n\}_{n\geq 0}$ be iteratively generated from an arbitrary $x_0 \in E$ and some $u_0 \in E$ by*

$$Kx_{n+1} = Kx_n + C_n(f - Kx_n - GKx_n + u_n), \quad n \geq 0 \tag{20}$$

Then the conclusions of Theorem 2 still hold.

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