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ON SOME CLASSES OF OPERATORS BY FUJII AND NAKAMOTO RELATED TO p-HYPONORMAL AND PARANORMAL OPERATORS

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ABSTRACT. Recently, we introduced class A as a new class of operators includind p-hyponormal and log-hyponormal operators. Class A is defined by an operator inequality, and also the definition of class A is similar to that of paranormality defined by a norm inequality. As generalizations of class A and paranormality, Fujii-Jung-S.H.Lee-M.Y.Lee-Nakamoto introduced class $\mathrm{A}(p,r)$ and Yamazaki-Yanagida introduced absolute-(p,r)-paranormality. Moreover, Fujii-Nakamoto introduced class $\mathrm{F}(p,r,q)$ and (p,r,q)-paranormality which are further generalizations of these classes.

In this paper, we shall show more precise inclusion relations among the families of class $\mathrm{F}(p,r,q)$ and (p,r,q)-paranormality than the results by Fujii-Nakamoto, and we shall also show several results on class $\mathrm{F}(p,r,q)$ and (p,r,q)-paranormality.

1. INTRODUCTION

In this paper, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx,x) \geq 0$ for all $x \in H$, and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

As extensions of hyponormal operators, i.e., $T^*T \geq TT^*$, p-hyponormal operators for $p > 0$ defined by $(T^*T)^p \geq (TT^*)^p$ and log-hyponormal operators defined by $\log T^*T \geq \log TT^*$ for an invertible operator T are well known. And also an operator T is p-quasihyponormal for $p > 0$ if T is p-hyponormal on $\overline{R(T)}$. It is easily obtained that every p-hyponormal operator is q-hyponormal for $p > q > 0$ by Löwner-Heinz theorem *"$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$,"* and every invertible p-hyponormal operator for $p > 0$ is log-hyponormal since $\log t$ is an operator monotone function. We remark that log-hyponormality is sometimes regarded as 0-hyponormality since $\frac{X^p - I}{p} \to \log X$ as $p \to +0$ for $X > 0$.

An operator T is paranormal if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$. It has been studied by many authors, so there are too many to cite their references, for instance, [2][12][17] and [22]. Ando [2] showed that *every p-hyponormal operator for $p > 0$ and log-hyponormal operator is paranormal.*

Recently, in [18], we introduced class A defined by $|T^2| \geq |T|^2$ where $|T| = (T^*T)^{\frac{1}{2}}$, and we showed that every log-hyponormal operator belongs to class A and every class A operator is paranormal. It turns out that these results contain another proof of Ando's result stated above. We remark that class A is defined by an operator inequality and paranormality is defined by a norm inequality, and their definitions appear to be similar forms.

And also we introduced two families of classes of operators based on class A and paranormality in [18] as follows: An operator T belongs to class $\mathrm{A}(k)$ for $k > 0$ if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq$

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$|T|^2$, and also an operator T is absolute-k-paranormal for $k>0$ if $\||T|^k Tx\| \geq \|Tx\|^{k+1}$ for every unit vector $x \in H$. Particularly an operator T is a class A (resp. paranormal) operator if and only if T is a class A(1) (resp. absolute-1-paranormal) operator.

On the other hand, Fujii-Izumino-Nakamoto [6] introduced p-paranormality for $p>0$ defined by $\||T|^p U|T|^p x\| \geq \||T|^p x\|^2$ for every unit vector $x \in H$, where $T=U|T|$ is the polar decomposition of T. We remark that 1-paranormality equals paranormality. As generalizations of class A(k), absolute-k-paranormality and p-paranormality, Fujii-Jung-S.H.Lee-M.Y.Lee-Nakamoto [10] introduced class A(p,r) and Yamazaki-Yanagida [32] introduced absolute-(p,r)-paranormality as follows:

Definition.

(1) *For each $p>0$ and $r>0$, an operator T belongs to class $A(p,r)$ if*

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r},$$

and let class $AI(p,r)$ be the class of all invertible class $A(p,r)$ operators.

(2) *For each $p>0$ and $r>0$, an operator T is absolute-(p,r)-paranormal if*

$$\||T|^p|T^*|^r x\|^r \geq \||T^*|^r x\|^{p+r} \tag{1.1}$$

for every unit vector $x \in H$.

It was pointed out that class A$(k,1)$ equals class A(k) in [30]. And also, in [32], it was shown that absolute-$(k,1)$-paranormality equals absolute-k-paranormality and absolute-(p,p)-paranormality equals p-paranormality. We should remark that the families of class AI(p,r) determined by operator inequalities and absolute-(p,r)-paranormality determined by norm inequalities constitute two increasing lines on $p>0$ and $r>0$ whose origin is log-hyponormality (see section 2).

Moreover, as a continuation of the discussion in [10], Fujii-Nakamoto [11] introduced the following classes of operators.

Definition.

(1) *For each $p>0$, $r\geq 0$ and $q\geq 1$, an operator T belongs to class $F(p,r,q)$ if*

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}}. \tag{1.2}$$

(2) *For each $p>0$, $r\geq 0$ and $q>0$, an operator T is (p,r,q)-paranormal if*

$$\||T|^p U|T|^r x\|^{\frac{1}{q}} \geq \||T|^{\frac{p+r}{q}} x\| \tag{1.3}$$

for every unit vector $x \in H$, where $T=U|T|$ is the polar decomposition of T.

We remark that class F$(p,r,\frac{p+r}{r})$ equals class A(p,r), and we shall show that $(p,r,\frac{p+r}{r})$-paranormality equals absolute-(p,r)-paranormality in the next section. Thus many reseachers have been discussed parallel families of classes of operators which are generalizations of class A and paranormality.

In this paper, firstly, we shall show more precise inclusion relations among the families of class F(p,r,q) and (p,r,q)-paranormality than the results by Fujii-Nakamoto in [11] from the view of monotonicity of class A(p,r) and absolute-(p,r)-paranormality. Secondly, we shall give a characterization of log-hyponormal operators via class F(p,r,q) and (p,r,q)-paranormality. Lastly, we shall show the result on powers of class F(p,r,q) operators.

2. Background and preliminaries

At the beginning of this section, we shall show another expression of (p, r, q)-paranormality without using U which appears in the polar decomposition of T, and it causes that $(p, r, \frac{p+r}{r})$-paranormality equals absolute-(p, r)-paranormality.

Proposition 1. *For each $p > 0$, $r > 0$ and $q \geq 1$, an operator T is (p, r, q)-paranormal if and only if*

$$\left\| |T|^p |T^*|^r x \right\|^{\frac{1}{q}} \geq \left\| |T^*|^{\frac{p+r}{q}} x \right\| \tag{2.1}$$

for every unit vector $x \in H$.

Corollary 2. *For each $p > 0$ and $r > 0$, $(p, r, \frac{p+r}{r})$-paranormality equals absolute-(p, r)-paranormality.*

Proof of Proposition 1. Let $T = U|T|$ be the polar decomposition of T. We remark that $|T^*|^t = U|T|^t U^*$ for any $t > 0$ and $T^* = U^*|T^*|$ is also the polar decomposition of T^*. Assume that T is (p, r, q)-paranormal for $p > 0$, $r > 0$ and $q \geq 1$, i.e.,

$$\left\| |T|^p U |T|^r x \right\|^{\frac{1}{q}} \geq \left\| |T|^{\frac{p+r}{q}} x \right\| \tag{1.3}$$

for every unit vector $x \in H$. (1.3) is equivalent to

$$\left\| |T|^p U |T|^r y \right\|^{\frac{1}{q}} \left\| y \right\|^{1-\frac{1}{q}} \geq \left\| |T|^{\frac{p+r}{q}} y \right\| \quad \text{for all } y \in H. \tag{2.2}$$

Then for every unit vector $x \in H$,

$$\begin{aligned} \left\| |T^*|^{\frac{p+r}{q}} x \right\| &= \left\| U|T|^{\frac{p+r}{q}} U^* x \right\| = (U^*U|T|^{\frac{p+r}{q}} U^* x, |T|^{\frac{p+r}{q}} U^* x)^{\frac{1}{2}} = \left\| |T|^{\frac{p+r}{q}} U^* x \right\| \\ &\leq \left\| |T|^p U |T|^r U^* x \right\|^{\frac{1}{q}} \left\| U^* x \right\|^{1-\frac{1}{q}} \leq \left\| |T|^p U |T|^r U^* x \right\|^{\frac{1}{q}} \left\| x \right\|^{1-\frac{1}{q}} = \left\| |T|^p |T^*|^r x \right\|^{\frac{1}{q}} \end{aligned}$$

and the first inequality holds by (2.2), so that we have (2.1). Conversely, assume that (2.1) holds. (2.1) is equivalent to

$$\left\| |T|^p |T^*|^r y \right\|^{\frac{1}{q}} \left\| y \right\|^{1-\frac{1}{q}} \geq \left\| |T^*|^{\frac{p+r}{q}} y \right\| \quad \text{for all } y \in H. \tag{2.3}$$

Then for every unit vector $x \in H$,

$$\begin{aligned} \left\| |T|^{\frac{p+r}{q}} x \right\| &= \left\| U^*|T^*|^{\frac{p+r}{q}} U x \right\| = (UU^*|T^*|^{\frac{p+r}{q}} U x, |T^*|^{\frac{p+r}{q}} U x)^{\frac{1}{2}} = \left\| |T^*|^{\frac{p+r}{q}} U x \right\| \\ &\leq \left\| |T|^p |T^*|^r U x \right\|^{\frac{1}{q}} \left\| U x \right\|^{1-\frac{1}{q}} \leq \left\| |T|^p U |T|^r U^* U x \right\|^{\frac{1}{q}} \left\| x \right\|^{1-\frac{1}{q}} = \left\| |T|^p U |T|^r x \right\|^{\frac{1}{q}} \end{aligned}$$

and the first inequality holds by (2.3), so that we have (1.3), i.e., T is (p, r, q)-paranormal. Hence the proof of Proposition 1 is complete. □

Proof of Corollary 2. By putting $q = \frac{p+r}{r} \geq 1$ in Proposition 1, $(p, r, \frac{p+r}{r})$-paranormality equals absolute-(p, r)-paranormality since (2.1) equals (1.1). □

Next, to explain the background of the classes of operators discussed in this paper, we have to state the following celebrated order preserving operator inequality.

Theorem F (Furuta inequality [13]).

If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

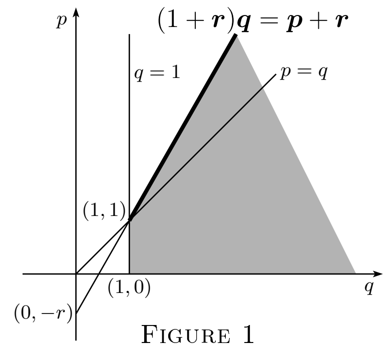

FIGURE 1

We remark that Theorem F yields Löwner-Heinz theorem when we put $r = 0$ in (i) or (ii) stated above. Alternative proofs of Theorem F were given in [4] and [25] and also an elementary one page proof in [14]. It was shown in [27] that the domain drawn for p, q and r in the Figure 1 is the best possible one for Theorem F.

Fujii-Nakamoto [11] observed that class $F(p, r, q)$ derives from Theorem F and (p, r, q)-paranormality corresponds to class $F(p, r, q)$, and also they showed the following Theorem A.1.

Theorem A.1 ([11]).

(i) *For a fixed $k > 0$, T is k-hyponormal if and only if T belongs to class $F(2kp, 2kr, q)$ for all $p > 0$, $r \geq 0$ and $q \geq 1$ with $(1+2r)q \geq 2(p+r)$, i.e., T belongs to class $F(p, r, q)$ for all $p > 0$, $r \geq 0$ and $q \geq 1$ with $(k+r)q \geq p+r$.*

(ii) *If T belongs to class $F(p_0, r_0, q_0)$ for $p_0 > 0$, $r_0 \geq 0$ and $q_0 \geq 1$, then T belongs to class $F(p_0, r, q_0)$ for any $r \geq r_0$.*

(iii) *If T belongs to class $F(p_0, r_0, q_0)$ for $p_0 > 0$, $r_0 \geq 0$ and $q_0 \geq 1$, then T belongs to class $F(p_0, r_0, q)$ for any $q \geq q_0$.*

(iv) *If T belongs to class $F(p, r, q)$ for $p > 0$, $r \geq 0$ and $q \geq 1$, then T is (p, r, q)-paranormal.*

(v) *If T is (p_0, r_0, q_0)-paranormal for $p_0 > 0$, $r_0 \geq 0$ and $q_0 > 0$, then T is (p_0, r_0, q)-paranormal for any $q \geq q_0$.*

(vi) *If T is $(p_0, r_0, 1)$-paranormal for $p_0 > 0$ and $r_0 \geq 0$, then T is $(p_0, r, 1)$-paranormal for any $r \geq r_0$.*

(vii) *If T is $(p, r, 1)$-paranormal for $p > 0$ and $r \geq 0$, then T is $\max\{p, r\}$-paranormal.*

On the other hand, chaotic order is defined by $\log A \geq \log B$ for positive and invertible operators A and B. Chaotic order is weaker than usual order $A \geq B$ since $\log t$ is an operator monotone function. As a characterization of chaotic order, the following Theorem B.1 was obtained by using Theorem F.

Theorem B.1 ([5][7][15][28]). *Let A and B be positive invertible operators. Then the following properties are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $(B^{\frac{p}{2}}A^pB^{\frac{p}{2}})^{\frac{1}{2}} \geq B^p$ *for all $p \geq 0$.*

(iii) $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ *for all $p \geq 0$ and $r \geq 0$.*

We remark that the equivalence between (i) and (ii) was shown in [3].

Noting that class $\mathrm{F}(p, r, \frac{p+r}{r})$ equals class $\mathrm{A}(p, r)$, we can verify that class $\mathrm{A}(p, r)$ derives from Theorem B.1. On class $\mathrm{A}(p, r)$ and absolute-(p, r)-paranormality, the following Theorem A.2 and Theorem A.3 were shown in [10] and [32], respectively. We remark that Figure 2 expresses the inclusion relations shown in Theorem A.2 and Theorem A.3.

Theorem A.2 ([10]).

(i) *T is log-hyponormal if and only if T belongs to class AI(p, r) for all $p > 0$ and $r > 0$.*
(ii) *If T belongs to class AI(p_0, r_0) for $p_0 > 0$ and $r_0 > 0$, then T belongs to class AI(p, r) for any $p \geq p_0$ and $r \geq r_0$.*
(iii) *If T belongs to class $A(p_0, r_0)$ for $p_0 > 0$ and $r_0 > 0$, then T belongs to class $A(p_0, r)$ for any $r \geq r_0$.*

Theorem A.3 ([32]).

(i) *T is log-hyponormal if and only if T is invertible and absolute-(p, r)-paranormal for all $p > 0$ and $r > 0$.*
(ii) *If T is absolute-(p_0, r_0)-paranormal for $p_0 > 0$ and $r_0 > 0$, then T is absolute-(p, r)-paranormal for any $p \geq p_0$ and $r \geq r_0$.*
(iii) *If T belongs to class $A(p, r)$ for $p > 0$ and $r > 0$, then T is absolute-(p, r)-paranormal.*
(iv) *If T is absolute-(p, r)-paranormal for $p > 0$ and $r > 0$, then T is normaloid, i.e., $\|T\| = r(T)$ where $r(T)$ is the spectral radius of T.*

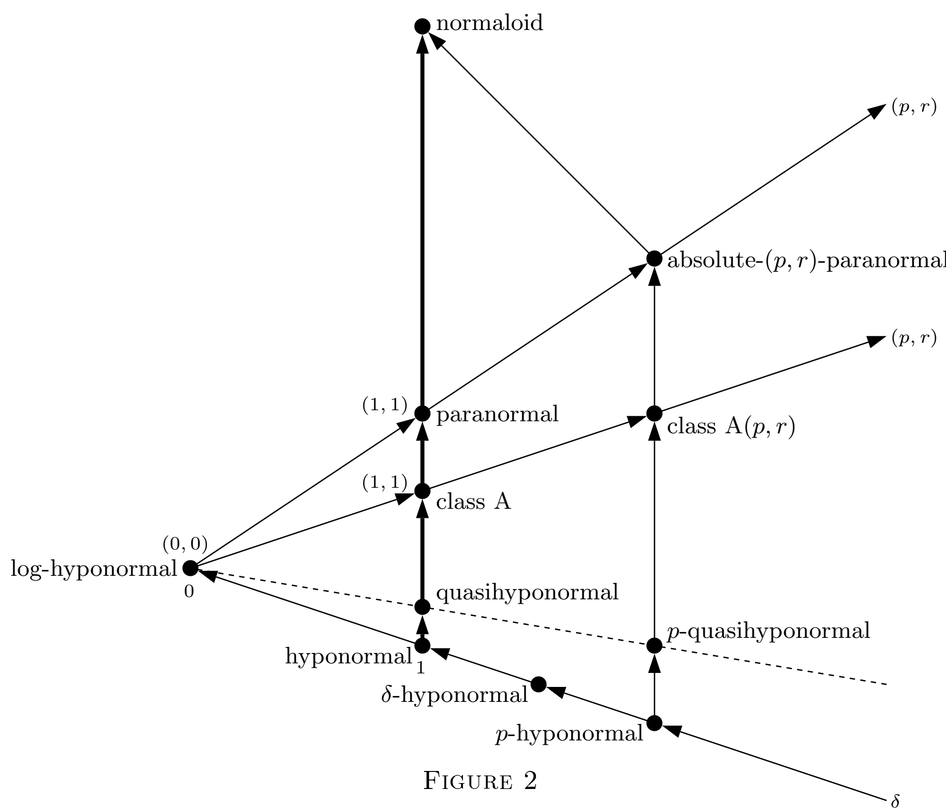

FIGURE 2

Theorem A.2 and Theorem A.3 state that the families of class AI(p, r) determined by operator inequalities and absolute-(p, r)-paranormality determined by norm inequalities have monotonicity on $p > 0$ and $r > 0$, and log-hyponormality regarded as class AI$(0,0)$ or absolute-$(0,0)$-paranormality, namely they constitute two increasing lines whose origin is log-hyponormality.

3. Inclusion relations

In this section, we shall show more precise inclusion relations than Theorem A.1 among the families of class F(p,r,q) and (p,r,q)-paranormality.

Firstly, we shall give expressions of p-quasihyponormality in terms of class F(p,r,q) and (p,r,q)-paranormality.

Proposition 3. *The following assertions hold for each $p > 0$ and $r > 0$:*

(i) *T is p-quasihyponormal if and only if T belongs to class $F(p,r,1)$ if and only if T is $(p,r,1)$-paranormal.*

(ii) *T is p-quasihyponormal if and only if T is $(p,0,1)$-paranormal.*

We remark that Proposition 1 does not hold for $r = 0$ and $q = 1$ since (2.1) holds for $p > 0$, $r = 0$ and $q = 1$, i.e., $\||T|^p x\| \geq \||T^*|^p x\|$ for every unit vector $x \in H$ if and only if T is p-hyponormal, but T is $(p,0,1)$-paranormal for $p > 0$ if and only if T is p-quasihyponormal by (ii) of Proposition 3.

Proof of Proposition 3.

Proof of (i). Noting that $\overline{R(T)} = N(T^*)^\perp = N(|T^*|)^\perp = N(|T^*|^r)^\perp = \overline{R(|T^*|^r)}$ for $r > 0$, T is p-quasihyponormal for $p > 0$ if and only if T is p-hyponormal on $\overline{R(|T^*|^r)}$ for $r > 0$. Hence T is p-quasihyponormal for $p > 0$ if and only if

$$|T^*|^r |T|^{2p} |T^*|^r \geq |T^*|^{2(p+r)},$$

i.e., T belongs to class F$(p,r,1)$ for $r > 0$ if and only if

$$\||T|^p |T^*|^r x\| \geq \||T^*|^{p+r} x\|$$

for every unit vector $x \in H$, i.e., T is $(p,r,1)$-paranormal for $r > 0$ by Proposition 1.

Proof of (ii). Noting that $\overline{R(T)} = N(T^*)^\perp = N(U^*)^\perp = \overline{R(U)}$, T is p-quasihyponormal for $p > 0$ if and only if $U^*|T|^{2p}U \geq U^*|T^*|^{2p}U = |T|^{2p}$ if and only if $\||T|^p Ux\| \geq \||T|^p x\|$ for every unit vector $x \in H$, that is, T is $(p,0,1)$-paranormal.

Hence the proof of Proposition 3 is complete. □

Secondly, we shall consider monotonicity of class F(p,r,q) and (p,r,q)-paranormality. Here we recall Theorem B.1.

Theorem B.1 ([5][7][15][28]). *Let A and B be positive invertible operators. Then the following properties are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $(B^{\frac{p}{2}} A^p B^{\frac{p}{2}})^{\frac{1}{2}} \geq B^p$ *for all* $p \geq 0$.

(iii) $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ *for all* $p \geq 0$ *and* $r \geq 0$.

In section 2, we verified that class A(p,r) derives from Theorem B.1, and also we explained that Theorem A.2 and Theorem A.3 state that the families of class AI(p,r) and absolute-(p,r)-paranormality constitute two increasing lines on $p>0$ and $r>0$ whose origin is log-hyponormality.

On the other hand, as a parallel result to Theorem B.1, Theorem F also leads to the following Theorem B.2.

Theorem B.2 ([8][9]). *For positive operators A and B, $A^\delta \geq B^\delta$ for a fixed $\delta>0$ if and only if*

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{\delta+r}{p+r}} \geq B^{\delta+r}$$

holds for all $p\geq\delta$ and $r\geq 0$.

Considering these matters, it seems natural that we rewrite class F(p,r,q) and (p,r,q)-paranormality by class F$(p,r,\frac{p+r}{\delta+r})$ and $(p,r,\frac{p+r}{\delta+r})$-paranormality when we discuss monotonicity of class F(p,r,q) and (p,r,q)-paranormality on p and r. In fact, we obtain the following results on monotonicity of class F$(p,r,\frac{p+r}{\delta+r})$ and $(p,r,\frac{p+r}{\delta+r})$-paranormality. And also the following Figure 3 represents the inclusion relations shown in this section.

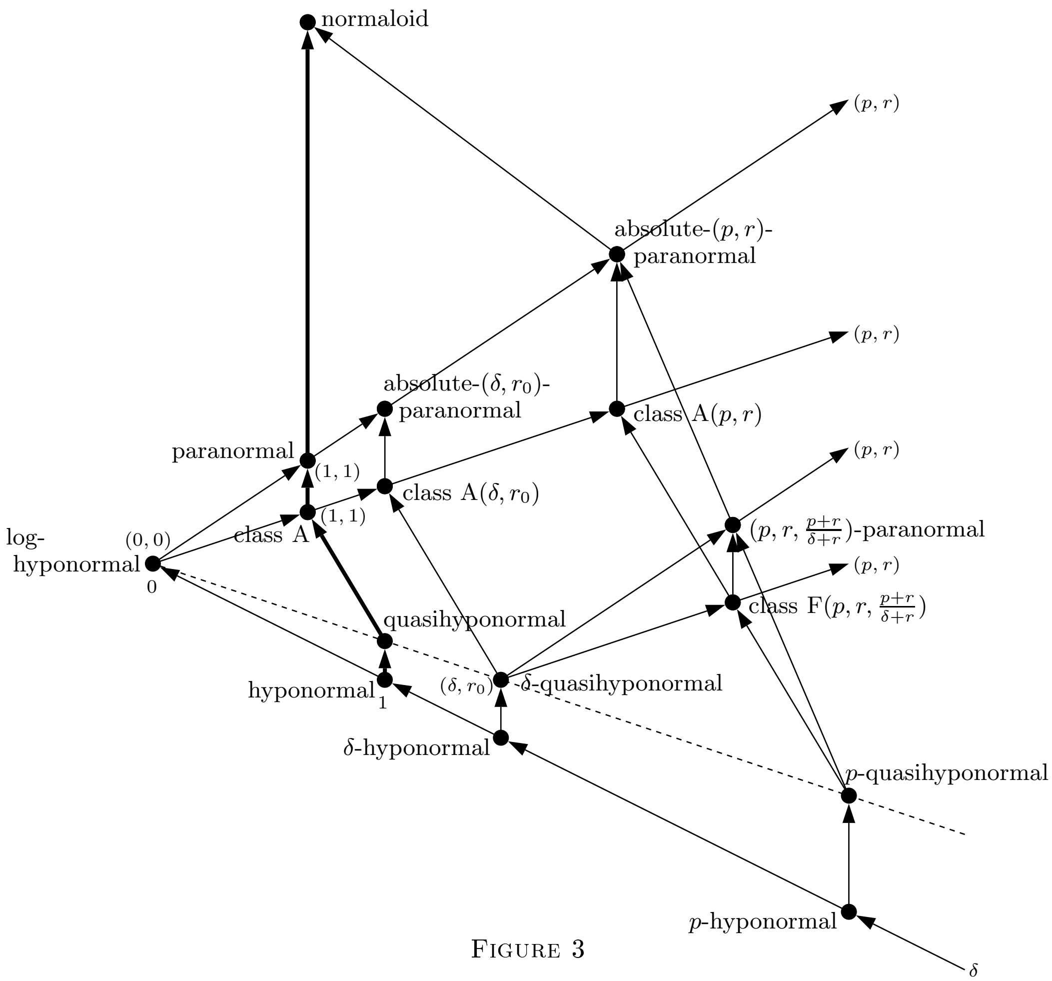

FIGURE 3

Theorem 4. *Let T be a class $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ operator for $p_0 > 0$, $r_0 \geq 0$ and $-r_0 < \delta \leq p_0$. Then the following assertions hold:*

(i) *T belongs to class $F(p_0, r, \frac{p_0+r}{\delta+r})$ for any $r \geq r_0$.*

(ii) *If T is invertible and $0 \leq \delta \leq p_0$, then T belongs to class $F(p, r, \frac{p+r}{\delta+r})$ for any $p \geq p_0$ and $r \geq r_0$.*

Theorem 5. *Let T be a $(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$-paranormal operator for $p_0 > 0$, $r_0 \geq 0$ and $\delta > -r_0$. Then the following assertions hold:*

(i) *If $-r_0 < \delta \leq p_0$, then T is $(p_0, r, \frac{p_0+r}{\delta+r})$-paranormal for any $r \geq r_0$.*

(ii) *If $0 \leq \delta$, then T is $(p, r_0, \frac{p+r_0}{\delta+r_0})$-paranormal for any $p \geq p_0$.*

(iii) *If $0 \leq \delta \leq p_0$, then T is $(p, r, \frac{p+r}{\delta+r})$-paranormal for any $p \geq p_0$ and $r \geq r_0$.*

Proposition 3, Theorem 4 and Theorem 5 assert that invertible class $\mathrm{F}(p, r, \frac{p+r}{\delta+r})$ and $(p, r, \frac{p+r}{\delta+r})$-paranormality for $\delta > 0$ constitute two increasing lines for $p \geq \delta > 0$ and $r \geq r_0 > 0$ which have δ-quasihyponormality as the origin since δ-quasihyponormality equals class $\mathrm{F}(\delta, r_0, 1)$ or $(\delta, r_0, 1)$-paranormality. And also, in case $\delta = 0$, (i) and (ii) of Theorem 4 means (iii) and (ii) of Theorem A.2, respectively, and Theorem 5 means (ii) of Theorem A.3. Therefore monotonicity of invertible class $\mathrm{F}(p, r, \frac{p+r}{\delta+r})$ and $(p, r, \frac{p+r}{\delta+r})$-paranormality for $\delta > 0$ is parallel to monotonicity of class $\mathrm{AI}(p, r)$ and absolute-(p, r)-paranormality since invertible δ-quasihyponormality (i.e., δ-hyponormality) approaches log-hyponormality as $\delta \to +0$.

We prepare the following result in order to prove Theorem 4.

Proposition 6. *Let A and B be positive operators such that $(B^{\frac{\beta_0}{2}} A^{\alpha_0} B^{\frac{\beta_0}{2}})^{\frac{\delta+\beta_0}{\alpha_0+\beta_0}} \geq B^{\delta+\beta_0}$ holds for fixed $\alpha_0 > 0$, $\beta_0 > 0$ and $-\beta_0 < \delta \leq \alpha_0$. Then the following assertions hold:*

(i) *$(B^{\frac{\beta}{2}} A^{\alpha_0} B^{\frac{\beta}{2}})^{\frac{\delta+\beta}{\alpha_0+\beta}} \geq B^{\delta+\beta}$ for all $\beta \geq \beta_0$.*

(ii) *If A and B are invertible and $0 \leq \delta \leq \alpha_0$, then $(B^{\frac{\beta}{2}} A^{\alpha} B^{\frac{\beta}{2}})^{\frac{\delta+\beta}{\alpha+\beta}} \geq B^{\delta+\beta}$ for all $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$.*

We need the following Theorem C.1 for the proof of Proposition 6.

Theorem C.1. *Let A and B be positive invertible operators such that $(B^{\frac{\beta_0}{2}} A^{\alpha_0} B^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}} \geq B^{\beta_0}$ holds for fixed $\alpha_0 > 0$ and $\beta_0 > 0$. Then for each fixed $\delta \geq -\beta_0$,*

$$f_\delta(\alpha, \beta) = B^{\frac{-\beta}{2}} (B^{\frac{\beta}{2}} A^{\alpha} B^{\frac{\beta}{2}})^{\frac{\delta+\beta}{\alpha+\beta}} B^{\frac{-\beta}{2}}$$

is increasing for $\alpha \geq \max\{\alpha_0, \delta\}$ and $\beta \geq \beta_0$.

Theorem C.1 is easily obtained as a variant from the following Theorem C.2 by putting $S = B^{-1}$ and $T = A^{-1}$.

Theorem C.2 ([19]). *Let S and T be positive invertible operators such that $S^{\beta_0} \geq (S^{\frac{\beta_0}{2}} T^{\alpha_0} S^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}}$ holds for fixed $\alpha_0 > 0$ and $\beta_0 > 0$. Then for each fixed $\delta \geq -\beta_0$,*

$$g_\delta(\alpha, \beta) = S^{\frac{-\beta}{2}} (S^{\frac{\beta}{2}} T^{\alpha} S^{\frac{\beta}{2}})^{\frac{\delta+\beta}{\alpha+\beta}} S^{\frac{-\beta}{2}}$$

is decreasing for $\alpha \geq \max\{\alpha_0, \delta\}$ and $\beta \geq \beta_0$.

Proof of Proposition 6.
Proof of (i). Assume that

$$(B^{\frac{\beta_0}{2}}A^{\alpha_0}B^{\frac{\beta_0}{2}})^{\frac{\delta+\beta_0}{\alpha_0+\beta_0}} \geq B^{\delta+\beta_0} \tag{3.1}$$

holds for fixed $\alpha_0 > 0$, $\beta_0 > 0$ and $-\beta_0 < \delta \leq \alpha_0$. Applying (i) of Theorem F to (3.1), we have

$$\{B^{\frac{(\delta+\beta_0)r_1}{2}}(B^{\frac{\beta_0}{2}}A^{\alpha_0}B^{\frac{\beta_0}{2}})^{\frac{(\delta+\beta_0)p_1}{\alpha_0+\beta_0}}B^{\frac{(\delta+\beta_0)r_1}{2}}\}^{\frac{1+r_1}{p_1+r_1}} \geq B^{(\delta+\beta_0)(1+r_1)} \tag{3.2}$$

for any $p_1 \geq 1$ and $r_1 \geq 0$. Putting $p_1 = \frac{\alpha_0+\beta_0}{\delta+\beta_0} \geq 1$ in (3.2), we have

$$(B^{\frac{\beta_0+(\delta+\beta_0)r_1}{2}}A^{\alpha_0}B^{\frac{\beta_0+(\delta+\beta_0)r_1}{2}})^{\frac{(\delta+\beta_0)(1+r_1)}{\alpha_0+\beta_0+(\delta+\beta_0)r_1}} \geq B^{(\delta+\beta_0)(1+r_1)} \quad \text{for any } r_1 \geq 0. \tag{3.3}$$

Put $\beta = \beta_0 + (\delta+\beta_0)r_1 \geq \beta_0$ in (3.3). Then we have

$$(B^{\frac{\beta}{2}}A^{\alpha_0}B^{\frac{\beta}{2}})^{\frac{\delta+\beta}{\alpha_0+\beta}} \geq B^{\delta+\beta} \quad \text{for } \beta \geq \beta_0.$$

Proof of (ii). Assume that

$$(B^{\frac{\beta_0}{2}}A^{\alpha_0}B^{\frac{\beta_0}{2}})^{\frac{\delta+\beta_0}{\alpha_0+\beta_0}} \geq B^{\delta+\beta_0} \tag{3.1}$$

holds for fixed $\alpha_0 > 0$, $\beta_0 > 0$ and $0 \leq \delta \leq \alpha_0$. By Löwner-Heinz theorem, (3.1) ensures

$$(B^{\frac{\beta_0}{2}}A^{\alpha_0}B^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}} \geq B^{\beta_0}.$$

Then we have

$$B^{\frac{-\beta}{2}}(B^{\frac{\beta}{2}}A^{\alpha}B^{\frac{\beta}{2}})^{\frac{\delta+\beta}{\alpha+\beta}}B^{\frac{-\beta}{2}} = f_\delta(\alpha,\beta) \geq f_\delta(\alpha_0,\beta_0) = B^{\frac{-\beta_0}{2}}(B^{\frac{\beta_0}{2}}A^{\alpha_0}B^{\frac{\beta_0}{2}})^{\frac{\delta+\beta_0}{\alpha_0+\beta_0}}B^{\frac{-\beta_0}{2}} \geq B^{\delta},$$

that is,

$$(B^{\frac{\beta}{2}}A^{\alpha}B^{\frac{\beta}{2}})^{\frac{\delta+\beta}{\alpha+\beta}} \geq B^{\delta+\beta}$$

for all $\alpha \geq \max\{\alpha_0, \delta\} = \alpha_0$ and $\beta \geq \beta_0$ by Theorem C.1 and (3.1).

Hence the proof of Proposition 6 is complete. □

Proof of Theorem 4. Since class $\mathrm{F}(p_0, 0, \frac{p_0}{\delta})$ for $0 < \delta \leq p_0$ equals δ-hyponormality and every δ-hyponormal operator belongs to class $\mathrm{F}(p, r, \frac{p+r}{\delta+r})$ for all $p \geq \delta$ and $r \geq 0$ by (i) of Theorem A.1, we may assume $r_0 > 0$. Suppose that T belongs to class $\mathrm{F}(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ for $p_0 > 0$, $r_0 > 0$ and $-r_0 < \delta \leq p_0$, i.e.,

$$(|T^*|^{r_0}|T|^{2p_0}|T^*|^{r_0})^{\frac{\delta+r_0}{p_0+r_0}} \geq |T^*|^{2(\delta+r_0)}. \tag{3.4}$$

Proof of (i). Applying (i) of Proposition 6, (3.4) ensures

$$(|T^*|^{r}|T|^{2p_0}|T^*|^{r})^{\frac{\delta+r}{p_0+r}} \geq |T^*|^{2(\delta+r)}$$

for any $r \geq r_0$, i.e., T belongs to class $\mathrm{F}(p_0, r, \frac{p_0+r}{\delta+r})$ for any $r \geq r_0$.

Proof of (ii). If T is invertible and $0 \leq \delta \leq p_0$, applying (ii) of Proposition 6, (3.4) ensures

$$(|T^*|^{r}|T|^{2p}|T^*|^{r})^{\frac{\delta+r}{p+r}} \geq |T^*|^{2(\delta+r)}$$

for any $p \geq p_0$ and $r \geq r_0$, i.e., T belongs to class $\mathrm{F}(p, r, \frac{p+r}{\delta+r})$ for any $p \geq p_0$ and $r \geq r_0$.

Hence the proof of Theorem 4 is complete. □

We need the following Theorem H-M in order to prove Theorem 5.

Theorem H-M (Hölder-McCarthy inequality [26]). *Let A be a positive operator. Then the following inequalities hold for all $x \in H$:*

(i) $(A^r x, x) \le (Ax, x)^r \|x\|^{2(1-r)}$ *or* $\|A^r x\| \le \|Ax\|^r \|x\|^{1-r}$ *for* $0 < r \le 1$.
(ii) $(A^r x, x) \ge (Ax, x)^r \|x\|^{2(1-r)}$ *or* $\|A^r x\| \ge \|Ax\|^r \|x\|^{1-r}$ *for* $r \ge 1$.

Proof of Theorem 5. Suppose that T is $(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$-paranormal for $p_0 > 0$, $r_0 \ge 0$ and $\delta \ge -r_0$, i.e.,

$$\big\||T|^{p_0}U|T|^{r_0}x\big\|^{\frac{\delta+r_0}{p_0+r_0}} \ge \big\||T|^{\delta+r_0}x\big\| \tag{3.5}$$

for every unit vector $x \in H$, where $T = U|T|$ is the polar decomposition of T. (3.5) holds if and only if

$$\big\||T|^{p_0}U|T|^{r_0}y\big\|^{\frac{\delta+r_0}{p_0+r_0}} \ge \big\||T|^{\delta+r_0}y\big\|\big\|y\big\|^{\frac{\delta+r_0}{p_0+r_0}-1} \tag{3.6}$$

holds for all $y \in H$.

Proof of (i). If $-r_0 < \delta \le p_0$, then for any $r \ge r_0$ and every unit vector $x \in H$,

$$\begin{aligned}
&\big\||T|^{p_0}U|T|^{r}x\big\|^{\frac{\delta+r}{p_0+r}}\\
=&\big\||T|^{p_0}U|T|^{r_0}\cdot|T|^{r-r_0}x\big\|^{\frac{\delta+r_0}{p_0+r_0}\cdot\frac{p_0+r_0}{\delta+r_0}\frac{\delta+r}{p_0+r}}\\
\ge&\left(\big\||T|^{\delta+r_0}\cdot|T|^{r-r_0}x\big\|\big\||T|^{r-r_0}x\big\|^{\frac{\delta+r_0}{p_0+r_0}-1}\right)^{\frac{p_0+r_0}{\delta+r_0}\frac{\delta+r}{p_0+r}} \qquad \text{by (3.6)}\\
\ge&\left(\big\||T|^{\delta+r}x\big\|\big\||T|^{\delta+r}x\big\|^{\frac{r-r_0}{\delta+r}(\frac{\delta+r_0}{p_0+r_0}-1)}\right)^{\frac{p_0+r_0}{\delta+r_0}\frac{\delta+r}{p_0+r}}\\
=&\big\||T|^{\delta+r}x\big\|,
\end{aligned}$$

and the last inequality holds by (i) of Theorem H-M for $\frac{r-r_0}{\delta+r} \in [0,1)$. Therefore T is $(p_0, r, \frac{p_0+r}{\delta+r})$-paranormal for any $r \ge r_0$.

Proof of (ii). If $0 \le \delta$, then for any $p \ge p_0$ and every unit vector $x \in H$,

$$\begin{aligned}
&\big\||T|^{p}U|T|^{r_0}x\big\|^{\frac{\delta+r_0}{p+r_0}}\\
\ge&\left(\big\||T|^{p_0}U|T|^{r_0}x\big\|^{\frac{p}{p_0}}\big\|U|T|^{r_0}x\big\|^{1-\frac{p}{p_0}}\right)^{\frac{\delta+r_0}{p+r_0}}\\
\ge&\left(\big\||T|^{p_0}U|T|^{r_0}x\big\|^{\frac{p}{p_0}}\big\||T|^{r_0}x\big\|^{1-\frac{p}{p_0}}\right)^{\frac{\delta+r_0}{p+r_0}} \qquad \text{since } U \text{ is partial isometry}\\
\ge&\left(\big\||T|^{\delta+r_0}x\big\|^{\frac{p_0+r_0}{\delta+r_0}\frac{p}{p_0}}\big\||T|^{\delta+r_0}x\big\|^{\frac{r_0}{\delta+r_0}(1-\frac{p}{p_0})}\right)^{\frac{\delta+r_0}{p+r_0}}\\
=&\big\||T|^{\delta+r_0}x\big\|,
\end{aligned}$$

and the first inequality holds by (ii) of Theorem H-M for $\frac{p}{p_0} \ge 1$ and the last inequality holds by (3.5) and (i) of Theorem H-M for $\frac{r_0}{\delta+r_0} \in [0,1]$. Therefore T is $(p, r_0, \frac{p+r_0}{\delta+r_0})$-paranormal for any $p \ge p_0$.

Proof of (iii). It is obvious by (i) and (ii).

Hence the proof of Theorem 5 is complete. □

Remark. Here we shall show that (i) of Theorem 4 ensures (ii) of Theorem A.1 and also Proposition 3 ensures (vi) and (vii) of Theorem A.1.

Proof of (ii) *of Theorem A.1.* Suppose that T belongs to class $\mathrm{F}(p_0, r_0, q_0)$ for $p_0 > 0$, $r_0 \geq 0$ and $q_0 \geq 1$. Let $q_0 = \frac{p_0+r_0}{\delta+r_0}$. Then class $\mathrm{F}(p_0, r_0, q_0)$ for $p_0 > 0$, $r_0 \geq 0$ and $q_0 \geq 1$ equals class $\mathrm{F}(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ for $p_0 > 0$, $r_0 \geq 0$ and $-r_0 < \delta \leq p_0$. By (i) of Theorem 4, T belongs to class $\mathrm{F}(p_0, r, \frac{p_0+r}{\delta+r})$ for $r \geq r_0$. Therefore, by (iii) of Theorem A.1, T belongs to class $\mathrm{F}(p_0, r, q_0)$ for $r \geq r_0$ since $q_0 \geq \frac{p_0+r}{\delta+r}$. □

Proof of (vi) *of Theorem A.1.* Obvious by Proposition 3. □

Proof of (vii) *of Theorem A.1.* Suppose that T is $(p, r, 1)$-paranormal for $p > 0$ and $r \geq 0$. By Proposition 3 and (v) of Theorem A.1, T is $(p, p, 2)$-paranormal, i.e., absolute-(p, p)-paranormal (or p-paranormal) for $p > 0$. Therefore T is absolute-(s, s)-paranormal where $s = \max\{p, r\}$, i.e., $\max\{p, r\}$-paranormal by (ii) of Theorem A.3. □

4. Log-hyponormality

As a characterization of log-hyponormal operators, the following Theorem D.1 was obtained.

Theorem D.1 ([10][31][32]). *Let T be an invertible operator. Then the following assertions are mutually equivalent:*

(i) *T is log-hyponormal.*
(ii) *T belongs to class $A(p, p)$, i.e., class $AI(p, p)$ for all $p > 0$.*
(iii) *T belongs to class $A(p, r)$, i.e., class $AI(p, r)$ for all $p > 0$ and $r > 0$.*
(iv) *T is p-paranormal for all $p > 0$.*
(v) *T is absolute-(p, r)-paranormal for all $p > 0$ and $r > 0$.*

(i)⇔(ii)⇔(iii) was obtained in [10], and also (i)⇔(iv) and (i)⇔(iv)⇔(v) were obtained in [31] and [32], respectively.

As an extension of Theorem D.1 via class $\mathrm{F}(p, r, q)$ and (p, r, q)-paranormality, we have the following Theorem 7.

Theorem 7. *Let T be an invertible operator. Then the following assertions are mutually equivalent for any fixed $\alpha \in (0, 1]$:*

(i) *T is log-hyponormal.*
(ii) *T belongs to class $F(p, p, \frac{2}{\alpha})$ for all $p > 0$.*
(iii) *T belongs to class $F(p, r, \frac{p+r}{r\alpha})$ for all $p > 0$ and $r > 0$.*
(iv) *T is $(p, p, \frac{2}{\alpha})$-paranormal for all $p > 0$.*
(v) *T is $(p, r, \frac{p+r}{r\alpha})$-paranormal for all $p > 0$ and $r > 0$.*

We remark that Theorem 7 ensures Theorem D.1 by putting $\alpha = 1$.

Proof of Theorem 7. (i)⇒(iii) holds by (i) of Theorem A.2 and (iii) of Theorem A.1. (iii)⇒(ii)⇒(iv) and (iii)⇒(v)⇒(iv) hold by putting $r = p$ and by (iv) of Theorem A.1. Therefore we have only to show (iv)⇒(i).

Suppose that T is invertible and $(p,p,\frac{2}{\alpha})$-paranormal for all $p>0$ and a fixed $\alpha\in(0,1]$. T is $(p,p,\frac{2}{\alpha})$-paranormal if and only if

$$\big\||T|^p|T^*|^p y\big\|^{\frac{\alpha}{2}}\big\|y\big\|^{1-\frac{\alpha}{2}}\geq\big\||T^*|^{p\alpha}y\big\| \tag{4.1}$$

for all $y\in H$ by Proposition 1. By (4.1) and the generalized arithmetic-geometric mean inequality, we have

$$\tfrac{\alpha}{2}\big\||T|^p|T^*|^p y\big\|^2+(1-\tfrac{\alpha}{2})\big\|y\big\|^2\geq\big\||T|^p|T^*|^p y\big\|^{2\cdot\frac{\alpha}{2}}\big\|y\big\|^{2(1-\frac{\alpha}{2})}\geq\big\||T^*|^{p\alpha}y\big\|^2 \tag{4.2}$$

for all $y\in H$ and (4.2) ensures

$$\tfrac{\alpha}{2}|T^*|^p|T|^{2p}|T^*|^p+(1-\tfrac{\alpha}{2})I\geq|T^*|^{2p\alpha}. \tag{4.3}$$

Since T is invertible, (4.3) holds if and only if

$$\frac{\alpha}{2}\frac{|T|^{2p}-I}{p}+\Big(1-\frac{\alpha}{2}\Big)\frac{|T^*|^{-2p}-I}{p}\geq\frac{|T^*|^{2p(\alpha-1)}-I}{p}. \tag{4.4}$$

By letting $p\to+0$ in (4.4), we have

$$\tfrac{\alpha}{2}\log|T|^2+(1-\tfrac{\alpha}{2})\log|T^*|^{-2}\geq\log|T^*|^{2(\alpha-1)},$$

that is, $\log|T|^2\geq\log|T^*|^2$. Hence T is log-hyponormal, so the proof of Theorem 7 is complete. $\square$

5. Powers of class F(p,r,q) operators

On powers of class A operators, we obtained the following result.

Theorem E.1 ([23])**.**

(i) *If T is an invertible class A operator, then T^n also belongs to class A for all positive integer n.*

(ii) *If T is an invertible class A operator, then*

$$|T|^2\leq|T^2|\leq\cdots\leq|T^n|^{\frac{2}{n}} \tag{5.1}$$

and

$$|T^*|^2\geq|T^{2^*}|\geq\cdots\geq|T^{n^*}|^{\frac{2}{n}} \tag{5.2}$$

hold for all positive integer n.

As an extension of (i) of Theorem E.1, Yamazaki [30] showed the following Theorem E.2.

Theorem E.2 ([30])**.** *Let T be a class AI(p,r) operator for $0<p\leq1$ and $0<r\leq1$. Then T^n belongs to class AI$(\frac{p}{n},\frac{r}{n})$ for all positive integer n.*

In this section, we shall show the following result on powers of class F(p,r,q) operators as a similar result to Theorem E.2.

Theorem 8. *Let T be an invertible class F(p,r,q) operator for $0<p\leq1$, $0\leq r\leq1$ and $q\geq1$ with $rq\leq p+r$. Then T^n belongs to class F$(\frac{p}{n},\frac{r}{n},q)$ for all positive integer n.*

We cite the following Lemma F in order to give a proof of Theorem 8.

Lemma F ([16]). *Let $A > 0$ and B be an invertible operator. Then*

$$(BAB^*)^{\lambda} = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

holds for any real number λ.

Proof of Theorem 8. Let T be an invertible class $\mathrm{F}(p,r,q)$ operator for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p+r$, i.e.,

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}}. \tag{1.2}$$

We remark that class $\mathrm{F}(p,r,q)$ operator T for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p+r$ belongs to class $\mathrm{F}(1,1,2)$, i.e., class A by (iii) of Theorem A.1 and (ii) of Theorem 4. By Lemma F, (1.2) holds if and only if

$$|T^*|^r|T|^p(|T|^p|T^*|^{2r}|T|^p)^{\frac{1}{q}-1}|T|^p|T^*|^r \geq |T^*|^{\frac{2(p+r)}{q}}$$

if and only if

$$|T|^p(|T|^p|T^*|^{2r}|T|^p)^{\frac{1}{q}-1}|T|^p \geq |T^*|^{2(\frac{p+r}{q}-r)}. \tag{5.3}$$

Then we have

$$\begin{aligned} |T|^p(|T|^p|T^{n^*}|^{\frac{2r}{n}}|T|^p)^{\frac{1}{q}-1}|T|^p &\geq |T|^p(|T|^p|T^*|^{2r}|T|^p)^{\frac{1}{q}-1}|T|^p \geq |T^*|^{2(\frac{p+r}{q}-r)} \\ &\geq |T^{n^*}|^{\frac{2}{n}(\frac{p+r}{q}-r)} \end{aligned} \tag{5.4}$$

for all positive integer n by (5.2) in Theorem E.1, (5.3) and Löwner-Heinz theorem since $\frac{1}{q} - 1 \in (-1, 0]$ and $\frac{p+r}{q} - r \in [0,1]$. (5.4) ensures

$$|T^{n^*}|^{\frac{r}{n}}|T|^p(|T|^p|T^{n^*}|^{\frac{2r}{n}}|T|^p)^{\frac{1}{q}-1}|T|^p|T^{n^*}|^{\frac{r}{n}} \geq |T^{n^*}|^{\frac{2}{n}\frac{p+r}{q}} \tag{5.5}$$

and (5.5) holds if and only if

$$(|T^{n^*}|^{\frac{r}{n}}|T|^{2p}|T^{n^*}|^{\frac{r}{n}})^{\frac{1}{q}} \geq |T^{n^*}|^{\frac{2}{n}\frac{p+r}{q}} \tag{5.6}$$

by Lemma F. Hence we have

$$(|T^{n^*}|^{\frac{r}{n}}|T^n|^{\frac{2p}{n}}|T^{n^*}|^{\frac{r}{n}})^{\frac{1}{q}} \geq (|T^{n^*}|^{\frac{r}{n}}|T|^{2p}|T^{n^*}|^{\frac{r}{n}})^{\frac{1}{q}} \geq |T^{n^*}|^{\frac{2}{q}(\frac{p}{n}+\frac{r}{n})}$$

for all positive integer n by (5.1) in Theorem E.1, (5.6) and Löwner-Heinz theorem, that is, T^n belongs to class $\mathrm{F}(\frac{p}{n}, \frac{r}{n}, q)$ for all positive integer n. □

Remark. On powers of p-hyponormal operators, Aluthge-Wang showed the following result in [1] (see also [20][21][24][29]).

Theorem E.3 ([1]). *Let T be a p-hyponormal operator for $0 < p \leq 1$. Then T^n is $\frac{p}{n}$-hyponormal for all positive integer n.*

We remark that Theorem 8 interpolates Theorem E.2 and Theorem E.3 in case T is invertible. In fact, Theorem 8 yields Theorem E.2 by putting $q = \frac{p+r}{r}$, and also Theorem 8 yields Theorem E.3 by putting $q = 1$ and $r = 0$.

6. Concluding remark

In [11], Fujii-Nakamoto introduced absolute (p,r)-paranormality in different form from (1.1) by Yamazaki-Yanagida as follows: T is absolute (p,r)-paranormal (in symbol: $T \in \mathrm{AP}(p,r)$) for $p>0$ and $r \geq 0$ if $\left\| |T|^p U |T|^r x \right\| \geq \left\| |T|x \right\|^{p+r}$ for every unit vector $x \in H$, where $T = U|T|$ is the polar decomposition of T. We note that we write $T \in \mathrm{AP}(p,r)$ if T is an absolute (p,r)-paranormal operator in the sense of Fujii-Nakamoto, and also $T \in \mathrm{AP}(p,r)$ for $p>0$ and $r \geq 0$ if and only if T is a $(p,r,p+r)$-paranormal operator. The following theorems are shown in [11].

Theorem F.1 ([11]).

(i) *If T belongs to class $F(p,r,1)$ for $p>0$ and $r \geq 0$ with $p+r \geq 1$, then $T \in AP(p,r)$.*
(ii) *If $T \in AP(p_0,r_0)$ for $p_0>0$ and $0 \leq r_0 \leq 1$, then $T \in AP(p,r_0)$ for any $p \geq p_0$.*
(iii) *If $T \in AP(p_0,r_0)$ for $p_0>0$ and $r_0 \geq 0$ with $p_0+r_0 \geq 1$, then $T \in AP(p_0,r)$ for any $r \geq r_0$.*

Theorem F.2 ([11]).

(i) *If T is (p,r,q)-paranormal for $p>0$, $r \geq 0$ and $q \geq 1$, then $T \in AP(p,s)$ for $s \geq 0$ with $r \leq s \leq 1+r$ and $p+r \geq q(1-s+r)$.*
(ii) *If $T \in AP(p,s)$ for $p>0$ and $s \geq 0$ with $p+s \geq 1$, then T is (p,r,q)-paranormal for $r \geq s$ and $q \geq 1$ with $q(1+r-s) \geq p+r \geq q(r-s)$.*
(iii) *If $T \in AP(p,s)$ for $p>0$ and $s \geq 0$ with $p+s \leq 1$, then T is (p,r,q)-paranormal for $r \geq s$ and $q \geq 1$ with $q(r-s) \geq p+r$.*

Here we shall show that Proposition 3 and Theorem 5 lead Theorem F.1 and Theorem F.2.

Proof of Theorem F.1.
Proof of (i). Suppose that T belongs to class $\mathrm{F}(p,r,1)$ for $p>0$ and $r \geq 0$ with $p+r \geq 1$. Then T is $(p,r,1)$-paranormal by Proposition 3. Hence T is $(p,r,p+r)$-paranormal for $p+r \geq 1$ by (v) of Theorem A.1, that is, $T \in \mathrm{AP}(p,r)$.

Proof of (ii). Suppose that $T \in \mathrm{AP}(p_0,r_0)$ for $p_0>0$ and $0 \leq r_0 \leq 1$, i.e., T is $(p_0, r_0, \frac{p_0+r_0}{(1-r_0)+r_0})$-paranormal. Since $0 \leq 1-r_0$, (ii) of Theorem 5 ensures that T is $(p, r_0, \frac{p+r_0}{(1-r_0)+r_0})$-paranormal for any $p \geq p_0$, that is, $T \in \mathrm{AP}(p,r_0)$.

Proof of (iii). Suppose that $T \in \mathrm{AP}(p_0,r_0)$ for $p_0>0$ and $r_0 \geq 0$ with $p_0+r_0 \geq 1$, i.e., T is $(p_0, r_0, \frac{p_0+r_0}{(1-r_0)+r_0})$-paranormal. Then T is $(p_0, r, \frac{p_0+r}{(1-r_0)+r})$-paranormal for any $r \geq r_0$ by (i) of Theorem 5 since $-r_0 < 1-r_0 \leq p_0$. Hence T is (p_0, r, p_0+r)-paranormal for $r \geq r_0$ by (v) of Theorem A.1, that is, $T \in \mathrm{AP}(p_0,r)$. □

Proof of Theorem F.2.
Proof of (i). Suppose that T is (p,r,q)-paranormal for $p>0$, $r \geq 0$ and $q \geq 1$. Let $\epsilon = \frac{p+r}{q}$ where $0 < \epsilon \leq p+r$. Then (p,r,q)-paranormality equals $(p, r, \frac{p+r}{(\epsilon-r)+r})$-paranormality. Since $-r < \epsilon - r \leq p$, (i) of Theorem 5 ensures that T is $(p, s, \frac{p+s}{(\epsilon-r)+s})$-paranormal for any $s \geq r$. Hence T is $(p,s,p+s)$-paranormal for $s \geq r$ and $(\epsilon - r)+s \geq 1$ by (v) of Theorem A.1, that is, $T \in \mathrm{AP}(p,s)$ for $s \geq r$ and $p+r \geq q(1-s+r)$.

Proof of (ii). Suppose that $T \in \mathrm{AP}(p,s)$ for $p>0$ and $s \geq 0$ with $p+s \geq 1$, i.e., T is $(p, s, \frac{p+s}{(1-s)+s})$-paranormal. Then T is $(p, r, \frac{p+r}{(1-s)+r})$-paranormal for any $r \geq s$ by (i) of Theorem 5 since $-s < 1-s \leq p$. Hence T is (p,r,q)-paranormal for $r \geq s$ and $q(1+r-s) \geq p+r$ by (v) of Theorem A.1.

Proof of (iii). Suppose that $T \in \mathrm{AP}(p,s)$ for $p>0$ and $s \geq 0$ with $p+s \leq 1$, i.e., T is $(p,s,p+s)$-paranormal. Then T is $(p,s,1)$-paranormal by (v) of Theorem A.1, and also $(p,s,1)$-paranormality equals $(p,r,1)$-paranormality for any $r \geq 0$ by Proposition 3. Hence T is (p,r,q)-paranormal for $r \geq 0$ and $q \geq 1$ by (v) of Theorem A.1. □

By scrutinizing the proof of Theorem F.2, we see that the condition of parameters in Theorem F.2 can be loosened as follows:

Theorem 9.

(i) *If T is (p,r,q)-paranormal for $p>0$, $r \geq 0$ and $q \geq 1$, then $T \in AP(p,s)$ for $s \geq r$ and $p+r \geq q(1-s+r)$.*

(ii) *If $T \in AP(p,s)$ for $p>0$ and $s \geq 0$ with $p+s \geq 1$, then T is (p,r,q)-paranormal for $r \geq s$ and $q(1+r-s) \geq p+r$.*

(iii) *If $T \in AP(p,s)$ for $p>0$ and $s \geq 0$ with $p+s \leq 1$, then T is (p,r,q)-paranormal for $r \geq 0$ and $q \geq 1$.*

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