ON RIGHT SIMPLE AND RIGHT 0-SIMPLE ORDERED GROUPOIDS-SEMIGROUPS

NIOVI KEHAYOPULU AND MICHAEL TSINGELIS

Received April 16, 2000

Abstract. The right (resp. left) simple and the simple ordered semigroups play an important role in the structure of ordered semigroups. In this note we prove that there is no essential difference between the right (resp. left) simple and the right (resp. left) 0-simple ordered semigroups. In this respect, we prove that an ordered groupoid S without zero is right (resp. left) simple if and only if the ordered groupoid S^0 arising from S by the adjunction of a zero is right (resp. left) 0-simple. Moreover, an ordered semigroup S with a zero element 0 is right (resp. left) 0-simple if and only if the set $S \setminus \{0\}$ is a right (resp. left) simple subsemigroup of S. The sufficient condition holds in ordered groupoids, in general. That is, if S is an ordered groupoid with zero and if the set $S \setminus \{0\}$ is a right (resp. left) simple subgroupoid of S, then S is right (resp. left) 0-simple.

If $(S, ., \leq)$ is an ordered groupoid, a zero element of S is an element of S, denoted by 0, such that $0 \leq x$ and $0x = x0 = 0 \ \forall \ x \in S$ (cf. also [1]). If $(S, ., \leq)$ is an ordered groupoid without zero and 0 an element which does not belong to S, we denote by $S^0 := S \cup \{0\}$ the ordered groupoid defined by the multiplication and the order below:

$$x * y := \begin{cases} xy & \text{if} \quad x, y \in S \\ 0 & \text{if} \quad x = 0 \text{ or } y = 0 \end{cases}$$

$$\leq_0 := \leq \cup \{(0, x) \mid x \in S^0\}.$$

This is the ordered groupoid arising from S by the adjunction of a zero element. In fact, the element 0 is the zero element of S^0. If $(S, ., \leq)$ is an ordered semigroup, then S^0 is an ordered semigroup, as well (cf. [7]).
If $(S, ., \leq)$ is an ordered groupoid, a right (resp. left) ideal of S is a non-empty subset I of S satisfying the conditions 1) $IS \subseteq I$ (resp. $SI \subseteq I$). 2) $a \in I$, $S \ni b \leq a$ imply $b \in I$. I is called an ideal of S if it both a left and a right ideal of S [3; Definition 1].

Definition 1. Let S be an ordered groupoid without zero. A subgroupoid T of S is called right (resp. left) simple if for every right (resp. left) ideal I of T we have $I = T$. (In other words, if T is the only right (resp. left) ideal of T) [cf. 4; Definition 3]. A subgroupoid T of S is called simple if for every ideal I of T we have $I = T$ (cf. also [5] and [6]).

Definition 2. Let S be an ordered groupoid with zero. A subgroupoid T of S is called right (resp. left) 0-simple if

1991 *Mathematics Subject Classification.* 06F05, 20N02.
Key words and phrases. Right (resp. left) simple, simple ordered groupoids (semigroups), right (left) 0-simple, 0-simple ordered groupoids (semigroups) .

1) $T^2 \neq \{0\}$.

2) The only right (resp. left) ideals of T are the sets $\{0\}$ and T.

A subgroupoid T of S is called 0-simple if $T^2 \neq \{0\}$ and the only ideals of T are the sets $\{0\}$ and T. □

An ordered groupoid S with 0 is called null if $S^2 = \{0\}$. Clearly, a right (resp. left) 0-simple ordered groupoid is not null, and a null ordered groupoid cannot be right or left 0-simple.

Remark 1. Let S be an ordered groupoid with 0. If R is a right (resp. left) ideal of S, then $0 \in R$ (resp. $0 \in L$). Moreover, the set $\{0\}$ is a right (resp. left) ideal of S.

Lemma (cf. [7; Proposition 1])**.** *Let $(S, ., \leq)$ be an ordered groupoid. If R is a right ideal of S, then $R \cup \{0\}$ is a right ideal of $(S^0, *, \leq)$. If R is a right ideal of $(S^0, *, \leq_0)$ and $R \neq \{0\}$, then $R \setminus \{0\}$ is a right ideal of $(S, ., \leq)$.*

Proposition 1. *Let $(S, ., \leq)$ be an ordered groupoid without zero. The following are equivalent:*

1) *$(S, ., \leq)$ is right (resp. left) simple.*

2) *$(S^0, *, \leq_0)$ is right (resp. left) 0-simple.*

Proof. 1) $\Longrightarrow$ 2). Let $a \in S$ ($S \neq \emptyset$). Since $a \in S \subset S^0$, we have $a * a \in S^0 * S^0$. Since $a \in S$, we have $a * a := aa \in S$. Since $0 \notin S$, we have $a * a \neq 0$. Thus $S^0 * S^0 \neq \{0\}$. Let R be a right ideal of S^0 such that $R \neq \{0\}$. By the Lemma, $R \setminus \{0\}$ is a right ideal of S. Since S is right simple, we have $R \setminus \{0\} = S$. Since R is a right ideal of S^0, we have $0 \in R$. Then we have
$$S^0 = S \cup \{0\} = (R \setminus \{0\}) \cup \{0\} = R \text{ (since } 0 \in R).$$
2) $\Longrightarrow$ 1). Let R be a right ideal of S. By the Lemma, $R \cup \{0\}$ is a right ideal of S^0. Then, by 2), $R \cup \{0\} = \{0\}$ or $R \cup \{0\} = S^0$. Let $R \cup \{0\} = \{0\}$. Let $a \in R$ ($R \neq \emptyset$). Then $a \in R \cup \{0\} = \{0\}$, and $S \ni a = 0$. Impossible. Thus $R \cup \{0\} = S^0 = S \cup \{0\}$. Then, since $0 \notin S$, we have $S \subseteq R$. Since R is a right ideal of S, we have $R = S$.

Corollary 1. *An ordered groupoid S without zero is simple if and only if the ordered semigroup S^0 arising from S by the adjunction of a zero element, is 0-simple.*

Proposition 2. *Let $(S, ., \leq)$ be an ordered semigroup with zero. The following are equivalent:*

1) *$(S, ., \leq)$ is right (resp. left) 0-simple.*

2) *$S \setminus \{0\}$ is a right (resp. left) simple subsemigroup of S.*

Proof. 1) $\Longrightarrow$ 2). Let S be right 0- simple. Then:

A) The set $S \setminus \{0\}$ is a subsemigroup of S. In fact:

Since S is right 0-simple, we have $S^2 \neq \{0\}$, then $S \neq \{0\}$. Since $0 \in S$, $\{0\} \subseteq S$. Then $\{0\} \subset S$ and $S \supseteq S\ \{0\} \neq \emptyset$.

Let $a, b \in S \setminus \{0\}$. Then $ab \in S \setminus \{0\}$. Indeed: Since $a, b \in S$, we have $ab \in S$. Let $ab = 0$. We consider the set

$$R := \{x \in S \mid ax = 0\}.$$

R is a right ideal of S. Indeed: $\emptyset \neq R \subseteq S$ (since $b \in R$). If $x \in R$ and $y \in S$, then $a(xy) = (ax)y = 0$, and $xy \in R$ (cf. also the proof of the Theorem 2.27 in [2]). Let $x \in R$, $S \ni y \leq x$. Since $x \in R$, $ax = 0$. Then $ay \leq ax = 0$. Since 0 is the zero of S, $0 \leq ay$. Then $ay = 0$, and $y \in R$.

Since R is a right ideal of S and S is right 0-simple, we have $R = \{0\}$ or $R = S$. Since $b \in R, b \neq 0$, we have $R \neq \{0\}$. Then $R = S$. We consider the set

$$(a] := \{t \in S \mid t \leq a\}.$$

(a] is a right ideal of S. Indeed: $\emptyset \neq (a] \subseteq S$ (since $a \in (a]$). Let $x \in (a], y \in S$. Then $x \leq a, xy \leq ay$. Since $y \in S = R$, we have $ay = 0$. Then $xy \leq 0 \leq a$, and $xy \in (a]$. Let $x \in (a], S \ni y \leq x$. Since $y \leq x \leq a$, we have $y \in (a]$.

Since $(a]$ is a right ideal of S and S is right 0-simple, we have $(a] = \{0\}$ or $(a] = S$. Since $a \in (a], a \neq 0$, we have $(a] \neq \{0\}$. Thus $(a] = S$. Hence we have $S^2 = SS = (a](a] \subseteq (a^2]$. Since $a \in S = R$, we have $a^2 = 0$. Thus we have $S^2 \subseteq (0] = \{0\}$. On the other hand, $0 = 00 \in S^2, \{0\} \subseteq S^2$. Thus $S^2 = \{0\}$. Impossible.

B) Let A be a right ideal of $S \setminus \{0\}$ ($\Longrightarrow \quad A = S \setminus \{0\}$?)

The set $A \cup \{0\}$ is a right ideal of S. Indeed:
$\emptyset \neq A \cup \{0\} \subseteq S$ (since $A \neq \emptyset$). If $a \in A \cup \{0\}, b \in S$, then $ab \in A \cup \{0\}$ (cf. the proof of the Theorem 2.27 in [2]).
Let $a \in A \cup \{0\}, S \ni b \leq a$ ($\Longrightarrow \quad b \in A \cup \{0\}$?)
If $b = 0$, then $b \in A \cup \{0\}$. If $b \neq 0$, then $a \neq 0$, and $a \in A$. Since $S \setminus \{0\} \ni b \leq a \in A$, A right ideal of $S \setminus \{0\}$, we have $b \in A \subset A \cup \{0\}$.

Since $A \cup \{0\}$ is a right ideal of S and S is right 0-simple, we have $A \cup \{0\} = \{0\}$ or $A \cup \{0\} = S$. Let $A \cup \{0\} = \{0\}$. Let $a \in A$ ($A \neq \emptyset$). Then $a \in S$ and $a = 0$. Impossible. Thus $A \cup \{0\} = S$, then $S \setminus \{0\} \subseteq A$. Besides, since A is a right ideal of $S \setminus \{0\}$, we have $A \subseteq S \setminus \{0\}$. Hence $A = S \setminus \{0\}$.

2) $\Longrightarrow$ 1). Let $S \setminus \{0\}$ be a right simple subsemigroup of S. Then:

A) We have $S^2 \neq \{0\}$. In fact:
Since $S \setminus \{0\}$ is a subsemigroup of S, we have $S \setminus \{0\} \neq \emptyset$. Let $a \in S \setminus \{0\}$. Then $a \in S$ and $a^2 \in S^2$. Since $S \setminus \{0\}$ is a subsemigroup of S, we have $a^2 \in S \setminus \{0\}$, then $a^2 \neq 0$. Since $a^2 \in S^2, a^2 \neq 0$, we have $S^2 \neq \{0\}$.

B) Let A be a right ideal of S and $A \neq \{0\}$ ($\Longrightarrow \quad A = S$?)

The set $A \setminus \{0\}$ is a right ideal of $S \setminus \{0\}$. Indeed:

Since A is a right ideal of S, we have $0 \in A$ (cf. the Remark). Then $\{0\} \subseteq A$. By hypothesis $A \neq \{0\}$. Thus $\{0\} \subset A$, and $A \setminus \{0\} \neq \emptyset$. Since $A \subseteq S$, we have $\emptyset \neq A \setminus \{0\} \subseteq S \setminus \{0\}$.
Let $a \in A \setminus \{0\}$, $b \in S \setminus \{0\}$ ($\Longrightarrow ab \in A \setminus \{0\}$?)

Since $a \in A, b \in S$, A right ideal of S, we have $ab \in A$. Since $a, b \in S \setminus \{0\}, S \setminus \{0\}$ subsemigroup of S, we have $ab \in S \setminus \{0\}$, so $ab \neq 0$.
Let $a \in A \setminus \{0\}, S \setminus \{0\} \ni b \leq a$ ($\Longrightarrow \quad b \in A \setminus \{0\}$?)
We have $S \ni b \leq a \in A$, A right ideal of S, so $b \in A$. Since $b \in S \setminus \{0\}$, $b \neq 0$.

Since $A \setminus \{0\}$ is a right ideal of $S \setminus \{0\}$ and $S \setminus \{0\}$ is a right simple subsemigroup of S, we have $A \setminus \{0\} = S \setminus \{0\}$. Then $A = S$. Indeed: Let $a \in S$. If $a = 0$, then, since $0 \in A$ (cf. the Remark), we have $a \in A$. If $a \neq 0$, then $a \in S \setminus \{0\} = A \setminus \{0\}$, and $a \in A$. On the other hand, since A is a right ideal of S, we have $A \subseteq S$.

Remark 2. The associativity does not play any role in the proof of condition 2) $\Longrightarrow$ 1) of Proposition 2 above. So condition 2) $\Longrightarrow$ 1) of Proposition 2 holds in ordered groupoids, in general. That is, if S is an ordered groupoid with zero and if the set the set $S \setminus \{0\}$ is a right (resp. left) simple subgroupoid of S, then S is right (resp. left) 0-simple.

Corollary 2. *Let S be an ordered groupoid with zero. If $S \setminus \{0\}$ is a simple subgroupoid of S, then S is 0-simple. In particular, an ordered semigroup S is 0-simple if and only if $S \setminus \{0\}$ is a simple subsemigroup of S.*

Remark 3. The sufficient condition in the Theorem 2.27 in [2] also holds. That is: If S is a semigroup and $S \setminus \{0\}$ a right (resp. left) simple subsemigroup of S, then S is a right (resp. left) simple semigroup. More generally, if S is a groupoid and if $S \setminus \{0\}$ a right (resp. left) simple subgroupoid of S, then S is a right (resp. left) simple groupoid.

References

[1] G. Birkhoff, *Lattice Theory,* Amer. Math. Soc. Coll. Publ. Vol. **XXV**, Providence, Rh. Island, 1967.

[2] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups* Vol. **I**, Amer. Math. Soc., Math. Surveys 7, Providence, Rh. Island, 1961.

[3] N. Kehayopulu, *On weakly prime ideals of ordered semigroups,* Mathematica Japonica, **35** (No. 6) (1990), 1051-1056.

[4] N. Kehayopulu, *On left regular ordered semigroups,* Mathematica Japonica, **35** (No. 6) (1990), 1057- 1060.

[5] N. Kehayopulu, *Note on Green's relations in ordered semigroups,* Mathematica Japonica, **36** (No. 2) (1991), 211-214.

[6] N. Kehayopulu, *On intra-regular ordered semigroups,* Semigroup Forum, **46** (1993), 271-278.

[7] N. Kehayopulu and M. Tsingelis, *A note on ordered groupoids, semigroups,* Scientiae Mathematicae, **3** (No. 2) (2000), 315-319.

University of Athens, Department of Mathematics
Mailinig (home) address: Nikomidias 18, 161 22 KESARIANI, GREECE
e-mail: nkehayop@cc.uoa.gr