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CHARACTERIZATIONS OF 0-SIMPLE ORDERED SEMIGROUPS

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Abstract. We characterize the 0-simple ordered semigroups that is the ordered semigroups S having a zero element, denoted by 0, in which $S^2 = \{0\}$ and $\{0\}$ and S are the only ideals of S.

When we speak about a simple ordered semigroup we assume that the ordered semigroup does not contain a zero element. Otherwise we speak about a 0-simple ordered semigroup. An ordered semigroup (without zero) is called simple if S is the only ideal of S. Following the terminology given by A. H. Clifford and G. B. Preston for algebraic semigroups in [2], an ordered semigroup (with 0) is called 0-simple if $S^2 = \{0\}$ and if the sets $\{0\}$ and S are the only ideals of S. An ordered semigroup is characterized in [4] as simple if $(SaS] = S$ for every $a \in S$. In this note a similar characterization of 0-simple ordered semigroups and some equivalent characterizations arising as corollaries are given.

If $(S, ., \leq)$ is an ordered semigroup, a zero of S, denoted by 0, is an element 0 of S such that $0x = x0 = 0$ and $0 \leq x$ for all $x \in S$ [1]. For an ordered semigroup S and $A \subseteq S$, we denote

$(A] := \{t \in S \mid t \leq a \text{ for some } a \in A\}$.

For $A = \{a\}$ $(a \in S)$, we write $(a]$ instead of $(\{a\}]$. If $(S, ., \leq)$ is an ordered semigroup, a non-empty subset I of S is called an ideal of S if 1) $IS \subseteq I$ and $SI \subseteq I$. 2) $a \in I$, $S \ni b \leq a$ imply $b \in I$ [3; Definition 1].

Definition (cf. also [5])**.** Let S be an ordered semigroup with 0. S is called 0-simple if

1) $S^2 \neq \{0\}$.
2) The only ideals of S are the sets $\{0\}$ and S.

As usual, we denote by $S \setminus \{0\}$ the complement of $\{0\}$ in S.

Proposition 1 (cf. also [6])**.** *Let $(S, ., \leq)$ be an ordered semigroup with 0. Then S is 0-simple if and only if $S \neq \{0\}$ and $(SaS] = S$ for every $a \in S \setminus \{0\}$.*

Proof. $\Longrightarrow$. Let S be 0-simple and let $S = \{0\}$. Then $S^2 = \{0\}$. Impossible.
Let $a \in \setminus \{0\}$ ($\Longrightarrow$ $(SaS] = S$?)
The set $(SaS]$ is an ideal of S. Indeed: $(SaS]S = (SaS](S] \subseteq (SaS^2] \subseteq (SaS]$, similarly $S(SaS] \subseteq (SaS]$. If $a \in (SaS]$ and $S \ni b \leq a$, then $b \in (SaS]$. Since S is 0- simple, we have $(SaS] = \{0\}$ or $(SaS] = S$.
Let $(SaS] = \{0\}$. We consider the set $I := \{x \in S \mid (SxS] = \{0\}\}$.

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I is an ideal of S. Indeed: $\emptyset \neq I \subseteq S$ (since $a \in I$).
Let $y \in S, x \in I$. Then $yx \in I$. Indeed: Since $x \in I, (SxS] = \{0\}$. Since $SyxS \subseteq SxS$, we have $(SyxS] \subseteq (SxS] = \{0\}$. Since $\emptyset \neq SyxS \subseteq (SyxS]$, we have $(SyxS] \neq \emptyset$. Thus $(SyxS] = \{0\}$, and $yx \in I$. Similarly $IS \subseteq I$.
Let $x \in I, S \ni y \leq x$. Then $y \in I$. Indeed: Since $x \in I, (SxS] = \{0\}$. Since $y \leq x$, we have $(SyS] \subseteq (SxS] = \{0\}$. Since $\emptyset \neq SyS \subseteq (SyS]$, we have $(SyS] \neq \emptyset$. Thus $(SyS] = \{0\}$, and $y \in I$.
Since S is 0-simple, we have $I = \{0\}$ or $I = S$. Since $a \in I$ and $a \neq 0$ $(a \in S \setminus \{0\})$, we have $I \neq \{0\}$. Thus $I = S$. Then $x \in I$ for every $x \in S$, and

$(SxS] = \{0\}$ for every $x \in S$.......... $(*)$

On the other hand, $S^3 = \bigcup_{x \in S} SxS$ then, by (*),

$$(S^3] = \bigcup_{x \in S} (SxS] = \{0\} \text{ .... } (**)$$

The set $(S^2]$ is an ideal of S, so $(S^2] = \{0\}$ or $(S^2] = S$. If $(S^2] = \{0\}$ then, since $\emptyset \neq S^2 \subseteq (S^2] = \{0\}$, we have $S^2 = \{0\}$. Impossible. Thus $(S^2] = S$.
Then we have

$S^2 = S(S^2] = (S](S^2] \subseteq (S^3] = \{0\}$ by (**).

Since $S^2 \neq \emptyset$, we have $S^2 = \{0\}$. Impossible.

$\Longleftarrow$. Suppose $S^2 = \{0\}$. Let $a \in S, a \neq 0$ $(S \neq \{0\})$. By hypothesis, $(SaS] = S$. Since $\emptyset \neq SaS \subseteq S^2 S = \{0\}S = \{0\}$, we have $SaS = \{0\}$, then $(SaS] = (0] = \{0\}$, and $S = \{0\}$. Impossible.
Let I be an ideal of S and $I \neq \{0\}$. Then $I = S$. Indeed: Let $a \in I, a \neq 0$. By hypothesis, $(SaS] = S$. Since $a \in I$, we have $SaS \subseteq SIS \subseteq I$, and $(SaS] \subseteq (I] = I$, so $S \subseteq I$, and $I = S$. □

By Proposition 1, we have the

Corollary 1. *Let $(S, ., \leq)$ be an ordered semigroup with 0 such that $S \neq \{0\}$. Then S is 0-simple if and only if $(SaS] = S$ for every $a \in S \setminus \{0\}$.*

Corollary 2. *Let $(S, ., \leq)$ be an ordered semigroup with 0 such that $S \neq \{0\}$. Then S is 0-simple if and only if*

For every $a, b \in S \setminus \{0\}$ there exist $x, y \in S$ such that $b \leq xay$.

Proof. $\Longrightarrow$. Let $a, b \in S \setminus \{0\}$. Since $a \in S \setminus \{0\}$ and S is 0-simple, by Corollary 1, we have $(SaS] = S$. Since $b \in S$, $b \in (SaS]$. Then there exist $x, y \in S$ such that $b \leq xay$.
$\Longleftarrow$. By Corollary 1, it is enough to prove that $a \in S \setminus \{0\}$ implies $(SaS] = S$.
Let $a \in S \setminus \{0\}$ and $b \in S$. If $b = 0$, then clearly $b \in (SaS]$. Let $b \in S \setminus \{0\}$. Since $a, b \in S \setminus \{0\}$, by hypothesis, there exist $x, y \in S$ such that $b \leq xay \in SaS$. Then $b \in (SaS]$.

Proposition 2. *Let $(S, ., \leq)$ be an ordered semigroup with 0 such that $S \neq \{0\}$. The following are equivalent:*

1) *For every $a, b \in S \setminus \{0\}$ there exist $x, y \in S$ such that $b \leq xay$.*
2) *For every $a \in S \setminus \{0\}$ and every $b \in S$ there exist $x, y \in S$ such that $b \leq xay$.*

Proof. 1) $\Longrightarrow$ 2). Let $a \in S \setminus \{0\}$, $b \in S$. If $b = 0$, then for the elements $x = y = 0 \in S$, we have $b \leq xay$. If $b \in S \setminus \{0\}$ then, by 1), there exist $x, y \in S$ such that $b \leq xay$.
2) $\Longrightarrow$ 1). It is clear.

Proposition 3. *Let $(S,.,\leq)$ be an ordered semigroup with 0 such that $S \neq \{0\}$. The following are equivalent:*

1) *For every $a,b \in S\setminus\{0\}$ there exist $x,y \in S$ such that $b \leq xay$.*
2) *For every $a,b \in S\setminus\{0\}$ there exist $x,y \in S\setminus\{0\}$ such that $b \leq xay$.*

Proof. 1) $\Longrightarrow$ 2). Let $a,b \in S\setminus\{0\}$. By 1), there exist $x,y \in S$ such that $b \leq xay$. If $x=0$ or $y=0$ then $xay=0$, and $b=0$. Impossible. Thus $x,y \in S\setminus\{0\}$.
2) $\Longrightarrow$ 1). It is clear.

Proposition 4. *Let $(S,.,\leq)$ be an ordered semigroup with 0 such that $S \neq \{0\}$. The following are equivalent:*

1) *For every $a \in S\setminus\{0\}$ and every $b \in S$ there exist $x,y \in S$ such that $b \leq xay$.*
2) *For every $a \in S\setminus\{0\}$ and every $b \in S$ there exist $x,y \in S\setminus\{0\}$ such that $b \leq xay$.*

Proof. 1) $\Longrightarrow$ 2). Let $a \in S\setminus\{0\}, b \in S$. If $b=0$, then for the elements $x=y=a \in S$, we have $0 \leq xay$. Let $b \in S\setminus\{0\}$. Since $a \in S\setminus\{0\}$ and $b \in S$, by hypothesis, there exist $x,y \in S$ such that $b \leq xay$. If $x=0$ or $y=0$, then $b=0$. Impossible. So $x,y \in S\setminus\{0\}$.
2) $\Longrightarrow$ 1). It is clear. □

By Corollary 1, Corollary 2, Proposition 2, Proposition 3 and Proposition 4, we have the following

Theorem 1. *Let $(S,.,\leq)$ be an ordered semigroup with 0 such that $S \neq \{0\}$. The following are equivalent:*

1) *S is 0-simple.*
2) *For every $a \in S\setminus\{0\}$, we have $(SaS]=S$.*
3) *For every $a,b \in S\setminus\{0\}$ there exist $x,y \in S$ such that $b \leq xay$.*
4) *For every $a \in S\setminus\{0\}$ and every $b \in S$ there exist $x,y \in S$ such that $b \leq xay$.*
5) *For every $a,b \in S\setminus\{0\}$ there exist $x,y \in S\setminus\{0\}$ such that $b \leq xay$.*
6) *For every $a \in S\setminus\{0\}$ and every $b \in S$ there exist $x,y \in S\setminus\{0\}$ such that $b \leq xay$.* □

For an ordered semigroup S, we denote by $\mathcal{I}$ the equivalence relation on S defined by

$\mathcal{I} := \{(a,b) \in S \times S \mid I(a)=I(b)\}$,

where $I(a)$ is the ideal of S generated by a $(a \in S)$. We have

$I(a) = (a \cup Sa \cup aS \cup SaS]$ (cf. [3]).

As usual, we denote by $(a)_{\mathcal{I}}$ the $\mathcal{I}$- class containing a and by $S/\mathcal{I}$ the set of all $(a)_{\mathcal{I}}$, $a \in S$.

Theorem 2. *Let $(S,.,\leq)$ be an ordered semigroup with 0. Then S is 0-simple if and only if*

1) *$S^2 \neq \{0\}$ and*
2) *$S/\mathcal{I} = \{\{0\}, S\setminus\{0\}\}$.*

Proof. $\Longrightarrow$. We have $S/\mathcal{I} \ni (0)_{\mathcal{I}} = \{0\}$. Indeed: Since $0 \in (0)_{\mathcal{I}}$, we have $\{0\} \subseteq (0)_{\mathcal{I}}$. Let $a \in (0)_{\mathcal{I}}$. Then $(a,0) \in \mathcal{I}$, $a \in I(a) = I(0) = \{0\}$, and $a=0$.
We have $S\setminus\{0\} \in S/\mathcal{I}$. Indeed: Since S is 0-simple, by Proposition 1, $S \neq \{0\}$, and $\{0\} \subset S$. Let $a \in S$, $a \neq 0$. Then $S/\mathcal{I} \ni (a)_{\mathcal{I}} = S\setminus\{0\}$. Indeed:

Let $b \in (a)_{\mathcal{I}}$. Then $(b,a) \in \mathcal{I}$, $a \in I(a) = I(b)$. If $b = 0$, then $I(b) = \{0\}$, and $a = 0$. Impossible. Thus $b \in S \setminus \{0\}$. If $b \neq 0$, then clearly $b \in S \setminus \{0\}$.
Let $b \in S \setminus \{0\}$. Then $b \in (a)_{\mathcal{I}}$. Indeed: Since $I(b)$ is an ideal of S, by hypothesis, we have $I(b) = \{0\}$ or $I(b) = S$. Since $b \in I(b)$, $b \neq 0$, we have $I(b) \neq \{0\}$. Thus $I(b) = S$. In a similar way, since $a \in S \setminus \{0\}$, we have $I(a) = S$. Since $I(a) = I(b)$, we have $(a,b) \in I$, and $b \in (a)_{\mathcal{I}}$.
Let $a \in S$. Then $(a)_{\mathcal{I}} \in \{\{0\}, S \setminus \{0\}\}$. Indeed: If $a = 0$, then $(a)_{\mathcal{I}} = (0)_{\mathcal{I}} = \{0\}$ (cf. the proof above). If $a \neq 0$, then $(a)_{\mathcal{I}} = S \setminus \{0\}$ (cf. the proof above).
$\Longleftarrow$. Let I be an ideal of S, $I \neq \{0\}$. Then $I = S$. Indeed:
We have $\{0\} \subset I$. Let $a \in I$, $a \neq 0$. Since $a \in S$, by hypothesis, we have $(a)_{\mathcal{I}} = \{0\}$ or $(a)_{\mathcal{I}} = S \setminus \{0\}$. If $(a)_{\mathcal{I}} = \{0\}$, then $a \in (a)_{\mathcal{I}} = \{0\}$, $a = 0$. Impossible. Thus $(a)_{\mathcal{I}} = S \setminus \{0\}$.......... $(*)$
Let $b \in S$. If $b = 0$, then clearly $b \in I$. Let $b \neq 0$. Since $b \in S \setminus \{0\} = (a)_{\mathcal{I}}$ (by (*)), we have $(b,a) \in \mathcal{I}$, and $b \in I(b) = I(a)$. Since $a \in I$, we have $I(a) \subseteq I$. Then $b \in I$.

Remark 1. If S is an ordered semigroup with 0 and

$$\mathcal{A} := \{I \mid \ I \text{ ideal of } S, I \neq \{0\}\},$$

then $\mathcal{A} \neq \emptyset$ if and only if $S \neq \{0\}$. In fact: Let $\mathcal{A} \neq \emptyset$, and let $I \in \mathcal{A}$. Then I is an ideal of S and $I \neq \{0\}$. Since I is an ideal of S, we have $0 \in I$, and $\{0\} \subseteq I$. Then $\{0\} \subset I \subseteq S$, and $S \neq \{0\}$.
If $S \neq \{0\}$, then clearly $S \in \mathcal{A}$.

Remark 2. Let S is an ordered semigroup with 0 such that $S^2 \neq \{0\}$. Then $S \neq \{0\}$, equivalently, $|S| \geq 2$ (as usual, we denote by $|S|$ the order of S).

Remark 3. Let S is an ordered semigroup with 0 such that the sets $\{0\}$ and S are the only ideals of S. If $S^2 = \{0\}$, then $|S| \leq 2$. In fact: Let $|S| > 2$ i.e $|S| \leq 3$. Since $0 \in S$, $|S \setminus \{0\}| \geq 2$. Let $a, b \in S \setminus \{0\}$, $a \neq b$. The set $(a]$ is an ideal of S. Indeed:

$$(a]S = (a](S] \subseteq (aS] \subseteq (S^2] = (0] = \{0\} \subseteq (a] \ \ (S^2 = \{0\})$$
$$x \in (a], S \ni y \leq x \text{ imply } y \in (a].$$

Thus $(a] = \{0\}$ or $(a] = S$. If $(a] = \{0\}$, then $a \in (a] = \{0\}$, $a = 0$. Impossible. Thus $(a] = S$. Similarly, since $S^2 = \{0\}$, the set $(b]$ is an ideal of S, then $(b] = S$. Then $a \in (a] = (b]$, and $a \leq b$. In a similar way, $b \leq a$, and $a = b$. Impossible.

Remark 4. Let S is an ordered semigroup with 0 in which the sets $\{0\}$ and S are the only ideals of S. Then S is 0-simple or we have $S^2 = \{0\}$ and $|S| \leq 2$. In fact: If $S^2 \neq \{0\}$, then S is 0-simple. If $S^2 = \{0\}$ then, by Remark 3, $|S| \leq 2$. □

Note. An ordered semigroup (or groupoid) S without zero is called simple if S is the only ideal of S. If S is an ordered semigroup (or groupoid) with zero, this is the natural definition for S to be simple: An ordered semigroup (or groupoid) with zero is called simple (0-simple in the terminology of Clifford) if

1) $S \neq \{0\}$.

2) The only ideals of S are the sets $\{0\}$ and S.

In this respect, we note the following: If $(S,.,\leq)$ is an ordered semigroup (or groupoid) with 0 such that $S^2=\{0\}$ then, for $a\in S$, the set $(a]:=\{x\in S \mid x\leq a\}$ is an ideal of S........$(*)$
Indeed: $\emptyset\neq(a]\subseteq S$ (since $a\in(a]$). $\emptyset\neq(a]S\subseteq S^2=\{0\}$ (since $a^2\in(a]S$), so $(a]S=\{0\}\subseteq(a]$ (since $0\leq a$). Similarly $S(a]\subseteq(a]$. If $x\in(a]$ and $S\ni y\leq x$, then $y\leq a$, so $y\in(a]$.

Remark A. If $(S,.,\leq)$ is an ordered semigroup (or groupoid) with 0, $|S|\geq 3$ and $S^2=\{0\}$, then there exists an ideal I of S such that $I\neq\{0\}$ and $I\neq S$. In fact: Let $a,b\in S$, $a\neq b$, $a\neq 0$, $b\neq 0$. Then: i) Let $b\in(a]$. By (*), the set $(b]$ is an ideal of S. We have $(b]\neq\{0\}$ and $(b]\neq S$. Indeed $b\in(b]$ and $b\neq 0$; $a\in S$ and $a\notin(b]$ (since $a\in(b]$ implies $a\leq b$. Moreover, since $b\in(a]$, we have $b\leq a$, then $b=a$. Impossible). ii) Let $b\in S\setminus\{a\}$. By (*), $(a]$ is an ideal of S. We have $(a]\neq\{0\}$ and $(a]\neq S$. Indeed: $a\in(a]$ and $a\notin\{0\}$; $b\in S$ and $b\notin(a]$.

Remark B. If $(S,.,\leq)$ is an ordered semigroup (or groupoid) with 0 and $|S|=2$, then the only ideals of S are the sets $\{0\}$ and S. In fact: Let $S=\{0,a\}$, $a\neq 0$. Let I be an ideal of S. Since $\emptyset\neq I\subseteq S$, we have $I=\{0\}$ or $I=\{a\}$ or $I=S$. If $I=\{0\}$, then I is an ideal of S. If $I=S$, then I is an ideal of S. Let $I=\{a\}$. Then $IS=\{a\}\{0,a\}=\{0,a^2\}\nsubseteq\{a\}$ (since $\{0,a^2\}\subseteq\{a\}$ implies $0=a$. Impossible). Thus the set $\{a\}$ is not an ideal of S. □

We moreover remark that: If $(S,.,\leq)$ is an ordered semigroup with 0 and $|S|=2$, that is, if $S=\{0,a\}$, $a\neq 0$, then S is one of the following:

A) S is the ordered semigroup with the multiplication and the order defined by:

.	0	a
0	0	0
a	0	0

$$\leq\,=\{(0,0)\}$$

which is not 0-simple (in the terminology of Clifford) since $S^2=\{0\}$.

B) S is the ordered semigroup with the multiplication and the order defined by:

.	0	a
0	0	0
a	0	a

$$\leq\,=\{(0,0),(0,a),(a,a)\}.$$

which is 0-simple (in the terminology of Clifford) since $S^2\neq\{0\}$, and for every ideal I of S, we have $I=\{0\}$ or $I=S$.

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