THE DIGITAL LINE AND OPERATION APPROACHES OF $T_{1/2}$-SPACES

H.Maki, H.Ogata, K.Balachandran, P.Sundaram and R.Devi

Received March 7, 2000

Abstract. The notion of operations on a topological space was introduced by S.Kasahara [8]. The notions of generalized continuous functions [1] and $T_{1/2}$-spaces [12] are further investigated using operation approaches [14]. As applications, it is shown that the Khalimsky line (=the digital line)[9] and two kinds of "digital picture" with a topology are typical examples of "Int∘Cl"-$T_{1/2}$ spaces.

1. Introduction . The concept of operations on topological spaces was introduced and investigated by S.Kasahara [8]. Using the concept of operations, D.S.Janković [7] has defined the concept of "α-closed sets" and investigated functions with "α-closed graphs". H.Ogata [14] introduced the notion of "operation-open sets" in topological spaces and "operation-separation axioms of topologiccal spaces".

In the present paper, we investigate "operation-generalized continuous functions", a characterization of "operation-$T_{1/2}$ spaces" and one-point compactification of some "operation-$T_{1/2}$ spaces". We obtain an example of the concept of the restriction to a subspace of operations in the sense of [16]. As application, we show that the Khalimsky line(=the digital line) (eg. [4,9]) and two kinds of "digital pictures" are typical examples of "Int∘Cl"-$T_{1/2}$ space. Several topological spaces that fail to be T_1 are often of importance in "topological digital topology". The digital line, the digital plane and the three-dimensional digital space are of great importance in the study of applications of point-set topology to computer graphics (eg.[9]). Articles [3,5,10] are topological approarches of digital spaces,i.e., "topological digital topology". The concept of low separation axioms and generalized concepts are used in the articles [3,5].

Throughout the present paper, (X,τ) and (Y,σ) (or simply X or Y) always denotes a topological space, on which no separation axioms are assumed unless explicitely stated.

2. Preliminaries . We begin by recalling some definitions and properties in [7],[8] and [14].

(2.1)([8],[14;Definition 2.1]) Let (X,τ) be a topological space. An operation γ on τ is a mapping from τ into the power set $P(X)$ of X such that $V \subset V^\gamma$ for each $V \in \tau$ where V^γ (or $\gamma(V)$) denotes the value of γ at V. It is denoted by $\gamma : \tau \to P(X)$. We note that H.Ogata used the term "operation γ" for the term "operation α" defined in [8].

The operations γ, γ' and γ'' defined by $\gamma(V) = V, \gamma'(V)$=Cl$(V)$ and $\gamma''(V)$ =Int (Cl(V)) for $V \in \tau$ are examples of operations on τ [7,8,14]. The operation γ (resp. γ', γ'') above

1991 *Mathematics Subject Classification.* Primary:54A05, 54A10, 54D10; Secondary:68U05..

Key words and phrases. operation, operation-closures, operation-open sets, operation-$T_{1/2}$ spaces, $T_{3/4}$-spaces, almost weakly Hausdorff spaces, the Khalimsky line, the digital line, digital circles, the digital plane.

is called the "identity operation" (resp. "closure operation", "interior-closure operation") and γ'' is denoted by Int∘Cl.

(2.2) [14] Let $\gamma : \tau \to P(X)$ be an operation. A non-empty subset A of X is called a *γ-open set* of (X,τ) if , for each point $x \in A$, there exists an open set U such that $x \in U$ and $U^\gamma \subset A$. We assume that the empty set is γ-open. τ_γ denotes the set of all γ-open sets in (X,τ). Clearly $\tau_\gamma \subset \tau$. A subset B of X is said to be *γ-closed* in (X,τ) if its complement $X \setminus B$ is γ-open in (X,τ).

(2.3) [14;Propositions 2.3, 2.9] For subset $A_i (i \in \nabla)$ of (X,τ), where ∇ is any index set, and operation $\gamma : \tau \to P(X)$, the following (i) and (ii) hold.

(i) If $A_i \in \tau_\gamma (i \in \nabla)$, then $\cup\{A_i | i \in \nabla\} \in \tau_\gamma$.

(ii) If $\gamma : \tau \to P(X)$ is regular, then τ_γ is a topology of X.

An operation γ is said to be *regular* [8] if, for every open neighbourhoods U and V of each $x \in X$, there exists an open neighbourhood W of x such that $W^\gamma \subset U^\gamma \cap V^\gamma$. If γ is monotone (i.e., $A^\gamma \subset B^\gamma$ for every $A \subset B$), then γ is regular.

(2.4) [7] For subsets A and B of (X,τ) and $\gamma : \tau \to P(X)$, γ-closedness in the sense of Janković and γ-closures are defined respectively as follows:

Let $\mathrm{Cl}_\gamma(B) = \{x | U^\gamma \cap B \neq \emptyset$ for every open neighbourhood U of $x\}$. This is called as the *γ-closure* of B. A subset A is said to be *γ-closed* (*in the sense of Janković*), if $\mathrm{Cl}_\gamma(A) = A$.

(2.5)[14;(3.4)] For a subset A of (X,τ), the following implications hold:
$A \subset \mathrm{Cl}(A) \subset \mathrm{Cl}_\gamma(A) \subset \tau_\gamma\text{-}\mathrm{Cl}(A)$, where $\tau_\gamma\text{-}\mathrm{Cl}(A) = \cap\{F | A \subset F$ and $X \setminus F \in \tau_\gamma\}$.

For a point $x \in X, x \in \tau_\gamma\text{-}\mathrm{Cl}(A)$ if and only if $V \cap A \neq \emptyset$ for any set $V \in \tau_\gamma$ containing x. In [14;Theorem 3.6(iii)], it is shown that if γ is open then $\mathrm{Cl}_\gamma(A) = \tau_\gamma\text{-}\mathrm{Cl}(A)$ for a subset A of (X,τ). Operation $\gamma : \tau \to P(X)$ is said to be *open* [14;Definition 2.6] if, for every open neighbourhood U of each $x \in X$, there exists a γ-open set S containing x such that $S \subset U^\gamma$. The "interior-closure operation" is open [17].

(2.6)[14;Theorem 3.7] For a subset A of (X,τ) and operation $\gamma : \tau \to P(X)$, the following properties are equivalent:

(a) A is γ-closed (in the sense of (2.2));

(b) A is γ-closed (in the sense of Janković) (i.e., $\mathrm{Cl}_\gamma(A) = A$);

(c) $\tau_\gamma\text{-}\mathrm{Cl}(A) = A$.

A subset A of (X,τ) is said to be *γ-generalized closed* (shortly γ-g.*closed*)[14;Definition 4.4], if $\mathrm{Cl}_\gamma(A) \subset U$ whenever $A \subset U$ and U is γ-open in (X,τ). Every γ-closed set is γ-generalized closed. The γ-g.closedness, where γ is the interior-closure operation, coincides with the δ-g*-closedness [2;Definition 4].

We have a property of operation-generalized closed sets.

Proposition 2.7. *Suppose that $\gamma : \tau \to P(X)$ is regular. If subsets A and B are γ-g.closed sets, then $A \cup B$ is γ-g.closed.* Λ

3. Operation-generalized continuous functions . Throughout this section , let γ (resp. β) be an operation on (X,τ) (resp. (Y,σ)).

Definition 3.1. Let $f : (X,\tau) \to (Y,\sigma)$ be a function.

(i) f is *(γ,β)-continuous* [14;Definition 4.12] if for each point x of X and every open set V containing $f(x)$ there exists an open set U containing x such that $f(U^\gamma) \subset V^\beta$,

(ii) f is *(γ,β)-irresolute* if the inverse image of each β-closed set in (Y,σ) is γ-closed in (X,τ),

(iii) f is *(γ,β)-generalized continuous* (shortly, (γ,β)-g.*continuous*), if the inverse image of each β-closed set is γ-g.closed set of (X,τ).

Proposition 3.2. *(i) Every (γ,β)-continuous function $f:(X,\tau)\rightarrow(Y,\sigma)$ is (γ,β)-irresolute.*

(ii) Every (γ,β)-irresolute function $f:(X,\tau)\rightarrow(Y,\sigma)$ is (γ,β)-g.continuous. Λ

The converse of Proposition 3.2 need not be true from the following examples. In Proposition 3.2(ii), let us take γ and β be the identity operations on τ and σ respectively. Then we have Proposition 1 in [1].

Example 3.3. Let $X=\{a,b,c\}$ and $\tau=\{\emptyset,\{a\},\{b\},\{a,b\},X\}$. Let $\gamma:\tau\rightarrow P(X)$ be the closure operation and $f:(X,\tau)\rightarrow(X,\tau)$ be a function defined by $f(a)=b, f(b)=c$ and $f(c)=a$. Then f is not (γ,γ)-continuous at a point $a\in X$. However, f is (γ,γ)-irresolute.

Example 3.4. Let $X=\{a,b,c\}, Y=\{p,q\}, \tau=\{\emptyset,\{a\},\{b\},\{a,b\},X\}$ and $\sigma=\{\emptyset,\{p\},Y\}$. Let $\gamma:\tau\rightarrow P(X)$ and $\beta:\sigma\rightarrow P(Y)$ be the closure operation and the identity operation respectively. Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be a function defined by $f(a)=f(c)=q$ and $f(b)=p$. Then f is not (γ,β)-irresolute. In fact, for a β-closed set $\{q\}, f^{-1}(\{q\})=\{a,c\}$ is not γ-closed. It is shown that f is (γ,β)-g.continuous.

4. γ-generalized closures and a characterization of γ-T$_{1/2}$ spaces . In this section we introduce a γ-generalized closure. Our goal in this section is to characterize γ-T$_{1/2}$ spaces [14] by using a family associated to the γ-generalized closure and to obtain some examples of "Int∘Cl"-T$_{1/2}$ space from topological digital topology (cf.Examples 5.1, 5.2, 5.3 and 5.4).

Definition 4.1. For a subset E of (X,τ) and an operation $\gamma:\tau\rightarrow P(X)$, let $\mathrm{Cl}^*_\gamma(E)$ denote the intersection of all γ-g.closed sets containing the set E.

For the identity operation $id{:}\tau\rightarrow P(X)$, the above Cl^*_{id}-closure of a set E coincides with the c*-closure in the sense of Dunham [6]:

(4.2) $\mathrm{Cl}^*_{id}(E)=c^*(E)$ for any set E of (X,τ).

Proposition 4.3. *Let E_1, E_2 and E be the subsets of (X,τ) and let $\gamma:\tau\rightarrow P(X)$ be an operation. Then the following hold.*

(i) $E\subset Cl^*_\gamma(E)\subset\tau_\gamma\text{-}Cl(E)$.

(ii) If $E_1\subset E_2$, *then* $Cl^*_\gamma(E_1)\subset Cl^*_\gamma(E_2)$.

(iii) $Cl^*_\gamma(E_1\cup E_2)\supset Cl^*_\gamma(E_1)\cup Cl^*_\gamma(E_2)$.

(iv) If $\gamma:\tau\rightarrow P(X)$ *is regular, then* $Cl^*_\gamma(E_1\cup E_2)=Cl^*_\gamma(E_1)\cup Cl^*_\gamma(E_2)$.

(v) $Cl^*_\gamma(Cl^*_\gamma(E))=Cl^*_\gamma(E)$. Λ

Definition 4.4. For a set A of (X,τ) and $\gamma:\tau\rightarrow P(X)$, we define the following family and a subset of X:

$\tau^*_\gamma=\{U\ |\mathrm{Cl}^*_\gamma(X\setminus U)=X\setminus U\}$ and

$\tau^*_\gamma\text{-Cl}(A)=\cap\{F:\ \ A\subset F, X\setminus F\in\tau^*_\gamma\}$.

Theorem 4.5. *(i)* $\tau^*_\gamma\supset\tau_\gamma$ *holds for any operation* γ.

(ii) If γ *is a regular operation, then* τ^*_γ *is a topology of* X *and*
$\tau^*_\gamma\text{-}Cl(A)=Cl^*_\gamma(A)$ *for any set* A *of* (X,τ).

Proof. (i) If A is γ-open, then $X\setminus A$ is γ-closed. Then $\mathrm{Cl}^*_\gamma(X\setminus A)=X\setminus A$ and hence $A\in\tau^*_\gamma$.

(ii) Using Proposition 4.3 and facts that $\tau^*_\gamma\text{-Cl}(\emptyset)=\emptyset$ and $\tau^*_\gamma\text{-Cl}(X)=X$, the closure operation $\tau^*_\gamma\text{-Cl}(\cdot)$ satisfies the Kuratowski closure axioms under the assumption. Therefore, τ^*_γ is a unique topology of X such that $\tau^*_\gamma\text{-Cl}(A)=\mathrm{Cl}^*_\gamma(A)$ for any subset A of (X,τ). Λ

Theorem 4.6. *(cf. [14;Proposition 4.10][18;Corollary 4.12]) For a topological space* (X,τ) *and an operation* $\gamma : \tau \to P(X)$*, the following properties are equivalent:*

(a) (X,τ) *is* γ-$T_{1/2}$ *(i.e., every* γ*-g.closed set is* γ*-closed);*

(b) every singleton $\{x\}$ *is* γ*-open or* γ*-closed;*

(c) $\tau_\gamma = \tau^*_\gamma$*.*

Proof. (a)$\Rightarrow$ (b) It is Proposition 4.10 in [14].

(b)$\Rightarrow$(c) By Theorem 4.5(i), it is enough to prove that $\tau^*_\gamma \subset \tau_\gamma$. Let $E \in \tau^*_\gamma$ and $E \neq X$. Suppose that $E \notin \tau_\gamma$. Then, $\mathrm{Cl}^*_\gamma(X \setminus E) = X \setminus E$ and $\mathrm{Cl}_\gamma(X \setminus E) \neq X \setminus E$. Since $\mathrm{Cl}_\gamma(X \setminus E) \neq \emptyset$, there exists a point $x \in X$ such that $x \in \mathrm{Cl}_\gamma(X \setminus E)$ and $x \notin X \setminus E$. Since $x \notin \mathrm{Cl}^*_\gamma(X \setminus E)$, there exists a γ-g.closed set A such that $x \notin A$ and $X \setminus E \subset A$. By the hypothesis, the singleton $\{x\}$ is γ-open or γ-closed.

Case 1. $\{x\}$ is γ-open: Since $X \setminus \{x\}$ is γ-closed and $X \setminus E \subset A \subset X \setminus \{x\}$, we have $\mathrm{Cl}_\gamma(X \setminus E) \subset \mathrm{Cl}_\gamma(A) \subset \mathrm{Cl}_\gamma(X \setminus \{x\}) = X \setminus \{x\}$,i.e., $x \notin \mathrm{Cl}_\gamma(X \setminus E)$.

Case 2. $\{x\}$ is γ-closed: Since $X \setminus \{x\}$ is γ-open set containing A and $A \supset X \setminus E$, we show $\mathrm{Cl}_\gamma(X \setminus E) \subset \mathrm{Cl}_\gamma(A) \subset X \setminus \{x\}$, i.e., $x \notin \mathrm{Cl}_\gamma(X \setminus E)$.

Hence in both cases, we have a contradiction to a fact that $x \in \mathrm{Cl}_\gamma(X \setminus E)$. We claim that $\tau^*_\gamma \subset \tau_\gamma$.

(c)$\Rightarrow$(a) Let A be a γ-g.closed set. Since $\mathrm{Cl}^*_\gamma(A) = A$, it follows from assumption that $X \setminus A \in \tau_\gamma$, that is, A is γ-closed. Λ

Recently, in [4], J.Dontchev and M.Ganster defined δ-generalized closed sets and investigated the class of $\mathrm{T}_{3/4}$-spaces, which is properly placed between the class of T_1-spaces and $\mathrm{T}_{1/2}$-spaces. A topological space (X,τ) is $\mathrm{T}_{3/4}$ if and only if every singleton $\{x\}$ is δ-open or closed [4;Theorem 4.3]. The notion of δ-openness was introduced by N.V.Velišcko [19]. It is shown that ,in (X,τ), a subset A is γ-open, where γ is the interior-closure operation, if and only if A is δ-open. That is, $\tau_{\text{"}Int\circ Cl\text{"}} = \tau_\delta$, where τ_δ (resp. $\tau_{\text{"}Int\circ Cl\text{"}}$) is the family of all δ-open sets (resp. "Int∘Cl"-open sets) in (X,τ). It is known that the family τ_δ is a topology of X and $\tau_\delta = \tau_s$ where τ_s is the so-called semi-regularization of τ. Every δ-open set is open in (X,τ). A space (X,τ) is called almost weakly Hausdorff (resp. weakly Hausdorff) if (X,τ_s) is $\mathrm{T}_{1/2}$ (resp. T_1) [4].

We obtain a relation between $\mathrm{T}_{3/4}$-spaces and "Int∘Cl"-$\mathrm{T}_{1/2}$ spaces. If $\gamma : \tau \to P(X)$ is the "Int∘Cl"-operation, the γ-$\mathrm{T}_{1/2}$ spaces are called as the "Int∘Cl"-$\mathrm{T}_{1/2}$ spaces.

Corollary 4.7. *(i) Every "Int∘Cl"-*$T_{1/2}$ *space is a* $T_{3/4}$*-space.*

(ii) A topological space is "Int∘Cl"-$T_{1/2}$ *(resp. "Int∘Cl"-*T_1 *) if and only if it is almost weakly Hausdorff (resp. weakly Hausdorff).*

Proof. (i) The proof follows from Theorem 4.6 and [4;Theorem 4.3].

(ii) Using properties $\tau_{\text{"}Int\circ Cl\text{"}} = \tau_\delta = \tau_s$ and Theorem 4.6 (resp. [14;Proposition 4.11]), the proof follows. Λ

Corollary 4.8. *Let* $f : (X,\tau) \to (Y,\sigma)$ *be a function and* $\gamma : \tau \to P(X)$ *and* $\beta : \sigma \to P(Y)$ *operations. If* $f : (X,\tau) \to (Y,\sigma)$ *is a bijective* (γ,β)*-irresolute function and* (Y,σ) *is* β-$T_{1/2}$*, then* (X,τ) *is* γ-$T_{1/2}$*.* Λ

Put $X^+ := X \cup \{\infty\}$, where ∞ just represents some point not in X. Let (X^+,τ^+) be the one-point compactification of a topological space (X,τ). Since X is open in (X^+,τ^+) and $\tau^+|X = \tau$ holds, we have the following properties: for a subset A of X,

τ-$\mathrm{Int}(A) = (\tau^+|X)$-$\mathrm{Int}(A) = (\tau^+$-$\mathrm{Int}(A)) \cap X$;

τ-$\mathrm{Cl}(A) = (\tau^+|X)$-$\mathrm{Cl}(A) = (\tau^+$-$\mathrm{Cl}(A)) \cap X$;

$((\tau\text{-Int}) \circ (\tau\text{-Cl}))(A) = ((\tau^+\text{-Int}) \circ (\tau^+\text{-Cl}))(A) \cap X$.

In [16;Definition 1.1] the notion of *restrictions* to an open subset of operations was introduced and investigated in general. Then, three operations from τ to $P(X)$, τ-Int$(\cdot)$, τ-Cl$(\cdot)$ and $((\tau\text{-Int}) \circ (\tau\text{-Cl}))(\cdot)$, are restrictions to an open set X of operations from τ^+ to $P(X^+)$, respectively, τ^+-Int$(\cdot)$, τ^+-Cl$(\cdot)$ and $((\tau^+\text{-Int}) \circ (\tau^+\text{-Cl}))(\cdot)$.

Theorem 4.9. *Suppose that (X,τ) satisfy the following property:*

$(*)$ for every $x \in X$ there exists an open neighbourhood U of the point x such that Cl(U) is a compact subspace of (X,τ). Then, the following properties hold.

(i) If (X,τ) is "$(\tau$-Int$)\circ(\tau$-Cl$)$"-$T_{1/2}$, then the one-point compactification (X^+,τ^+) is "$(\tau^+$-Int$)\circ(\tau^+$-Cl$)$"-$T_{1/2}$.

(ii) If (X,τ) is "$(\tau$-Int$)\circ(\tau$-Cl$)$"-T_1, then (X^+,τ^+) is "$(\tau^+$-Int$)\circ(\tau^+$-Cl$)$"-T_1.

Proof. For a point $x \neq \infty$, there exists an open neighbourhood of x, say $U(x)$, such that Cl$(U(x))$ is a compact subspace of (X,τ). Let $S(x) = (X\setminus \text{Cl}(U(x))) \cup \{\infty\}$. Then we note that the following properties hold:

$(**) x \in U(x), U(x) \in \tau$ and $\tau^+\text{-Cl}(U(x)) \cap \{\infty\} = \emptyset$ (especially, $\tau^+\text{-Cl}(\{x\}) \cap \{\infty\} = \emptyset$) and

$(***)$ $\infty \in S(x), S(x) \in \tau^+$ and $\tau^+\text{-Cl}(S(x)) \subset X^+ \setminus \{x\}$.

(i) Let $\{x\}$ be a singleton of X^+. We claim that, in (X^+,τ^+), if $x \neq \infty$ then $\{x\}$ is "$(\tau^+$-Int$)\circ(\tau^+$-Cl$)$"-open or "$(\tau^+$-Int$)\circ(\tau^+$-Cl$)$"-closed and if $x = \infty$ then $\{\infty\}$ is "$(\tau^+$-Int$)\circ(\tau^+$-Cl$)$"- closed. Using assumption that (X,τ) is "Int$\circ$Cl"-T$_{1/2}$ and Theorem 3.6, we consider the following three cases. We abbreviate τ-Int$(\cdot)$, τ-Cl$(\cdot)$ by Int$(\cdot)$, Cl$(\cdot)$,respectively,in the proof below.

Case 1. $x \neq \infty$ and $\{x\}$ is "Int$\circ$Cl"-open in (X,τ): In this case, $\{x\}$ is a unique nonempty open set contained in $\{x\}$, Int(Cl($\{x\}$))=$\{x\}$ holds in (X,τ) and also $\{x\} \in \tau^+$. By $(**)$ it is shown that

$\tau^+\text{-Cl}(\{x\}) = (\tau^+\text{-Cl}(\{x\})) \cap X^+ = (\tau^+\text{-Cl}(\{x\})) \cap (X \cup \{\infty\}) = \{(\tau^+\text{-Cl}(\{x\})) \cap X\} \cup \{(\tau^+\text{-Cl}(\{x\})) \cap \{\infty\}\} = \{(\tau^+\text{-Cl}(\{x\})) \cap X\} \cup \emptyset = (\tau^+|X)\text{-Cl}(\{x\}) = \tau\text{-Cl}(\{x\})$ and so

$\tau^+\text{-Int}(\tau^+\text{-Cl}(\{x\})) = \tau^+\text{-Int}(\text{Cl}(\{x\})) = \tau^+\text{-Int}(\text{Cl}(\{x\}))\cap X = (\tau^+|X)\text{-Int}(\text{Cl}(\{x\})) = \tau\text{-Int}(\tau\text{-Cl}(\{x\}))=\text{Int}(\text{Cl}(\{x\}) = \{x\}$.

Thus we show that $\{x\}$ is "$(\tau^+$-Int$)\circ(\tau^+$-Cl$)$"-open in (X^+,τ^+) because of $\{x\} \in \tau^+$.

Case 2. $x \neq \infty$ and $\{x\}$ is "Int$\circ$Cl"-closed in (X,τ): Let $y \in X^+ \setminus \{x\}$. First we suppose that $y \neq \infty$. Then $y \in X \setminus \{x\}$ and there exists a subset $V \in \tau$ such that Int(Cl(V)) $\subset X \setminus \{x\}$ and $y \in V$.

Then, $\tau^+\text{-Int}(\tau^+\text{-Cl}(V)) =\{ \tau^+\text{-Int}(\tau^+\text{-Cl}(V))\} \cap X^+ \subset \{\tau^+\text{-Int}(\tau\text{-Cl}(V))\} \cap X^+ = \{(\tau^+\text{-Int}(\tau\text{-Cl}(V))) \cap X\} \cup \{(\tau^+\text{-Int}(\tau\text{-Cl}(V))) \cap \{\infty\}\} \subset \{((\tau^+|X)\text{-Int}(\tau\text{-Cl}(V))\} \cup \{(\tau\text{-Cl}(V)) \cap \{\infty\} \} =\text{Int}(\text{Cl}(V)) \cup \emptyset=\text{Int}(\text{Cl}(V))$ and so we have $\tau^+\text{-Int}(\tau^+\text{-Cl}(V)) \subset \text{Int}(\text{Cl}(V)) \subset X \setminus \{x\}$. Then, for a point $y \neq \infty$ and $y \in X^+ \setminus \{x\}$, there exists a subset $V \in \tau^+$ such that $y \in V$ and $\tau^+\text{-Int}(\tau^+\text{-Cl}(V)) \subset X^+ \setminus \{x\}$.

Next, we suppose that $y = \infty$. Since $x \neq \infty$, by using $(***)$ for x, there exists a subset $S(x) \in \tau^+$ such that $y = \infty \in S(x)$ and $\tau^+\text{-Int}(\tau^+\text{-Cl}(S(x))) \subset X^+ \setminus \{x\}$.

Therefore, in this case, $X^+ \setminus \{x\}$ is "$(\tau^+$-Int$)\circ(\tau^+$-Cl$)$"-open and hence $\{x\}$ is "$(\tau^+$-Int$)\circ(\tau^+$-Cl$)$"-closed in (X^+,τ^+).

Case 3. $x = \infty$: Let $y \in X^+\setminus\{\infty\}$. Since $y \neq \infty$, by $(**)$ for y above, there exists a subset $U(y) \in \tau$ containing y such that $\tau^+\text{-Cl}(U(y)) \subset X^+ \setminus \{\infty\}$. Therefore, we have $\tau^+\text{-Int}(\tau^+\text{-Cl}(U(y))) \subset \tau^+\text{-Cl}(U(y)) \subset X^+ \setminus \{\infty\}$ and hence $X^+ \setminus \{\infty\}$ is "$(\tau^+$-Int$)\circ(\tau^+$-Cl$)$"-open (i.e., $\{x\}$ is "$(\tau^+$-Int$)\circ(\tau^+$-Cl$)$"-closed in (X^+,τ^+)).

(ii) The proof is omitted. Λ

5. Examples:the digital line and two classes of digital circles .

Example 5.1. The Khalimsky line or the so called the digital line (eg.[4,9]) is the set Z of the integers equipped with the topology κ having $\{\{2n\text{-}1, 2n, 2n+1\} \mid n \in \mathrm{Z}\}$ as a subbase and denoted by (Z, κ). It was shown, in [4], that (Z, κ) is $\mathrm{T}_{3/4}$ and, in [2], that (Z, κ) is almost weakly Hausdorff. By Corollary 4.7, (Z, κ) is a typical example of "Int∘Cl"-$\mathrm{T}_{1/2}$ spaces; it is not T_1. We note that every singleton $\{2n+1\}$,$n \in \mathrm{Z}$, is "Int∘Cl"-open and every singleton $\{2m\}$,$m \in \mathrm{Z}$, is "Int∘Cl"-closed in (Z, κ).

Example 5.2. Let R be an equivalence relation on Z defined by xRy if and only if $x \equiv y$ mod 8 and Z/8 denote the set of all equivalence classes $[m]$, where m=0,1,2,...,7. Let $\pi : \mathrm{Z} \to \mathrm{Z}/8$ the natural projection defined by $\pi(x) = [x]$. Let κ_8 be the quotient topology on Z/8 with respect to π. Then, $(\mathrm{Z}/8, \kappa_8)$ denotes the quotient space of the digital line (Z, κ). It is known in [5] that the space $(\mathrm{Z}/8, \kappa_8)$ is homeomorphic to a subspace of the product space $(\mathrm{Z} \times \mathrm{Z}, \kappa \times \kappa)$, say $(Y_8, (\kappa \times \kappa)|Y_8)$, where $Y_8 = (\{-1, 0, 1\} \times \{-1, 1\}) \cup (\{-1, 1\} \times \{-1, 0, 1\})$. The space $(Y_8, (\kappa \times \kappa)|Y_8)$ is an example of "Int∘Cl"-$\mathrm{T}_{1/2}$ spaces; it is not T_1. Since there exists a homeomorphism $f : (\mathrm{Z}/8, \kappa_8) \to (Y_8, (\kappa \times \kappa)|Y_8)$ and every homeomorphism between topological spaces is ("Int∘Cl","Int∘Cl")-irresolute, by Corollary 4.8 the space $(\mathrm{Z}/8, \kappa_8)$ is also "Int∘Cl"-$\mathrm{T}_{1/2}$. We note that Y_8 is called as the *8-neighbours* of (0,0) (cf.[11]). It is probably unexpected that $\mathrm{Cl}(Y_8) \neq Y_8$ holds in $(\mathrm{Z} \times \mathrm{Z}, \kappa \times \kappa)$. So we get the following example.

Example 5.3. Let S be an equivalence relation on Z defined by xSy if and only if $x \equiv y$ mod 16 and Z/16 denote the set of all equivalence classes $[m]$, where m=0,1,2,...,15. Let $\pi : \mathrm{Z} \to \mathrm{Z}/16$ the natural projection defined by $\pi(x) = [x]$. Let κ_{16} be the quotient topology on Z/16 with respect to π. Then, $(\mathrm{Z}/16, \kappa_{16})$ denotes the quotient space of the digital line (Z, κ). The space $(\mathrm{Z}/16, \kappa_{16})$ is homeomorphic to a subspace of the product space $(\mathrm{Z} \times \mathrm{Z}, \kappa \times \kappa)$, say $(Y_{16}, (\kappa \times \kappa)|Y_{16})$, where $Y_{16} = \{(n, j) : |n| = 2, |j|\ 5\ 2\} \cup \{(i, m) : |m| = 2, |i|\ 5\ 2\}$. We hope to call that the space $(Y_{16}, (\kappa \times \kappa)|Y_{16})$ is a *"finite digital circle"*, because $\mathrm{Cl}(Y_{16}) = Y_{16}$ holds in $(\mathrm{Z} \times \mathrm{Z}, \kappa \times \kappa)$. The spaces $(Y_{16}, (\kappa \times \kappa)|Y_{16})$ and $(\mathrm{Z}/16, \kappa_{16})$ are "Int∘Cl"-$\mathrm{T}_{1/2}$.

Example 5.4. The one-point compactification $(\mathrm{Z} \cup \{\infty\}, \kappa^+)$ of the digital line (Z, κ) is one of typical examlpes of "Int∘Cl"-$\mathrm{T}_{1/2}$ spaces, where ∞ is a point not in Z. This is obtained by Theorem 4.9 and Example 5.1 because the property $(*)$ in Theorem 4.9 is valid for (Z, κ). We note that the singleton $\{\infty\}$ and every singleton $\{2n\}$,$n \in \mathrm{Z}$, are "(κ^+-Int)∘(κ^+-Cl)"- closed and every singleton $\{2m+1\}$,$m \in \mathrm{Z}$, is "(κ^+-Int)∘(κ^+-Cl)"-open. We hope to call that the space $(\mathrm{Z} \cup \{\infty\}, \kappa^+)$ is the *"infinite digital circle"*.

Example 5.5. The one point compactification of the digital plane (Z^2, κ^2) is denoted by S^2_∞, where $\mathrm{Z}^2 = \mathrm{Z} \times \mathrm{Z}$ and $\kappa^2 = \kappa \times \kappa$. We hope to call that the space $S^2_\infty = (\mathrm{Z}^2 \cup \{\infty\}, (\kappa^2)^+)$ is the *"infinite digital sphere"*. The spaces (Z^2, κ^2) and S^2_∞ are not $\mathrm{T}_{1/2}$ (cf. [3] for further properties of (Z^2, κ^2)).

Remark 5.6. The "Int∘Cl"-$\mathrm{T}_{1/2}$-axiom is independent of the T_1-separation axiom. In fact, the digital line (Z, κ) is not T_1; it is "Int∘Cl"-$\mathrm{T}_{1/2}$ (cf. Example 5.1). The real line with the cofinite topology is an example of a T_1-space [4;p.26], which is not "Int∘Cl"-$\mathrm{T}_{1/2}$.

From Corollary 4.7, Remark 5.6, [15;$(*)$,Theorem 1 and Remark 4] and [4;Corollary 4.4, Example 4.6, Corollary 4.7 and Example 4.8], we have the following implications:

"Int∘Cl"-T_1	$\to$	T_1	$\to$	$\mathrm{T}_{3/4}$	$\to$	$\mathrm{T}_{1/2}$
$\searrow$			$\nearrow$			
	"Int∘Cl"-$\mathrm{T}_{1/2}$					

where $A \to B$ presents that A implies B. Also we observe that none of the implication is reversible.

Example 5.7. We consider the following one point unions: $S^1_f \vee S^1_f, S^1_f \vee S^1_\infty$ and $S^1_f \vee S^2_\infty$ and so on, where S^1_f, S^1_∞ and S^2_∞ are a *finite digital circle* $((Y_{16}, (\kappa \times \kappa)|Y_{16})$ in Example 5.3, the *infinite digital circle* in Example 5.4 and the *infinite digital sphere* in Example 5.5, respectively. We have interesting examples of digital pictures with topologies over one point unions above (cf [13; p.20]). Let E_1 (resp. E_2) be *finite digital circles* (resp. *infinite digital circles*) attached at the integer points (i.e., $\{16s|s \in \mathrm{Z}\}$) of mod 16 of (Z, κ) and E_3 *infinite digital spheres* attached at the integer points of mod 16 of (Z, κ). Then E_1 (resp. E_2, E_3) is a natural and interesting digital pictures with topologies over $S^1_f \vee S^1_f$ (resp. $S^1_f \vee S^1_\infty, S^1_f \vee S^2_\infty$) (cf. [5][13;p.20]).

Acknowledgement: The authors wish to express their thanks for the referee who has kindly suggested many valuable comments to the previous version.

References

[1] K.Balachandran, P.Sundaram and H.Maki, *On generalized continuous maps in topological spaces*, Mem.Fac.Sci.Kochi Univ.(Math.) **12** (1991), 5-13.

[2] J.Dontchev and I.Arokiarani and K.Balachandran, *On generalized δ-closed sets and almost weakly Hausdorff spaces*, (preprint)..

[3] J.Dontchev,M.Fujimoto, H.Maki, T.Noiri and S.Takigawa, *The topological structures and groups of digital n-spaces*, (in preparation).

[4] J.Dontchev and M.Ganster, *On δ-generalized closed sets and $T_{3/4}$-spaces*, Mem.Fac.Sci. Kochi Univ. (Math.) **17** (1996), 15-31.

[5] J.Dontchev and H.Maki, *Groups of θ-generalized homeomorphisms and the digital line*, Topology Appl. **95** (1999), 113-128.

[6] W.Dunham, *A new closure operator for non-T_1 topologies*, Kyungpook Math.J. **22** (1982), 55-60.

[7] D.S.Janković, *On functions with α-closed graphs*, Glas. Mat. Ser.3 **18(38)** (1983), 141-148.

[8] S.Kasahara, *Operation-compact spaces*, Math. Japon. **24** (1979), 97-105.

[9] E.D.Khalimsky, R.Kopperman and P.R.Meyer, *Computer graphics and connected topologies on finite ordered sets*, Topology Appl. **36** (1990), 1-17.

[10] T.Y.Kong, R.Kopperman and P.R.Meyer, *A topological approach to digital topology*, Amer. Math. Monthly **98** (1991), 901-917.

[11] T.Y.Kong and A.Rosenfeld, *Digital Topology: Introductions and survey*, Computer Vision, Graphics and Image Processing **48** (1989), 357-393.

[12] N.Levine, *Generalized closed sets in Topology*, Rend.Circ.Mat.Palermo **19** (1970), 89-96.

[13] S.P.Novikov, *Topology I*, Encyclopadia of Mathmatical Sciences (Springer,Berlin) **12** (1996).

[14] H.Ogata, *Operations on topological spaces and associated topology*, Math. Japon. **36** (1991), 175-184.

[15] H.Ogata, *Remarks on some operation-separation axioms*, Bul. Fukuoka Univ. Ed. Part 3 **40** (1991), 41-43.

[16] H.Ogata and H.Maki, *Operation-subspace topologies*, Meetings on topological spaces theory and it's applications (held at Yatsushiro College of Technology) **August** (1996), 3-5.

[17] J.Umehara and H.Maki, *Operator approaches of weakly Hausdorff spaces*, Mem. Fac. Sci. Kochi Univ. (Math.) **11** (1990), 65-73.

[18] J.Umehara, H.Maki and T.Noiri, *Bioperations on topological spaces and some separation axioms*, Mem. Fac. Sci. Kochi Univ. (Math.) **13** (1992), 45-59.

[19] N.V.Velišcko, *H-closed topological spaces*, Amer. Math. Soc. Transl. (2) **78** (1968), 103-118.

H.Maki, Wakagidai 2-10-13, Fukuma-cho, Munakata-gun, Fukuoka-ken 811-31, JAPAN

Department of Mathematics, Kyushu Kyoritsu University, Kitakyushu, Fukuoka 807,JAPAN

Department of Mathematics,Bharathiar University,Coimbatore 641046,Tamil Nadu, INDIA

Department of Mathematics, Nallumuthu Gounder Mahalingam College, Pollachi 642001, Tamil Nadu, INDIA

Department of Mathematics, Kongu Nadu Arts and Science College, Coimbatore 641029, Tamil Nadu, INDIA

E-mail address: makih@pop12.odn.ne.jp