CONDITIONS FOR A BCK-ALGEBRA TO BE n-FOLD POSITIVE IMPLICATIVE

Young Bae Jun and Kyung Ho Kim

Received June 27, 2000

Abstract. Some conditions for a BCK-algebra to be n-fold positive implicative are investigated.

1. Introduction

In 1999, Huang and Chen [1] introduced the notion of n-fold positive implicative ideals and n-fold positive implicative BCK-algebras, and investigated some properties. In this short note, we find some conditions for a BCK-algebra to be n-fold positive implicative.

2. Preliminaries

For any x and y of a BCK-algebra X, let $x * y^n$ denote $(\cdots((x*y)*y)*\cdots)*y$ in which y occurs n times.

A BCK-algebra X is said to be *n-fold positive implicative* (see Huang and Chen [1]) if there exists a natural number n such that $x*y^{n+1} = x*y^n$ for all $x, y \in X$.

A subset I of a BCK-algebra X is called an *ideal* of X if

(I1) $0 \in I$,

(I2) $x*y \in I$ and $y \in I$ imply $x \in I$.

A subset I of a BCK-algebra X is called an *n-fold positive implicative ideal* of X (see Huang and Chen [1]) if it satisfies (I1) and

(I3) for every $x, y, z \in X$, there exists a natural number n such that $(x*y^{n+1})*z \in I$ and $z \in I$ imply $x*y^n \in I$.

3. Conditions for a BCK-algebra to be n-fold positive implicative

In what follows let X denote a BCK-algebra unless otherwise specified. For any $a \in X$ let $A(a)$ denote the set of all elements of X which are less than or equal to a, i.e.,

$$A(a) := \{x \in X \mid x \le a\}.$$

Note that $0 \in A(a)$, and $A(a)$ is not an ideal of X. Iséki and Tanaka gave a condition for the set $A(a)$ to be an ideal of X as follows:

Proposition 3.1 (Iséki and Tanaka [2, Proposition 2]). *Any set $A(a)$ is an ideal of X if and only if $x*y \le z$ and $y \le z$ imply $x \le z$ for all $x, y, z \in X$.*

2000 Mathematics Subject Classification. 06F35, 06B10.

Key words and phrases. n-fold positive implicative BCK-algebra, n-fold positive implicative ideal.

Lemma 3.2 (Huang and Chen [1, Theorem 1.7]). *A subset I of X is an n-fold positive implicative ideal of X if and only if*

(i) $0 \in I$,

(ii) $(x * y) * z^n \in I$ *and* $y * z^n \in I$ *imply* $x * z^n \in I$.

We give a condition for the set $A(a)$ to be an n-fold positive implicative ideal.

Theorem 3.3. *For any $a \in X$ the set $A(a)$ is an n-fold positive implicative ideal of X if and only if $x * z^{n+1} = 0$ whenever $(x * y) * z^{n+1} = 0$ and $y * z^{n+1} = 0$.*

Proof. For every $a \in X$ suppose that $A(a)$ is an n-fold positive implicative ideal of X and let $x, y, z \in X$ be such that $(x * y) * z^{n+1} = 0$ and $y * z^{n+1} = 0$. Then $(x * y) * z^n \in A(z)$ and $y * z^n \in A(z)$. It follows from Lemma 3.2 that $x * z^n \in A(z)$, i.e., $x * z^{n+1} = 0$.

Conversely, consider $A(z)$ for any $z \in X$. Let $x, y, z \in X$ be such that $(x * y) * z^n \in A(z)$ and $y * z^n \in A(z)$. Then $(x * y) * z^n \leq z$ and $y * z^n \leq z$, i.e., $(x * y) * z^{n+1} = 0$ and $y * z^{n+1} = 0$. Using the hypothesis, we get $x * z^{n+1} = 0$, i.e., $x * z^n \leq z$ and so $x * z^n \in A(z)$. Therefore $A(z)$ is an n-fold positive implicative ideal of X by Lemma 3.2. □

Since every n-fold positive implicative ideal of X is an ideal (see Huang and Chen [1, Proposition 1.2]), we have the following corollary which is a partial generalization of Proposition 3.1.

Corollary 3.4. *If a BCK-algebra X satisfies the condition*

$$(x * y) * z^{n+1} = 0 \text{ and } y * z^{n+1} = 0 \text{ imply } x * z^{n+1} = 0, \tag{3.1}$$

then $A(a)$ is an ideal of X.

Lemma 3.5 (Huang and Chen [1, Theorem 1.5]). *An ideal I of X is n-fold positive implicative if and only if $x * y^{n+1} \in I$ implies $x * y^n \in I$.*

Note that the zero ideal of X is not an n-fold positive implicative ideal of X (see Huang and Chen [1, Example 1.3]). We provide a condition for the zero ideal to be n-fold positive implicative.

Theorem 3.6. *For every $a \in X$ if $A(a)$ is an ideal of X, then the zero ideal $\{0\}$ is n-fold positive implicative.*

Proof. Let $x, y \in X$ be such that $x * y^{n+1} = 0$. Then $(x * y^{n-1}) * y = x * y^n \leq y$, and so $(x * y^{n-1}) * y \in A(y)$. Since $y \in A(y)$, it follows from (I2) that $x * y^{n-1} \in A(y)$, i.e., $x * y^{n-1} \leq y$ and thus $x * y^n = 0$. Hence, by Lemma 3.5, we conclude that $\{0\}$ is an n-fold positive implicative ideal of X. □

Corollary 3.7. *If a BCK-algebra X satisfies the condition* (3.1)*, then the zero ideal of X is n-fold positive implicative.*

Combining Proposition 3.1 and Theorem 3.6, we have the following corollary.

Corollary 3.8. *If a BCK-algebra satisfies the condition*

$$x * y \leq z \text{ and } y \leq z \text{ imply } x \leq z, \tag{3.2}$$

then the zero ideal $\{0\}$ is n-fold positive implicative.

Lemma 3.9 (Huang and Chen [1, Theorem 1.11]). *A BCK-algebra X is n-fold positive implicative if and only if the zero ideal of X is n-fold positive implicative.*

Using Theorem 3.6 and Lemma 3.9, we give a condition for a BCK-algebra to be n-fold positive implicative.

Theorem 3.10. (i) *For any $a \in X$ if $A(a)$ is an ideal of X, then X is n-fold positive implicative.*

(ii) *If a BCK-algebra X satisfies the condition* (3.1) *or* (3.2), *then X is n-fold positive implicative.*

Lemma 3.11 (Huang and Chen [1, Corollary 1.12]). *A BCK-algebra X is n-fold positive implicative if and only if all ideals are n-fold positive implicative.*

Combining Theorem 3.10 and Lemma 3.11, we obtain the following corollary

Corollary 3.12. (i) *For any $a \in X$ if $A(a)$ is an ideal of X, then all ideals are n-fold positive implicative.*

(ii) *If a BCK-algebra X satisfies the condition* (3.1) *or* (3.2), *then all ideals are n-fold positive implicative.*

References

[1] Y. Huang and Z. Chen, *On ideals in BCK-algebras*, Math. Japonica **50(2)** (1999), 211-226.
[2] K. Iséki and S. Tanaka, *Ideal theory of BCK-algebras*, Math. Japonica **21** (1976), 351-366.

Young Bae Jun
Department of Mathematics Education
Gyeongsang National University
Jinju 660-701, Korea
E-mail: ybjun@nongae.gsnu.ac.kr

Kyung Ho Kim
Department of Mathematics
Chungju National University
Chungju 380-702, Korea
E-mail: ghkim@gukwon.chungju.ac.kr