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ON FACTORIZATION THEOREM FOR TRANSFINITE DIMENSION $\dim_C$

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Abstract. Let $f : X \to Z$ be a continuous mapping from a compact $C-$space X onto a compact space Z. Then there are a compact $C-$space Y and continuous mappings $g : X \to Y$ and $h : Y \to Z$ such that $\dim_C Y \le \dim_C X, wY \le wZ$ and $f = hg$.
Here $\dim_C$ is Borst's transfinite extension of the covering dimension dim in the class of compact C-spaces.

1 Introduction Throughout this note all spaces are assumed to be normal T_1. The necessary information about notions and notations we use can be found in [1] and [6, 7].

A space X is called a *C-space* if for every sequence of open covers $\{\alpha_i\}_{i=1}^{\infty}$ of X there exist disjoint open families β_i, $i = 1, 2, ...$ such that $X = \cup_{i=1}^{\infty}\beta_i$ and $\beta_i \succ \alpha_i$ for every i.

A space X is called *A-weakly infinite-dimensional (S-weakly infinite-dimensional)* if for every sequence $\{A_i, B_i\}_{i=1}^{\infty}$ of pairs of disjoint closed sets in X there exist open sets $V_i, i = 1, 2, ...$ such that
$A_i \subset V_i \subset [V_i] \subset X \setminus B_i$ and
$\cap_{i=1}^{\infty} BdV_i = \emptyset$ ($\cap_{i=1}^{n} BdV_i = \emptyset$ for some positive integer n).

It is known (cf. [6]) that every C-space is A-weakly infinite-dimensional and the space $W = [0, \omega_1)$ of all ordinal numbers less than the first uncountable ordinal number ω_1 with the order topology is S-weakly infinite-dimensional ($\dim W = 0$) but it is not a C-space (there exists a cover of W which has not a σ-disjoint open refinement). The space W is not paracompact.

In [2] P. Borst gave the definition of finite C-spaces.
A space X is called *a finite C-space* if for every sequence of finite open covers $\{\alpha_i\}_{i=1}^{\infty}$ of X there exist disjoint open families β_i, $i = 1, 2, ..n$, for some n such that $X = \cup_{i=1}^{n}\beta_i$ and $\beta_i \succ \alpha_i$ for every i.
Every finite C-space is S-weakly infinite-dimensional. Every finite-dimensional (in the sense of dim) space is a finite C-space, in particular the space W is a finite C-space.

Observe that a paracompact space X is S-weakly infinite-dimensional (a finite C-space) if and only if the space X is A-weakly infinite-dimensional (a C-space) and there exists a compact subspace K of X such that $\dim F < \infty$ for every closed subspace F with $F \cap K = \emptyset$ (see [2] for a proof for a metrizable space X).

One can ask
Question 1.
a) Is every A-weakly infinite-dimensional paracompact space a C-space?

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b) Is every S-weakly infinite-dimensional (paracompact) space a finite C-space?

Recall that the coincidence of weakly infinite-dimensionality and C-property for metrizable compacta is one of the main open problems in Infinite Dimension Theory.

In [2, 3] P. Borst classified the class of S-weakly infinite-dimensional spaces with help of transfinite dimension function $\dim_{\mathrm{W}}$ and the class of finite C-spaces with help of transfinite dimension function $\dim_{\mathrm{C}}$ (in fact he considered only the metrizable case, for the case of normal spaces see part 3 of the paper), and proved that $\dim_{\mathrm{W}} X \leq \dim_{\mathrm{C}} X$ for a space X. Both transfinite dimensions $\dim_{\mathrm{W}}$ and $\dim_{\mathrm{C}}$ are natural transfinite extensions of dimension dim.

Let us recall

Question 2 ([2]). Does the equality $\dim_{\mathrm{W}} X = \dim_{\mathrm{C}} X$ hold for every space X?

In [4, 10] K. Yokoi and the author independently proved the factorization theorem for $\dim_{\mathrm{W}}$ for compact spaces. In [5] the author announced the factorization theorem for $\dim_{\mathrm{C}}$ for compact spaces. Here this theorem is proved.

2 Function Ord

Let us recall now the definition of the ordinal function Ord [2].

Let L be an arbitrary set. By $\mathrm{Fin}\, L$ we will denote the collection of all finite, non-empty subsets of L.

Let M be a subset of $\mathrm{Fin}\, L$. For $\sigma \in \{\emptyset\} \cup \mathrm{Fin}\, L$ we put
$M^\sigma = \{\tau \in \mathrm{Fin}\, L | \quad \sigma \cup \tau \in M \text{ and } \sigma \cap \tau = \emptyset\}$. M^a abbreviates $M^{\{a\}}$.

Define the ordinal number $\mathrm{Ord} M$ inductively as follows
$\mathrm{Ord} M = 0$ iff $M = \emptyset$,
$\mathrm{Ord} M \leq \alpha$ iff for every $a \in L$ $\mathrm{Ord} M^a < \alpha$,
$\mathrm{Ord} M = \alpha$ iff $\mathrm{Ord} M \leq \alpha$ and $\mathrm{Ord} M < \alpha$ is not true, and
$\mathrm{Ord} M = \infty$ (does not exist) iff $\mathrm{Ord} M > \alpha$ for every ordinal number α.

Recall some propositions about Ord.

Lemma 2.1 ([2]). Let $M, N \subset \mathrm{Fin}\, L$. Then
(a) if $\sigma, \tau \in \mathrm{Fin}\, L$ and $\sigma \cap \tau = \emptyset$, then $(M^\sigma)^\tau = M^{\sigma \cup \tau}$;
(b) if $\sigma, \tau \in \mathrm{Fin}\, L$ then $\sigma \in M^\tau$ iff $\tau \in M^\sigma$;
(c) If $N \subset M$ and $\mathrm{Ord} M$ exists then $\mathrm{Ord} N$ exists and $\mathrm{Ord} N \leq \mathrm{Ord} M$.
(d) if $n \geq 0$ then $\mathrm{Ord} M \leq n$ iff $|\sigma| \leq n$ for every $\sigma \in M$.

Lemma 2.2 ([2]). Let $\Phi : L \mapsto L_1$ be a function from a set L to a set L_1 and $M \subset \mathrm{Fin}\, L$ and $M_1 \subset \mathrm{Fin}\, L_1$ be such that for every $\sigma \in M$ we have
(1) $\Phi(\sigma) \in M_1$, and
(2) $|\Phi(\sigma)| = |\sigma|$.
Then $\mathrm{Ord} M \leq \mathrm{Ord} M_1$.

A subset M of $\mathrm{Fin}\, L$ is said to be *inclusive* if for every $\sigma, \sigma_1 \in \mathrm{Fin}\, L$ such that $\sigma \in M$ and $\sigma_1 \subset \sigma$ also $\sigma_1 \in M$.

Lemma 2.3 ([2]). Let M be an inclusive subset of $\mathrm{Fin}\, L$. Then $\mathrm{Ord} M = \infty$ iff there exists a sequence $\{a_i\}_{i=1}^\infty$ of distinct elements of L such that $\sigma_n = \{a_i\}_{i=1}^n \in M$ for every $n = 1, 2, ...$

Let $a \in L$ and $M \subset \operatorname{Fin} L$. We put $M[\bar{a}] = \{\tau \in M : a \notin \tau\}$.

Lemma 2.4 ([5]). Let L_1, L_2 be sets such that $L_1 \subset L_2$. Let $M_i \subset \operatorname{Fin} L_i, i = 1, 2$, $\emptyset \neq M_1 \subset M_2$ and M_1, M_2 be inclusive. Let also the following three conditions hold:
(1) for every $a_2 \in L_2 \wedge M_2$ we can find infinitely many distinct elements $a_1 \in L_1 \wedge M_1$ such that $a_1 \neq a_2$ and $M_2^{a_2}[\bar{a}_1] \subset M_2^{a_1}$;
(2) for every $\sigma_1 \in M_1$ we have $M_2^{\sigma_1} \neq \emptyset$ iff $M_1^{\sigma_1} \neq \emptyset$;
(3) for any different elements $a_1, ..., a_k \in L_1$ if $\sigma = \{a_1, ..., a_k\} \in M_2$ then $\sigma \in M_1$.
Then $\operatorname{Ord} M_1 = \operatorname{Ord} M_2$.

3 Dimension $\dim_C$ Here we will use some ideas from [2] where the metrizable case was considered.

Let X be a space. A family γ of nonempty open subsets of X is *a cover of* X if the union of all members of γ is equal to X. Some members of a cover can be equal. For two families α and β of subsets of X the notation $\beta \succ \alpha$ means that for each element $V \in \beta$ there exists an element $U \in \alpha$ such that $V \subset U$. If A is a subspace of X then the notation $[A]_X$ means the closure of A in X.

A sequence $\sigma = \{\alpha_1, ..., \alpha_n\}$ of covers of X is said to be *inessential* if there exist disjoint open families $\beta_i, i = 1, ..., n$, such that $X = \cup_{i=1}^n \beta_i$ and $\beta_i \succ \alpha_i$. Otherwise σ is called *essential*.

Lemma 3.1. Let $\sigma = \{\alpha_1, ..., \alpha_n\}$, $\tau = \{\beta_1, ..., \beta_m\}$ be finite sequences of covers of X, $m \leq n$ and $\beta_i \succ \alpha_i$ for every $i = 1, ..., m$. If σ is essential then τ is also essential.

By *a shrinking* of a finite cover $\sigma = \{A_1, ..., A_n\}$ of X we mean any finite cover $\tau = \{B_1, ..., B_n\}$ of X such that $[B_i]_X \subset A_i$ for every i.

Lemma 3.2. Let $\dim X > 0$. Then
(a) for every finite cover α of X there exists an essential finite cover β of X such that $\beta \succ \alpha$;
(b) if α is an essential finite cover of X and β is a shrinking of α then $\alpha \neq \beta$;

Proof. (a) Since $\dim X > 0$ then there exists an essential finite cover γ of X. Note that $\gamma \wedge \alpha = \{G \cap A : G \in \gamma, A \in \alpha\}$ is a finite cover of X and $\gamma \wedge \alpha \succ \alpha, \gamma$. By lemma 3.1 the cover $\gamma \wedge \alpha$ is essential.
(b) Let $\alpha = \{U_1, ..., U_k\}, \beta = \{V_1, ..., V_k\}$ and $[V_i]_X \subset U_i$ for every $i = 1, ..., k$. If $\alpha = \beta$ then $U_i = V_{j_i}$ for every $i = 1, ..., k$, where $P = \begin{pmatrix} 1 & \dots & k \\ j_1 & \dots & j_k \end{pmatrix}$ is a permutation. Let us decompose the permutation P into the product of independent cycles. Consider one of the cycles, for example we have the cycle (123). It means $U_1 = V_2, U_2 = V_3, U_3 = V_1$. We have $U_1 \supset [V_1]_X \supset V_1 = U_3 \supset [V_3]_X \supset V_3 = U_2 \supset [V_2]_X \supset V_2 = U_1$. It is clear that $U_i = [U_i]_X$, $i = 1, 2, 3$. So the cover α consists of clopen sets. We can construct a finite cover γ of X such that the cover γ consists of open disjoint sets and $\gamma \succ \alpha$. But α is essential. It is a contradiction.

We put $T(X) = \{O : O$ is a non-empty open subset of $X\}$ and
$K(X) = \{\alpha \in \operatorname{Fin} T(X) : \alpha$ is a cover of $X\}$.
If $K \subset K(X)$ then we put $M_K = \{\sigma \in \operatorname{Fin} K : \sigma$ is essential $\}$.

Let α be a finite cover of X. (Recall that α may have repeating elements.) By $\alpha_{K(X)}$ we denote the element from $K(X)$ which consists of the same open nonempty sets as α does.

(Observe that $\alpha_{K(X)}$ does not have repeating elements.) The notation $\alpha \in K(X) \wedge M_{K(X)}$ means that α is an essential finite open cover without repeating elements.

Lemma 3.3. For every essential finite open cover α of X there exists a sequence $\{\beta_i\}_{i=1}^{\infty}$ of distinct elements from $K(X) \wedge M_{K(X)}$ such that $\alpha \prec \beta_1 \prec \beta_2$ ...
In particular if $\dim X > 0$ then for every finite open cover γ of X we can find a sequence $\{\beta_i\}_{i=1}^{\infty}$ of distinct elements from $K(X) \wedge M_{K(X)}$ such that $\gamma \prec \beta_1 \prec \beta_2$ ...

Proof. Put $\beta_1 = \alpha_{K(X)} \in K(X) \wedge M_{K(X)}$. Consider a shrinking γ_1 of β_1. By Lemma 3.2 (b) we have $\gamma_1 \neq \beta_1$. It is clear that $\beta_2 = (\gamma_1)_{K(X)} \in K(X) \wedge M_{K(X)}$, $\beta_2 \succ \beta_1$ and $\beta_1 \neq \beta_2$. Thus we can obtain a sequence $\{\beta_i\}_{i=1}^{\infty}$ of distinct elements from $K(X) \wedge M_{K(X)}$ such that $\gamma \prec \beta_1 \prec \beta_2$ ...
Observe that by Lemma 3.2 (a) there exists an essential finite cover α of X such that $\alpha \succ \gamma$.

Lemma 3.4 ([2], the metrizable case). Let $n \geq 0$. Then $\dim X \leq n$ iff for every element $\sigma \in M_{K(X)}$ $|\sigma| \leq n$.

Proof. Let $\dim X \leq n$ and $\sigma = \{\alpha_i\}_{i=1}^{m} \in M_{K(X)}$. Assume that $m > n$. Put $\tau = \{\alpha_i\}_{i=1}^{n+1}$ and $\beta = \{\cap_{i=1}^{n+1} U_i : U_i \in \alpha_i, i = 1, ..., n+1\}$. By Ostrand's theorem (cf. [6], p. 184) there exists a cover ν of X which can be represented as the union of $n+1$ disjoint open families $\nu_1, ..., \nu_{n+1}$ such that $\nu \succ \beta$. It is clear that $\nu_i \succ \alpha_i$ for every i. Hence the sequence τ is inessential. By Lemma 3.1 we have that the sequence σ is inessential. It is a contradiction. Consequently $m \leq n$.
For the sufficiency let α be a finite cover of X and $\dim X > 0$. By Lemma 3.3 there exist (n+1) distinct elements $\beta_1, ..., \beta_{n+1}$ from $K(X) \wedge M_{K(X)}$ such that $\beta_i \succ \alpha$ for every i. Observe that $\{\beta_1, ..., \beta_{n+1}\} \in \mathrm{Fin}\, K(X)$ and this finite sequence is inessential by assumption. Hence there exist finite disjoint open families $\mu_i, i = 1, ..., n+1$, such that $X = \cup_{i=1}^{n+1} \mu_i$ and $\mu_i \succ \beta_i \succ \alpha$ for every i. The finite cover $\mu = \{V : V \in \mu_i, i = 1, ..., n+1\}$ is a finite refinement of the cover α of order $\leq n+1$.

Theorem 3.5 ([2], the metrizable case). Let $n \geq 0$. Then $\mathrm{Ord} M_{K(X)} \leq n$ iff $\dim X \leq n$.

Proof. $\mathrm{Ord} M_{K(X)} \leq n$ iff (by Lemma 2.1 (d)) for every element $\sigma \in M_{K(X)}$ $|\sigma| \leq n$ iff (by Lemma 3.4) $\dim X \leq n$. We are done.

Definition 3.6 ([2]). We set $\dim_{\mathrm{C}} X = \mathrm{Ord} M_{K(X)}$.

Observe that by Theorem 3.5 the transfinite function $\dim_{\mathrm{C}}$ is a natural transfinite extension of dim in the class of normal T_1-spaces.

Lemma 3.7. Let $\dim X > 0$. Then the space X is a finite C-space iff for every sequence $\{\alpha_i\}_{i=1}^{\infty}$ of distinct elements of $K(X)$ there exists some n such that $X = \cup_{i=1}^{n} \nu_i$, where ν_i is a disjoint finite open family and $\nu_i \succ \alpha_i$ for every $i = 1, ..., n$.

Proof. We need to prove only the sufficiency. Let $\{\alpha_i\}_{i=1}^{\infty}$ be a sequence of finite covers of X and $\dim X > 0$. Consider the sequence $\sigma = \{(\alpha_i)_{K(X)}\}_{i=1}^{\infty}$ of elements from $K(X)$. There are two cases. The first one is that there exists a subsequence $\{\beta_i\}_{i=1}^{\infty}$ of σ which consists of distinct elements. Then we can find some n such that $X = \cup_{i=1}^{n} \nu_i$, where ν_i is a disjoint finite open family and $\nu_i \succ \beta_i$ for every $i = 1, ..., n$. Hence the space X is a finite C-space. The second one is that there exists a subsequence $\{\beta_i\}_{i=1}^{\infty}$ of σ such that $\beta_1 = \beta_2 = ...$ By Lemma 3.3 we can find a sequence $\{\gamma_i\}_{i=1}^{\infty}$ of distinct elements from $K(X) \wedge M_{K(X)}$ such that $\beta_i \prec \gamma_1 \prec \gamma_2$ ... for every i. There exists some n such that $X = \cup_{i=1}^{n} \nu_i$, where ν_i is a disjoint finite open family and $\nu_i \succ \gamma_i \succ \beta_i$ for every $i = 1, ..., n$.

Hence the space X is again a finite C-space.

Theorem 3.8 ([2], the metrizable case). A space X is a finite C-space iff $\dim_C X$ exists.

Proof. $\dim_C X = \infty$ iff (by Definition 3.6) $\mathrm{Ord} M_{K(X)} = \infty$ iff (Lemma 2.3) there exists a sequence $\{\alpha_i\}_{i=1}^{\infty}$ of distinct elements of $K(X)$ such that $\sigma_n = \{\alpha_i\}_{i=1}^{n} \in M_{K(X)}$ for every $n = 1, 2, ...$ iff (by Lemma 3.7) the space X is not a finite C-space.

Definition 3.9. A set $S \subset K(X)$ is *a countably inscribed system in* $K(X)$ if for every element $\gamma \in K(X) \wedge M_{K(X)}$ there exists a sequence $\{\beta_i\}_{i=1}^{\infty}$ of distinct elements from S such that $\gamma \prec \beta_i$ for every i.

Observe that by Lemma 3.3 for every space X always there exists a countably inscribed system in $K(X)$ (for example, if $\dim X = 0$ then put $S = \emptyset$, if $\dim X > 0$ then put $S = K(X)$).

Theorem 3.10. Let S be a countably inscribed system in $K(X)$. Then $\dim_C X = \mathrm{Ord} M_S$.

Proof. If $\dim X = 0$ then the theorem is evident. Let $\dim X > 0$. We will prove that $\mathrm{Ord} M_S = \mathrm{Ord} M_{K(X)}$. Put $L_1 = S,\ L_2 = K(X),\ M_1 = M_S,\ M_2 = M_{K(X)}$. Let us verify the conditions of Lemma 2.4. It is evident that $\emptyset \neq M_1 \subset M_2$ and M_1, M_2 are inclusive.
Condition (1): Let $\alpha \in L_2 \wedge M_2$. By Definition 3.9 there exists a sequence $\{\beta_i\}_{i=1}^{\infty}$ of distinct elements of $L_1 \wedge M_1$ such that $\alpha \prec \beta_i$ and $\alpha \neq \beta_i$ for every i. It is evident that $M_2^{\alpha}[\bar{\beta}_i] \subset M_2^{\beta_i}$ for every i.
Condition (2): Let $\sigma = \{\alpha_i\}_{i=1}^{n} \in M_1$ and $M_2^{\sigma} \neq \emptyset$. So there exists $\tau = \{\gamma_i\}_{i=1}^{m} \in M_2^{\sigma}$, $\tau \cap \sigma = \emptyset$ and $\tau \cup \sigma \in M_2$. By Definition 3.9 for every element $\gamma_i \in \tau$ (recall that $\gamma_i \in L_2 \wedge M_2$) we can find a sequence $\{\theta_j^i\}_{j=1}^{\infty}$ of distinct elements of $M_1 \wedge L_1$ such that $\gamma_i \prec \theta_j^i$ for every j.
Now it is easy to find a finite sequence $\eta = \{\nu_i\}_{i=1}^{n}$ of distinct elements of $M_1 \wedge L_1$ such that $\nu_i \succ \gamma_i, i = 1, ..., n$, and $\eta \cap \sigma = \emptyset$. By Lemma 3.1 we have $\eta \cup \sigma \in M_1$. Hence $\eta \in M_1^{\sigma}$ and thus $M_1^{\sigma} \neq \emptyset$.
Condition (3) is evident.
So by Lemma 2.4 we have $\mathrm{Ord} M_S = \mathrm{Ord} M_{K(X)}$ and hence $\dim_C X = \mathrm{Ord} M_S$.

Proposition 3.11. Let X be a compact space. Then there exists a countably inscribed system S in $K(X)$ with $|S| \leq wX$.

Proof. Let B be a base in X with $|B| = wX$. Put $U_B = \{A \in T(X) : A = \cup\sigma$ for some element $\sigma \in \mathrm{Fin}\, B\} \subset T(X)$ and $S = \mathrm{Fin}\, U_B \wedge K(X)$. Observe that $|S| \leq wX$. Consider $\gamma \in K(X) \wedge M_{K(X)}$. By Lemma 3.3 we can find a sequence $\{\beta_i\}_{i=1}^{\infty}$ of distinct elements from $K(X) \wedge M_{K(X)}$ such that $\gamma \prec \beta_1 \prec \beta_2$ ... It is easy to see that the covers $\beta_i, i = 1, 2, ...$ can be chosen from S because the space X is compact.

4 Cech-Stone compactification theorem for $\dim_C$ Let X, Y be spaces and $X \subset Y$. By *a blowing up* of an element $\alpha = \{A_s : s \in S\} \in K(X)$ in Y we mean any element $\beta = \{B_s : s \in S \cup S_1\} \in K(Y)$ such that
(1) $B_s \cap X = A_s$ for every $s \in S$, and
(2) $B_s \cap X = \emptyset$ for every $s \in S_1$.
The set S_1 can be empty.

Observe that a blowing up exists not always. For example, let $X = [0,1) \cup (1,2], Y = [0,2]$ and $\alpha = \{A_1, A_2\}$, where $A_1 = [0,1), A_2 = (1,2]$. The cover α of X has not any blowing up.

Lemma 4.1. Let there exist a mapping $f : K(X) \to K(Y)$ such that $f(\alpha)$ is a blowing up of α in Y for every element $\alpha \in K(X)$. Then $\mathrm{Ord}M_{K(X)} \leq \mathrm{Ord}M_{K(Y)}$.

Proof. Let us verify the conditions of Lemma 2.2. Consider $\sigma = \{\alpha_i\}_{i=1}^k \in M_{K(X)}$.
Condition (1): Let $f(\sigma) \notin M_{K(Y)}$. Then there exist open disjoint families $\beta_i, i = 1, ..., k$, such that $Y = \cup_{i=1}^k \beta_i$ and $\beta_i \succ f(\alpha_i)$. Note that $\gamma_i = \beta_i \wedge X = \{U \cap X : U \in \beta_i\}$ is an open disjoint family in $X, i = 1, ..., k$, such that $X = \cup_{i=1}^n \gamma_i$ and $\gamma_i \succ \alpha_i, i = 1, ..., k$. Consequently $\sigma \notin M_{K(X)}$. It is a contradiction.
Condition (2): It is evident that $|f(\sigma)| = |\sigma|$ because f is injective.
Hence by Lemma 2.2 we have $\mathrm{Ord}M_{K(X)} \leq \mathrm{Ord}M_{K(Y)}$.

Theorem 4.2. Let F be a closed subset of a finite C-space X. Then $\dim_{\mathrm{C}} F \leq \dim_{\mathrm{C}} X$.

Proof. Define a mapping $f : K(F) \to K(X)$ as follows. Let $\alpha = \{O_i\}_{i=1}^k \in K(F)$. Then there exists $V_i \in T(X)$ such that $V_i \cap F = O_i$, i = 1,..,k. Put $f(\alpha) = \{V_1, ..., V_k, V_{k+1} = X \setminus F\}$. It is evident $f(\alpha)$ is a blowing up of α in X. By Lemma 4.1 we get $\dim_{\mathrm{C}} F \leq \dim_{\mathrm{C}} X$.

Let X be a space, O be an open set in X and βX be Cech-Stone compactification of X. Put $exO = \beta X \setminus [X \setminus O]_{\beta X}$. Note that $exO \cap X = O$.

Lemma 4.3. Let $O, V \in T(X)$ and $W \in T(\beta X)$. Then
(a) $ex(O \cup V) = exO \cup exV$;
(b) $ex(O \cap V) = exO \cap exV$,
in particular $exO \cap exV = \emptyset$ iff $O \cap V = \emptyset$;
(c) $exO \subset exV$ iff $O \subset V$;
(d) $ex(W \cap X) \subset [W]_{\beta X}$;

Proof. (a)-(c) see for a proof (cf. [7], p. 476).
(d) By definition $ex(W \cap X) = \beta X \setminus [X \setminus (W \cap X)]_{\beta X}$. We will prove $[X \setminus (W \cap X)]_{\beta X} \supset (\beta X \setminus [W]_{\beta X})$. Let $x \in \beta X \setminus [W]_{\beta X}$. Then there exists an open in βX set U such that $x \in U \subset \beta X \setminus [W]_{\beta X}$. It is evident that $(U \cap X) \cap (W \cap X) = \emptyset$. Consequently $U \cap X \subset X \setminus (W \cap X)$ and $[U \cap X]_{\beta X} \subset [X \setminus (W \cap X)]_{\beta X}$. Note that $x \in U \subset [U]_{\beta X} = [U \cap X]_{\beta X} \subset [X \setminus (W \cap X)]_{\beta X}$.

For $\alpha = \{O_i\}_{i=1}^k \in K(X)$ put $ex(\alpha) = \{exO_i\}_{i=1}^k$. For $\beta = \{O_i\}_{i=1}^k \in K(\beta X)$ put $\beta \wedge X = \{O_i \cap X\}_{i=1}^k$.

Lemma 4.4. (a) $ex(\alpha)$ is a blowing up of α in βX.
(b) The cover $\beta \wedge X$ of X is essential if the cover $ex(\beta \wedge X)$ of βX is essential.
(c) Let $\dim \beta X > 0$. Then for every element $\alpha \in K(\beta X) \wedge M_{K(\beta X)}$ we can find a sequence $\{\beta_i\}_{i=1}^\infty$ of distinct elements from $K(X) \wedge M_{K(X)}$ such that $\alpha \prec ex(\beta_1) \prec ex(\beta_2)$ ... (Observe that the sequence $\{ex(\beta_i)\}_{i=1}^\infty$ consists of distinct elements of $K(\beta X) \wedge M_{K(\beta X)}$).

Proof. (a) Apply Lemma 4.3 (a).
(b) Let $\beta \wedge X$ be inessential. Then there exists a finite cover μ of X such that $\mu \succ \beta \wedge X$ and μ is a disjoint family. By Lemma 4.3 (a) $ex(\mu)$ is a finite cover of βX such that $ex(\mu) \succ ex(\beta \wedge X)$ (by Lemma 4.3 (c)) and $ex(\mu)$ is a disjoint family (by Lemma 4.3 (b)). Hence the cover $ex(\beta \wedge X)$ of βX is inessential. It is a contradiction.

(c) Consider a shrinking ν of α. Observe that $\nu \wedge X$ is a finite cover of X such that $ex(\nu \wedge X) \succ \alpha$ (by Lemma 4.3 (d)) and hence $ex(\nu \wedge X)$ is essential. By the point (b) we have that the cover $\nu \wedge X$ of X is essential. By Lemma 3.3 there exists a sequence $\{\beta_i\}_{i=1}^{\infty}$ of distinct elements from $K(X) \wedge M_{K(X)}$ such that $\nu \wedge X \prec \beta_1 \prec \beta_2$ ... By Lemma 4.3 (c) we have that $ex(\nu \wedge X) \prec ex(\beta_1) \prec ex(\beta_2)$ ...

Put $K_{eX} = \{ex(\alpha) \in K(\beta X) : \alpha \in K(X)\} \subset K(\beta X)$.

Lemma 4.5. Let $\dim X > 0$. Then
(a) $\{\alpha_i\}_{i=1}^{k} \in M_{K(X)}$ iff $\{ex\alpha_i\}_{i=1}^{k} \in M_{K_{eX}}$.
In particular, we have $\mathrm{Ord}M_{K(X)} = \mathrm{Ord}M_{K_{eX}}$.
(b) $\mathrm{Ord}M_{K_{eX}} = \mathrm{Ord}M_{K(\beta X)}$.

Proof. (a) The necessity was proved in Lemma 4.1. Let $\sigma = \{ex(\alpha_i)\}_{i=1}^{k} \in M_{K_{eX}}$. Note that $ex^{-1}\sigma = \{\alpha_i\}_{i=1}^{k}$. Let $ex^{-1}\sigma \notin M_{K(X)}$. Then there exist finite open disjoint families $\beta_i, i = 1, ..., k$, in X such that $X = \cup_{i=1}^{n}\beta_i$ and $\beta_i \succ \alpha_i$. Note that $ex(\beta_i)$ is an open disjoint family in βX (by Lemma 4.3 (b)), $i = 1, ..., k$, such that $\beta X = \cup_{i=1}^{n} ex(\beta_i)$ and $ex(\beta_i) \succ ex(\alpha_i)$ (by Lemma 4.3 (c)), , $i = 1, ..., k$. Hence $\sigma \notin M_{K_{eX}}$. It is a contradiction. Hence $ex^{-1}\sigma \in M_{eX}$. In order to prove the equality consider the mappings ex and ex^{-1} and apply Lemma 2.2.
(b) By Lemma 4.4 (c) the set K_{eX} is a countably inscribed system in $K(\beta X)$. So by Theorem 3.10 $\mathrm{Ord}M_{K_{eX}} = \mathrm{Ord}M_{K(\beta X)}$.

Theorem 4.6. (a) $\dim_{\mathrm{C}} X = \dim_{\mathrm{C}} \beta X$.
(b) X is a finite C-space iff βX is a C-space.

Proof. (a) If $\dim X = 0$ then the theorem is evident. If $\dim X > 0$ then apply Lemma 4.5.
(b) Observe that X is a finite C-space iff (by Theorem 3.8) $\mathrm{Ord}M_{K(X)} \neq \infty$ iff (by the point (a)) $\mathrm{Ord}M_{K(\beta X)} \neq \infty$ iff (by Theorem 3.8) βX is a finite C-space iff (it is evident) βX is a C-space.

Observe that the result from Theorem 4.6 (b) was recently generalized in [9] to Tychonoff spaces.

5 Factorization theorem for $\dim_{\mathrm{C}}$ Here we will use some ideas from [1] (see the proof of the factorization theorem for the transfinite dimension trInd, p. 554). In this section by covers of a space X we will mean only elements from $K(X)$.

Let X be a compact space.

Lemma 5.1 (cf. [6], p. 188). Let $\alpha \in K(X)$. Then there exist a metrizable compactum $N(\alpha)$, a cover $\beta \in K(N(\alpha))$ and a continuous onto mapping $f : X \to N(\alpha)$ such that $f^{-1}(\beta) \succ \alpha$.

Let $f : X \to Z$ be a continuous mapping from X onto a compactum Z, $L \subset K(Z)$ and $|L| \leq wZ$. Put $B = B(f : X \to Z, L) = \{\sigma \in M_L : f^{-1}\sigma$ is inessential $\}$. Let $Y = Y(f : X \to Z, L)$ be a compact space and $g = g(f : X \to Z, L) : X \to Y, h = h(f : X \to Z, L) : Y \to Z$ be continuous onto mappings such that
(1) $f = hg, wY \leq wZ$, and
(2) for every $\sigma \in B$ we have $h^{-1}\sigma$ is inessential.

Lemma 5.2. Compact space $Y = Y(f : X \to Z, L)$ and mappings $g = g(f : X \to Z, L), h = h(f : X \to Z, L)$ exist.

Proof. If $B = \emptyset$ then put $Y = Z, g = f$ and $h = id_Z$.
Let $B \neq \emptyset$ and $\sigma = \{\alpha_i\}_{i=1}^n \in B$. Then $f^{-1}\sigma \notin M_{K(X)}$. Consequently there exist finite disjoint open systems $\beta_i, i = 1, ..., n$, in X such that $\beta_i \succ f^{-1}\alpha_i$ for every i and $\cup_{i=1}^n \beta_i \in K(X)$.
For the cover $\beta_\sigma = \cup_{i=1}^n \beta_i$ of X by Lemma 5.1 there are a metrizable compactum $N(\beta_\sigma)$, a continuous onto mapping $g_\sigma : X \mapsto N(\beta_\sigma)$ and $\gamma \in K(N(\beta_\sigma))$ such that $g_\sigma^{-1}\gamma \succ \beta_\sigma$. It is clear that the cover γ can be chosen so that $\gamma = \cup_{i=1}^n \gamma_i$, where γ_i are finite disjoint open systems in $N(\beta_\sigma)$ such that $g_\sigma^{-1}\gamma_i \succ \beta_i \succ f^{-1}\alpha_i$ for every i.
Consider the diagonal product $g : X \to Z \times \sqcap\{N(\beta_\sigma) : \sigma \in B\}$ of mappings f and $g_\sigma,\ \sigma \in B$. Put $Y = gX$ and $h : Y \to Z,\ p_\sigma : Y \to N(\beta_\sigma), \sigma \in B$, are projections. It is evident that $g_\sigma = p_\sigma g,\ \sigma \in B,\ f = hg,\ wY \leq wZ$.
Consider an element $\sigma = \{\alpha_i\}_{i=1}^k \in B$. As it was already proved there exists a cover γ of $N(\beta_\sigma)$ such that
(1) $\gamma = \cup_{i=1}^n \gamma_i$, where γ_i are finite disjoint open systems in $N(\beta_\sigma)$;
(2) $g^{-1}(p_\sigma^{-1}\gamma_i) = g_\sigma^{-1}\gamma_i \succ f^{-1}\alpha_i = g^{-1}(h^{-1}\alpha_i), i = 1, ..., n$.
So we get $p_\sigma^{-1}\gamma_i \succ h^{-1}\alpha_i, i = 1, ..., n$. Note that $p_\sigma^{-1}\gamma_i, i = 1, ..., n$, are finite disjoint open systems in Y and $Y = \cup_{i=1}^n p_\sigma^{-1}\gamma_i$. Consequently $h^{-1}\sigma$ is inessential.

Theorem 5.3. Let $f : X \to Z$ be a continuous mapping from a compact C-space X onto a compact space Z. Then there exist a compact C-space Y and continuous mappings $g : X \to Y$ and $h : Y \to Z$ such that $\dim_C Y \leq \dim_C X, wY \leq wZ$ and $f = hg$.

Proof. If the compact space X is finite-dimensional in the sense of dim then the statement is valid due to Mardesic's theorem (cf. [1], p. 300) and Theorem 3.5. If $\dim X = \infty$ & $\dim Z < \infty$ then put $Y = Z,\ g = f,\ h = id_Z$. Observe that the inequality $\dim_C X \geq \dim_C Y = \dim Y$ holds in this case.
Let $\dim X = \infty$ & $\dim Z = \infty$. By Proposition 3.11 there exists a countably inscribed system S in $K(Z)$ such that $|S| \leq wZ$.
If $B(f : X \to Z, S) = \emptyset$ then define a mapping $F : S \to K(X)$ as follows: $F(\alpha) = f^{-1}\alpha$ for any $\alpha \in S$. Observe that by our assumption for any $\sigma \in M_S$ $F(\sigma) = f^{-1}(\sigma) \in M_{K(X)}$ and $|F(\sigma)| = |\sigma|$. So by Lemma 2.2 we have $\mathrm{Ord} M_{K(X)} \geq \mathrm{Ord} M_S$. Put $Y = Z,\ g = f,\ h = id_Z$ and recall that by Theorem 3.10 $\mathrm{Ord} M_S = \dim_C Z$.
If $B(f : X \to Z, S) \neq \emptyset$ then put $Y_0 = Z,\ g_0 = f,\ S_0 = S$. By Lemma 5.2 there exist the compactum $Y_1 = Y(g_0 : X \to Y_0, S_0)$ and the mappings $g_1 = g(g_0 : X \to Y_0, S_0) : X \to Y_1$, $h_0^1 = h(g_0 : X \to Y_0, S_0) : Y_1 \to Y_0$. We can assume that $\dim Y_1 = \infty$ (otherwise put $Y = Y_1, g = g_1, h = h_0^1$).
By Proposition 3.10 we can find a countably inscribed system S_1 in $K(Y_1)$ such that $|S_1| \leq wZ$. We can also assume that for every $\alpha \in S_0$ $(h_0^1)^{-1}\alpha \in S_1$.
By induction for every $i = 2, 3, ...$ we can construct compacta $Y_i, wY_i \leq wZ$, countably inscribed systems S_i in $K(Y_i)$ and mappings $g_i : X \to Y_i, h_{i-1}^i : Y_i \to Y_{i-1}$ such that
(1) $Y_i = Y(g_{i-1} : X \to Y_{i-1}, S_{i-1}),\ g_i = g(g_{i-1} : X \to Y_{i-1}, S_{i-1}),\ h_{i-1}^i = h(g_{i-1} : X \to Y_{i-1}, S_{i-1}) : Y_i \to Y_{i-1}$;
(2) $|S_i| \leq wZ$ and for every $\alpha \in S_{i-1}$ $(h_{i-1}^i)^{-1}\alpha \in S_i$;
(3) $\dim Y_i = \infty$ for every i.

Put Y is the limit of the inverse system $W = \{Y_i, h_{i-1}^i\}, i = 0, 1, ...$
Let $h_i : Y \to Y_i, i = 0, 1, ...$ be the projections and $g : X \to Y$ be the natural mapping from X to Y such that $g_i = h_i g$. It is clear that Y is a compactum with $wY \leq wZ$. Put

$h = h_0 : Y \to Z = Y_0$. We get $f = hg$. If the compact space Y is finite-dimensional then it is nothing to prove more (in this case $\dim_C X \geq \dim_C Y = \dim Y$).
Let $\dim Y = \infty$. Put $S = \cup_{i=1}^{\infty} h_i^{-1} S_i \subset K(Y)$. Observe that the system S is a countably inscribed system in $K(Y)$. Really, let $\alpha \in K(Y) \wedge M_{K(Y)}$. Then there are an integer m and $\gamma \in K(Y_m)$ such that $h_m^{-1}\gamma \succ \alpha$. Note that $\gamma \in M_{K(Y_m)}$. By Definition 3.9 there exists a sequence $\{\beta_i\}_{i=1}^{\infty}$ of distinct elements from S_m such that $\gamma \prec \beta_i$ for every i. Note that the sequence $\{h_m^{-1}\beta_i\}_{i=1}^{\infty}$ consists of distinct elements from S such that $\alpha \prec h_m^{-1}\beta_i$ for every i. So by Definition 3.9 S is a countably inscribed system in $K(Y)$. Observe that by Theorem 3.10 $\mathrm{Ord} M_S = \dim_C Y$.
Now we will prove that $\mathrm{Ord} M_S \leq \mathrm{Ord} M_{K(X)}$. Define a mapping $F : S \to K(X)$ such that $F(\alpha) = g^{-1}\alpha$ for any $\alpha \in S$. Let $\sigma = \{\alpha_i\}_{i=1}^{n} \in M_S$. Then there exists an integer m such that $\alpha_i = h_m^{-1}\xi_i$, where $\xi_i \in S_m, i = 1, ..., n$. Put $\xi = \{\xi_i\}_{i=1}^{n}$. Suppose $F(\sigma) = g^{-1}(\sigma) = g_m^{-1}(\xi) \notin M_{K(X)}$. By the construction we have $(h_m^{m+1})^{-1}(\xi) \notin M_{K(Y_{m+1})}$. Thus $\sigma = h_m^{-1}\xi = h_{m+1}^{-1}(h_m^{m+1})^{-1}(\xi) \notin M_S$. It is a contradiction. Consequently $F(\sigma) \in M_{K(X)}$. It is clear $|F(\sigma)| = |\sigma|$. By Lemma 2.2 we have $\mathrm{Ord} M_S \leq \mathrm{Ord} M_{K(X)}$. So $\dim_C Y \leq \dim_C X$.

Let us note that a collective (for countably many closed subsets) factorization theorem for C-compacta without any mention of $\dim_C$ has recently been proved in [8].

Theorem 5.4. Let X be a finite C-space. Then there is a compact C-space Y such that $[X]_Y = Y, wY = wX$ and $\dim_C Y \leq \dim_C X$.

Proof. Let $f : X \to I^\tau$ be a homeomorphic embedding, where $\tau = wX$. There exists the extension $\beta f : \beta X \to I^\tau$ of f. By Theorem 5.3 there are a compact space Y_1 and mappings $g : \beta X \to Y_1$ and $h : Y_1 \to I^\tau$ such that $\dim_C Y_1 \leq \dim_C \beta X = \dim_C X$ (by Theorem 4.6), $wY_1 \leq wI^\tau = \tau$ and $f = hg$.
The mapping $g|_X$ is a homeomorphic embedding. So we can put $X = gX$. The compactum $[X]_{Y_1}$ is the required compactification Y of the space X because by Theorem 4.2 we have $\dim_C Y = \dim_C [X]_{Y_1} \leq \dim_C Y_1 \leq \dim_C X$ and $wY = wX$.

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