SOME CLASSIFICATIONS OF HYPERK-ALGEBRAS OF ORDER 3

M.M.Zahedi, R.A. Borzoei, H. Rezaei

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Abstract. In this paper first we give some definitions and examples on hyper$K$-algebras. Then we give some theorems and obtain some results which are needed to state and prove the main theorems of this manuscript (Theorems 3.16 and 3.21). In these theorems we give some classifications of hyper$K$-algebras of order 3 which satisfy the normal condition or simple condition. Finally we give two open problems.

1. Introduction The hyper algebraic structure theory was introduced by F. Marty in 1934 [8]. Since then many researchers have worked on this area. Imai and Iseki in 1966 [4] introduced the notion of a BCK-algebra. Recently [1,7,10] Borzoei, Jun and Zahedi et al applied the hyper structures to BCK-algebras and introduced the concept of hyper$K$-algebra which is a generalization of BCK-algebra. Now we follow [1,2,10] and give some classifications of hyper$K$-algebras of order 3 which satisfy the normal condition or the simple condition.

2. Preliminaries

Definition 2.1.[1]. Let H be a nonempty set and "$\circ$" be a hyper operation on H, that is $\circ$ is a function from $H \times H$ to $P^*(H) = P(H) - \{\emptyset\}$. Then H is called a hyper$K$-algebra if it contains a constant "0" and satisfies the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$
(HK3) $x < x$
(HK4) $x < y, y < x \implies x = y$
(HK5) $0 < x$,

for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$, $\exists b \in B$ such that $a < b$.

Note that if $A, B \subseteq H$, then by $A \circ B$ we mean the subset $\bigcup\limits_{a \in A, b \in B} a \circ b$ of H.

Example 2.2. (i) Define the hyper operation "$\circ$" on $H = [0, +\infty)$ as follows:

$$xoy = \begin{cases} [0, x] & \text{if } x \le y \\ (0, y] & \text{if } x > y \neq 0 \\ \{x\} & \text{if } y = 0 \end{cases}$$

for all $x, y \in H$. Then $(H, \circ, 0)$ is a hyper$K$-algebra.

(ii) Let $H = \{0, 1, 2\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{2\}$	$\{0,1,2\}$

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(iii) Let $H = \{0, 1, 2\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0, 2\}$
1	$\{1\}$	$\{0, 1\}$	$\{1, 2\}$
2	$\{2\}$	$\{0, 2\}$	$\{0, 2\}$

Theorem 2.3[1]. Let $(H, \circ, 0)$ be a hyper$K$-algebra. Then for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H the following hold:

(i) $(A \circ B) \circ C = (A \circ C) \circ B$,
(ii) $x \circ y < z \Leftrightarrow x \circ z < y$,
(iii) $A \circ B < C \Leftrightarrow A \circ C < B$,
(iv) $(x \circ z) \circ (x \circ y) < y \circ z$,
(v) $(A \circ C) \circ (B \circ C) < A \circ B$,
(vi) $A \subseteq B$ implies $A < B$,
(vii) $x \circ y < x$,
(viii) $A \circ B < A$,
(ix) $x \in x \circ 0$,
(x) $A \subseteq A \circ 0$

Definition 2.4[1]. Let I be a nonempty subset of a hyper$K$-algebra $(H, \circ)$. Then I is called a *weak hyperK-ideal* of H if
(WHKI1) $0 \in I$
(WHKI2) $x \circ y \subseteq I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

Definition 2.5[1]. Let I be a nonempty subset of a hyper$K$-algebra $(H, \circ)$. Then I is said to be a *hyperK-ideal* of H if
(HKI1) $0 \in I$,
(HKI2) $x \circ y < I$ and $y \in I$ imply that $x \in I$, for all $x, y \in H$.

Definition 2.6[2]. Let H be a hyper$K$-algebra. An element $a \in H$ is called to be a *left(resp. right) scalar* if $|a \circ x| = 1$(resp. $|x \circ a| = 1$) for all $x \in H$. If $a \in H$ is both left and right scalar,we say that a is an *scalar element*.

Definition 2.7[2]. Let H be a hyper$K$-algebra and I be a nonempty subset of H such that $0 \in I$. Then I is said to be a *positive implicative hyperK-ideal* of

(i) type 1, if for all $x, y, z \in H$, $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ implies that $x \circ z \subseteq I$
(ii) type 2, if for all $x, y, z \in H$, $(x \circ y) \circ z < I$ and $y \circ z \subseteq I$ implies that $x \circ z \subseteq I$.
(iii) type 3, if for all $x, y, z \in H$, $(x \circ y) \circ z < I$ and $y \circ z < I$ implies that $x \circ z \subseteq I$.
(iv) type 4, if for all $x, y, z \in H$, $(x \circ y) \circ z < I$ and $y \circ z < I$ implies that $x \circ z < I$.

Definition 2.8[2]. The hyper$K$-algebra H is called to be a *positive implicative hyperK-algebra*, if it satisfies the following condition,

$$(x \circ z) \circ (y \circ z) = (x \circ y) \circ z$$

for all $x, y, z \in H$.

Theorem 2.9[2]. Let H be a positive implicative hyper$K$-algebra. Then any weak hyper$K$-ideal of H is a positive implicative hyper$K$-ideal of type 1.

Theorem 2.10[2]. Let $0 \in H$ be a right scalar element. If I is a positive implicative hyper$K$-ideal of type 2 (type 3), then I is a hyper$K$-ideal of H.

Definition 2.11[2]. Let I be a nonempty subset of H. We say that I satisfies the *additive condition*, if $x < y$ and $y \in I$ implies that $x \in I$, for all $x, y \in H$.

Theorem 2.12[2]. Let I be positive implicative hyper$K$-ideal of type 4 such that satisfies the additive condition. Then I is a hyper$K$-ideal of H.

Definition 2.13[10]. Let $(H_1, \circ_1, 0_1)$ and $(H_2, \circ_2, 0_2)$ be two hyper$K$-algebras and $f : H_1 \longrightarrow H_2$ be a function. Then f is said to be a homomorphism iff

(i) $f(0_1) = 0_2$

(ii) $f(x \circ_1 y) = f(x) \circ_2 f(y), \quad \forall x, y \in H_1$.

If f is 1-1 (or onto) we say that f is a monomorphism (or epimorphism). And if f is both 1-1 and onto, we say that f is an isomorphism.

3. Main results

Note: Throughout this paper we let always H be a hyper$K$-algebra of order 3 and usually we use the set $\{0, 1, 2\}$ for showing the elements of H.

Definition 3.1. We say that H satisfies the *normal condition* if one of the conditions $1 < 2$ or $2 < 1$ holds. If no one of these conditions hold, then we say that H satisfies the *simple condition*.

Note: Clearly the conditions $1 < 2$ and $2 < 1$ can not hold simultaneously, because $1 < 2$ and $2 < 1$ imply that $1 = 2$ which is impossible. Clearly H always satisfies the normal condition or simple condition.

Example 3.2.(i) The hyper$K$-algebras which have been defined in Example 2.2(ii) and (iii) satisfy the normal condition.

(ii) Let $(H, \circ_1)$ and $(H, \circ_2)$ are defined as follows:

$\circ_1$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0\}$

$\circ_2$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1\}$	$\{1,2\}$
2	$\{2\}$	$\{1,2\}$	$\{0,1\}$

Then $(H, \circ_1)$ and $(H, \circ_2)$ are hyper$K$-algebras and they satisfy the simple condition.

Definition 3.3. A nonempty subset I of H is called *proper*, if $I \neq \{0\}$ and $I \neq H$.

Theorem 3.4. Let H satisfies the normal condition. Then there is at most one proper hyper$K$-ideal of H .

Proof. The only possible cases are $I_1 = \{0, 1\}$ and $I_2 = \{0, 2\}$. We show that at least one of I_1 or I_2 is not a hyper$K$-ideal of H . Since H satisfies the normal condition,then $1 < 2$ or $2 < 1$. If $1 < 2$, then I_2 is not a hyper$K$-ideal of H . Since, otherwise $1 \circ 2 < I_2$ and $2 \in I_2$ implies that $1 \in I_2$ which is not true. Also if $2 < 1$, then by a similar way we show that I_1 is not a hyper$K$-ideal of H.

Example 3.5.(i) Consider Example 2.2(ii). Then H satisfies the normal condition and $I_1 = \{0, 1\}$ is a hyper$K$-ideal of H. Where $I_2 = \{0, 2\}$ is not a hyper$K$-ideal of H .

(ii) Consider Example 2.2(iii). Then H satisfies the normal condition . And both of $I_1 = \{0, 1\}$, $I_2 = \{0, 2\}$ are not hyper$K$-ideals of H.

(iii) Consider Example 3.2(ii). Then $(H, \circ_1)$ satisfies the simple condition. Thus H does not satisfy the normal condition . Moreover both of $I_1 = \{0, 1\}$ and $I_2 = \{0, 2\}$ are hyper$K$-ideals of H. Therefore the normal condition in Theorem 3.4 is necessary.

Theorem 3.6. Let H satisfies the normal condition. Then there is at least one proper positive implicative hyper$K$-ideal of type 4.

Proof. Since H satisfies the normal condition, then $1 < 2$ or $2 < 1$. If $1 < 2$, then for all $x, z \in H, x \circ z < I = \{0, 2\}$, since $0 \in 0 \circ x, 0 \in x \circ x$ for all $x \in H$, $1 \in 1 \circ 0, 2 \in 2 \circ 0, 0 \in 1 \circ 2$ and 1 or $2 \in 2 \circ 1$. Therefore $I = \{0, 2\}$ is a positive implicative hyper$K$-ideal of type 4. Moreover if $2 < 1$, similarly we conclude that $\{0, 1\}$ is a positive implicative hyper$K$-ideal of type 4.

Example 3.7. (i) Consider Example 2.2(ii). Then $I_1 = \{0, 1\}$ is a proper positive implicative hyper$K$-ideal of type 4, but $I_2 = \{0, 2\}$ is not.

(ii) Consider Example 2.2(iii). Then both of $I_1 = \{0,1\}$ and $I_2 = \{0,2\}$ are proper positive implicative hyper$K$-ideal of type 4.

(iii) Consider Example 3.2(ii). Then $(H, \circ_2)$ satisfies the simple condition and has not proper positive implicative hyper$K$-ideal of type 4. Therefore the normal condition in Theorem 3.6 is necessary.

Theorem 3.8. Let H satisfies the normal condition and I be a proper subset of H. Then I is a hyper$K$-ideal of H if and only if I is a positive implicative hyper$K$-ideal of type 4 and satisfies the additive condition.

Proof. ($\Rightarrow$) We know that the only possible cases for having proper hyper$K$-ideals are $I_1 = \{0,1\}$ and $I_2 = \{0,2\}$. Let $I_1 = \{0,1\}$ be a hyper$K$-ideal of H . We claim that $2 \circ 1 \not< I_1$ and $2 \circ 0 \not< I_1$. Let $2 \circ 1 < I_1$ or $2 \circ 0 < I_1$. Since $1, 0 \in I_1$ and I_1 is a hyper$K$-ideal of H then $2 \in I$ which is not true. Therefore we have $2 \circ 1 = \{2\}$ and $2 \circ 0 = \{2\}$. Now we show that I_1 is a positive implicative hyper$K$-ideal of type 4. On the contrapositive, we show that if $x \circ z \not< I_1$ then for all $y \in H$, $(x \circ y) \circ z \not< I_1$ or $y \circ z \not< I_1$. Since $2 \circ 1 = \{2\}$ and H satisfies the normal condition, then $2 \not< 1$ and so $1 < 2$. Hence $0 \in 1 \circ 2$ and therefore $1 \circ 2 < I_1$. Since $0 \in 0 \circ x$ and $0 \in x \circ x$ for all $x \in H$ and $1 \in 1 \circ 0$, thus the only cases which $x \circ z \not< I_1$ are $2 \circ 1 = \{2\}$ and $2 \circ 0 = \{2\}$. In the first case we let $x = 2$ and $z = 1$. We show that $y \circ 1 \not< I_1$ or $(2 \circ y) \circ 1 \not< I_1$, for all $y \in H$.
If $y = 0$, then

$$(2 \circ y) \circ 1 = (2 \circ 0) \circ 1 = \{2\} \circ 1 = \{2\} \not< I_1$$

If $y = 1$, then

$$(2 \circ y) \circ 1 = (2 \circ 1) \circ 1 = \{2\} \circ 1 = \{2\} \not< I_1$$

If $y = 2$, then

$$y \circ 1 = 2 \circ 1 = \{2\} \not< I_1$$

The proof of the second case is similar. Therefore we prove that $I_1 = \{0,1\}$ is a positive implicative hyper$K$-ideal of type 4. Moreover since $2 \not< 1$ and $2 \not< 0$ we can conclude that I_1 satisfies the additive condition. By a similar way we can show that $I_2 = \{0,2\}$ is a positive implicative hyper$K$-ideal of type 4 and satisfies the additive condition.

($\Leftarrow$) Let I be a proper positive implicative hyper$K$-ideal of type 4 and satisfies the additive condition. Then by Theorem 2.12, I is a hyper$K$-ideal of H .

Example 3.9. Consider Example 2.2(iii). Then $I = \{0\}$ is a hyper$K$-ideal of H , which is not a positive implicative hyper$K$-ideal ,while it is additive. Since $(1 \circ 2) \circ 2 = \{0,1,2\} < \{0\}$ and $2 \circ 2 = \{0,1,2\} < \{0\}$ but $1 \circ 2 = \{1,2\} \not< \{0\}$. Thus in Theorem 3.8, I must be proper and we can not omit this condition.

Theorem 3.10. Let H satisfies the normal condition. If $I_1 = \{0,1\}$ ($I_2 = \{0,2\}$) is a hyper$K$-ideal of H , then $I_2 = \{0,2\}$ ($I_1 = \{0,1\}$) is a positive implicative hyper$K$-ideal of type 4.

Proof. Let $I_1 = \{0,1\}$ ($I_2 = \{0,2\}$) is a hyper$K$-ideal of H . Then we have $2 \circ 1 = \{2\} \not< I_1 (1 \circ 2 = \{1\} \not< I_2)$. Thus $2 \not< 1 (1 \not< 2)$, and since H satisfies the normal condition , we get that $1 < 2 (2 < 1)$. Hence by the proof of Theorem 3.6, $I_2(I_1)$ is a positive implicative hyper$K$-ideal of type 4.

Theorem 3.11. Let H satisfies the normal condition, $I_1 = \{0,1\}$ and $I_2 = \{0,2\}$. Then I_1 or I_2 is a hyper$K$ ideal of H if and only if I_1 and I_2 both are positive implicative hyper$K$-ideals of type 4.

Proof. ($\Rightarrow$) Let $I_1(I_2)$ be a hyper$K$-ideal of H . Then by Theorems 3.8 and 3.10, $I_1(I_2)$ and $I_2(I_1)$ both are positive implicative hyper$K$-ideals of type 4.

($\Leftarrow$) Let I_1 and I_2 both are positive implicative hyper$K$-ideals of type 4. Since H satisfies the normal condition , then $1 \not< 2$ or $2 \not< 1$. If $1 \not< 2$, then we can get that I_2 satisfies the additive condition and so by Theorem 3.8, I_2 is a hyper$K$-ideal of H . If $2 \not< 1$, then I_1 satisfies the normal condition and so by Theorem 3.8, I_1 is a hyper$K$-ideal of H .

Example 3.12. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1,2\}$
2	$\{2\}$	$\{1\}$	$\{0,1,2\}$

We can show that $I_2 = \{0, 2\}$ is a positive implicative hyper$K$-ideal of type 4 but $I_1 = \{0, 1\}$ is not. Moreover H has not any proper hyper$K$-ideal.

Theorem 3.13. Let H satisfies the normal condition and H has a proper positive implicative hyper$K$-ideal I of type 3. Then

(i) $1 \circ 0 = \{1\}, 2 \circ 0 = \{2\}$

(ii) $x \circ y \neq \{0, 1, 2\}$, for all $x, y \in H$.

(iii) If $I = \{0, 1\}(I = \{0, 2\})$, then $2 \circ 1 = \{2\}(1 \circ 2 = \{1\})$ and $x \circ y \neq \{0, 2\}(x \circ y \neq \{0, 1\})$, for all $x, y \in H$.

Proof.(i) Let $1 \circ 0 \neq \{1\}$. Since $1 \in 1 \circ 0$ and $0 \notin 1 \circ 0$, thus $1 \circ 0 = \{1, 2\}$. Hence $\{1, 2\} = 1 \circ 0 \subseteq (1 \circ 0) \circ 0$, by Theorem 2.3(x). Thus $(1 \circ 0) \circ 0 < I$ and $0 \circ 0 < I$, but $1 \circ 0 = \{1, 2\} \nsubseteq I$ which is a contradiction. Therefore, we conclude that $1 \circ 0 = \{1\}$. By a similar way, we can show that $2 \circ 0 = \{2\}$.

(ii) If there exists $x, y \in H$ such that $x \circ y = \{0, 1, 2\}$, then by Theorem 2.3(ix), $\{0, 1, 2\} = x \circ y \subseteq (x \circ 0) \circ y < I$ and $0 \circ y < I$. Thus $\{0, 1, 2\} = x \circ y \subseteq I$, which is a contradiction.

(iii) Let $I = \{0, 1\}$. If $2 \circ 1$ is equal to one of the subsets $\{0\}, \{1\}, \{0, 1\}, \{0, 2\}$ and $\{0, 1, 2\}$ of H, then $(2 \circ 1) \circ 0 < I$ and $1 \circ 0 < I$. So $\{2\} = 2 \circ 0 \subseteq I$, which is a contradiction. Therefore we have $2 \circ 1 = \{2\}$. Moreover, let there are $x, y \in H$ such that $x \circ y = \{0, 2\}$. Then $\{0, 2\} = x \circ y \subseteq (x \circ 0) \circ y < I, 0 \circ y < I$ and $\{0, 2\} = x \circ y \nsubseteq I = \{0, 1\}$. Thus for all $x, y \in H, x \circ y \neq \{0, 2\}$. The proof of the case $I = \{0, 2\}$, is similar.

Example 3.14. Let $H = \{0, 1, 2\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

We can show that $I_2 = \{0, 2\}$ is a positive implicative hyper$K$-ideal of type 2, but H has not any proper positive implicative hyper$K$-ideal of type 3. Moreover there is $x, y \in H$, such that $x \circ y = \{0, 1, 2\}$.

Corollary 3.15. There is not a hyper$K$-algebra of order 3 which satisfies the normal condition and both of $I_1 = \{0, 1\}$ and $I_2 = \{0, 2\}$ be positive implicative hyper$K$-ideals of type 3.

Proof. On the contrary let there is a hyper$K$-algebra H of order 3 such that both of I_1 and I_2 be positive implicative hyper$K$-ideals of type 3. Then by Theorem 3.13(ii), (iii) we have $2 \circ 1 = \{2\}$ and $1 \circ 2 = \{1\}$. So $2 \not< 1$ and $1 \not< 2$ which means that H does not satisfy the normal condition.

Theorem 3.16. There are 17 non-isomorphism hyper$K$- algebras of order 3 which satisfy the normal condition and each of them has only one proper positive implicative hyper$K$-ideal of type 3.

Proof. Let $H = \{0, 1, 2\}$ and the following table shows a probable hyper$K$-algebra structure of H.

H	0	1	2
0	a_{11}	a_{12}	a_{13}
1	a_{21}	a_{22}	a_{23}
2	a_{31}	a_{32}	a_{33}

Let $I = \{0,1\}$ be a proper positive implicative hyper$K$-ideal of type 3. By Theorem 3.13,

$$a_{21} = 1 \circ 0 = \{1\}\ ,\ a_{31} = 2 \circ 0 = \{2\}\ ,\ a_{32} = 2 \circ 1 = \{2\}$$

$$\{0,2\} \neq a_{ij} \neq \{0,1,2\} \quad , \quad \forall\ \ 1 \leq i,j \leq 3$$

Also, by (HK3) and (HK5) we have

$$0 \in a_{11} \cap a_{12} \cap a_{13} \cap a_{22} \cap a_{33}$$

Since H satisfies the normal condition and $2 \circ 1 = \{2\}$, then $2 \not< 1$ and so $1 < 2$. This means that $0 \in 1 \circ 2 = a_{23}$. Therefore the only cases for $a_{11}, a_{12}, a_{13}, a_{22}, a_{23}$ and a_{33} are $\{0\}$ and $\{0,1\}$. That is there are $2^6 = 64$ cases for H. By some manipulation we can conclude that the following 17 cases are the requested hyper$K$-algebras. In fact each of the other cases (i.e. 47 cases) is not a hyper$K$-algebra, since does not satisfy the condition (HK2).

H_1	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0\}$

H_2	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0\}$

H_3	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0,1\}$
2	$\{2\}$	$\{2\}$	$\{0\}$

H_4	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{2\}$	$\{0\}$

H_5	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

H_6	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

H_7	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{2\}$	$\{0\}$

H_8	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

H_9	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

H_{10}	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

H_{11}	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0\}$	$\{0,1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

H_{12}	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

H_{13}	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

H_{14}	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

H_{15}	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

H_{16}	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

H_{17}	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

Now we show that the above 17 hyper$K$-algebras are determined uniquely by isomorphism. Clearly H_1 is not isomorphic to any $H_i, 2 \leq i \leq 17$ Because in $H_1, x \circ y$ is a singleton for all $x, y \in H_1$, while in each $H_i, 2 \leq i \leq 17$, there exist $x, y \in H_i$ such that $x \circ y = \{0,1\}$. Now we show that there is not an isomorphism between H_i and H_j for all $2 \leq i \neq j \leq 17$. Suppose $f : H_i \to H_j$ be an isomorphism. Since the hyper operation on H_i and H_j are different, so f is not identity. Therefore we must have:

$$f(0) = 0 \quad , \quad f(1) = 2 \quad , \quad f(2) = 1$$

Now, since there are $x, y \in H_i$ such that $x \circ y = \{0,1\}$, thus

$$f(x) \circ f(y) = f(x \circ y) = f(\{0,1\}) = \{0,2\}$$

It means that there are $t, z \in H_j$ such that $t \circ z = \{0,2\}$, which is impossible. Hence f is not an isomorphism. In other words each H_i is determined uniquely by isomorphism.

Note that we can find 17 hyper$K$-algebras of order 3 which satisfy the normal condition and each of them has only one proper positive implicative hyper$K$-ideal $\{0,2\}$ of type 3. But it is easy to check that each of these hyper$K$-algebras is isomorphic to one of the above mentioned hyper$K$-algebras.

Lemma 3.17. Let H satisfies the simple condition. Then

(i) $1 \circ 2 \neq \{2\}$ and $2 \circ 1 \neq \{1\}$.

(ii) $1 \circ 0 = \{1\}$ and $2 \circ 0 = \{2\}$.

Proof.(i) On the contrary let $1 \circ 2 = \{2\}$. By Theorem 2.3(vii), $1 \circ 2 < 1$ and this implies that $2 < 1$ which is impossible, since H satisfies the simple condition. Thus $1 \circ 2 \neq \{2\}$. Similarly, we can show that $2 \circ 1 \neq \{1\}$.

(ii) On the contrary let $1 \circ 0 \neq \{1\}$. Since $1 \in 1 \circ 0$ and $0 \notin 1 \circ 0$, then we must have $1 \circ 0 = \{1,2\}$. Since $2 \in 1 \circ 0$, then $1 \circ 0 < 2$. By Theorem 2.3(ii), $1 \circ 2 < 0$ and so there is $a \in 1 \circ 2$ such that $a < 0$. From (HK4) and (HK5) we get that $a = 0$ and so $0 \in 1 \circ 2$, that is , $1 < 2$ which is impossible. Therefore $1 \circ 0 = \{1\}$. Similarly, we can show that $2 \circ 0 = \{2\}$.

Theorem 3.18. Let H satisfy the simple condition. Then $I_1 = \{0,1\}(I_2 = \{0,2\})$ is a positive implicative hyper$K$-ideal of type 4 if and only if, $2 \circ 1 = \{2\}(1 \circ 2 = \{1\})$.

Proof.($\Rightarrow$) Let $I_1 = \{0,1\}$ be a positive implicative hyper$K$-ideal of type 4 and on the contrary let $2 \circ 1 \neq \{2\}$. Since H satisfies the simple condition, then $0 \notin 2 \circ 1$. Moreover, by Lemma 3.17, $2 \circ 1 \neq \{1\}$. Thus we have $2 \circ 1 = \{1,2\}$. So $(2 \circ 1) \circ 0 = \{1,2\} \circ 0 = \{1,2\} < I_1$ and $1 \circ 0 = \{1\} < I_1$. Since I_1 is of type 4 we conclude that $2 \circ 0 < I_1$. But by Lemma 3.17 we have $2 \circ 0 = \{2\} \not< I_1$, which is a contradiction. Hence $2 \circ 1 = \{2\}$.

For the case of, $I = \{0,2\}$, the proof is similar.

($\Leftarrow$) Let $2 \circ 1 = \{2\}(1 \circ 2 = \{1\})$. On the contrapositive we show that if there are $x, z \in H$ such that $x \circ z \not< I_1(I_2)$, then for all $y \in H, y \circ z \not< I_1(I_2)$ or $(x \circ y) \circ z \not< I_1(I_2)$. Since $x \circ z \not< I_1$, we conclude $x \circ z = \{2\}$. So by Lemma 3.17 we must have $2 \circ 1 = \{2\}(1 \circ 2 = \{1\})$ and $2 \circ 0 = \{2\}(1 \circ 0 = \{1\})$. Now if we do similar to the proof of Theorem 3.8, we can get that $I_1(I_2)$ is a positive implicative hyper$K$-ideal of type 4.

Theorem 3.19. Let H satisfies the simple condition and I be a proper positive implicative hyper$K$-ideal of type 3. Then

(i) $1 \circ 2 = \{1\}$ and $2 \circ 1 = \{2\}$.

(ii) $x \circ y \neq \{0,1,2\}$ for all $x, y \in H$.

(iii) If $I = \{0,1\}(I = \{0,2\})$, then $x \circ y \neq \{0,2\}(x \circ y \neq \{0,1\})$ for all $x, y \in H$.

Proof. (i) Let $1 \circ 2 \neq \{1\}$. By Lemma 3.17(i) and simple condition, we will have $1 \circ 2 = \{1,2\}$. Thus $(1 \circ 2) \circ 2 = \{1,2\} \circ 2 = \{0,1,2\} < I$ and $2 \circ 2 < I$, therefore $1 \circ 2 \subseteq I$. But $1 \circ 2 = \{1,2\} \nsubseteq I$, which is a contradiction. Hence $1 \circ 2 = \{1\}$. Similarly we can show that $2 \circ 1 = \{2\}$.

(ii) Let there are $x, y, \in H$ such that $x \circ y = \{0,1,2\}$. Then $(x \circ 0) \circ y = \{0,1,2\} < I$ and $0 \circ y < I$, thus $x \circ y \subseteq I$. But $x \circ y = \{0,1,2\} \nsubseteq I$, which is a contradiction. Thus $x \circ y \neq \{0,1,2\}$ for all $x, y \in H$.

(iii) Let $I = \{0,1\}$ and there are $x, y \in H$ such that $x \circ y = \{0,2\}$. Then $(x \circ 0) \circ y < I$ and $0 \circ y < I$, so $x \circ y \subseteq I$. But $x \circ y = \{0,2\} \nsubseteq I$, which is a contradiction. Therefore $x \circ y \neq \{0,2\}$ for all $x, y \in H$.

For the case $I = \{0,2\}$, the proof is similar.

Theorem 3.20. There is only one hyper$K$- algebra of order 3 which have two proper positive implicative hyper$K$-ideals of type 3.

Proof. Let $H' = \{0,1,2\}$ and the following table show a probable hyper$K$-algebra structure of H'.

H'	0	1	2
0	a_{11}	a_{12}	a_{13}
1	a_{21}	a_{22}	a_{23}
2	a_{31}	a_{32}	a_{33}

Let $I_1 = \{0,1\}$ and $I_2 = \{0,2\}$ are proper positive implicative hyper$K$-ideals of type 3. By Theorems 3.17 and 3.19,

$$a_{21} = 1 \circ 0 = \{1\} \ , \ a_{31} = 2 \circ 0 = \{2\} \ , \ a_{23} = 1 \circ 2 = \{1\} \ , \ a_{32} = 2 \circ 1 = \{2\}$$

$$a_{ij} \neq \{0,1,2\} \quad , \quad a_{ij} \neq \{0,1\} \quad , \quad a_{ij} \neq \{0,2\} \quad , \quad \forall \ \ 1 \leq i,j \leq 3$$

Since by (HK3) and (HK5)

$$0 \in a_{11} \cap a_{12} \cap a_{13} \cap a_{22} \cap a_{33}$$

Then we conclude that

$$a_{11} = a_{12} = a_{13} = a_{22} = a_{33} = \{0\}$$

Therefore the only hyper$K$-algebra which contains two proper positive implicative hyper$K$-ideals of type 3 is as follows:

H_1'	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0\}$

Theorem 3.21. There is 11 non-isomorphism hyper$K$- algebras of order 3 with simple condition, such that they have just one proper positive implicative hyper$K$-ideal of type 3.

Proof. Let $H' = \{0,1,2\}$ and suppose that the following table shows a probable hyper$K$-algebra structure of H'.

H'	0	1	2
0	a_{11}	a_{12}	a_{13}
1	a_{21}	a_{22}	a_{23}
2	a_{31}	a_{32}	a_{33}

Let $I = \{0, 1\}$ be a positive implicative hyper$K$-ideal of type 3. By Theorems 3.17 and 3.19,

$$a_{21} = 1 \circ 0 = \{1\}\ ,\ a_{31} = 2 \circ 0 = \{2\}\ ,\ a_{23} = 1 \circ 2 = \{1\}\ ,\ a_{32} = 2 \circ 1 = \{2\}$$

$$\{0, 2\} \neq a_{ij} \neq \{0, 1, 2\}\ ,\ \ \forall\ \ 1 \leq i, j \leq 3$$

By (HK3) and (HK5) we have

$$0 \in a_{11} \cap a_{12} \cap a_{13} \cap a_{22} \cap a_{33}$$

Then the only cases for $a_{11}, a_{12}, a_{13}, a_{22}$ and a_{33} are $\{0\}$ and $\{0, 1\}$. That is, there is $2^5 = 32$ cases for $H^{'}$. By some manipulation we show that 11 cases are hyper$K$-algebras. These hyper$K$-algebras are shown by $H_2^{'}, H_3^{'}, ..., H_{12}^{'}$. Each of the other cases (i.e 21 cases) is not a hyper$K$-algebra because they do not satisfy the condition (HK2).

$H_2^{'}$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0\}$

$H_3^{'}$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

$H_4^{'}$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0\}$

$H_5^{'}$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

$H_6^{'}$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

$H_7^{'}$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

$H_8^{'}$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

$H_9^{'}$	0	1	2
0	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

$H_{10}^{'}$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

$H_{11}^{'}$	0	1	2
0	$\{0\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

$H_{12}^{'}$	0	1	2
0	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

Now similar to the proof of Theorem 3.16 we can see that each of the above hyper$K$-algebras is determined by isomorphism.

Theorem 3.22. Let H satisfies the simple condition and I be a proper subset of H. Then I is a hyper$K$-ideal of H if and only if I is a positive implicative hyper$K$-ideal of type 4

Proof.($\Rightarrow$) Let I be a hyper$K$-ideal of H and $I = \{0, 1\}(I = \{0, 2\})$. Then we must have $2 \circ 1 = \{2\}(1 \circ 2 = \{1\})$, because in the other case we receive to a contradiction. So by Theorem 3.18, $I = \{0, 1\}(I = \{0, 2\})$ is a positive implicative hyper$K$-ideal of type 4.

($\Leftarrow$) Let I be a positive implicative hyper$K$-ideal of type 4. If $I = \{0, 1\}(I = \{0, 2\})$, then by Theorem 3.18 and Lemma 3.17, $2 \circ 1 = \{2\}(1 \circ 2 = \{1\})$ and $2 \circ 0 = \{2\}(1 \circ 0 = \{1\})$. But this implies that $I = \{0, 1\}(I = \{0, 2\})$ is a hyper$K$-ideal of H .

Corollary 3.23. Let $0 \in H$ be a right scalar element. If I is a proper positive implicative hyper$K$-ideal of type 2, then I is a positive implicative hyper$K$-ideal of type 4.

Proof. The proof follows from Theorems 2.10, 3.8 and 3.22.

Example 3.24. (i) The converse of Corollary 3.23 is not correct in general. To show this consider Example 2.2(iii). Then $I_1 = \{0,1\}$ is a positive implicative hyper$K$-ideal of type 4 but it is not of type 2, since $(2 \circ 0) \circ 0 = \{2\} < \{0,1\} = I$ and $0 \circ 0 = \{0\} \subseteq I$ but $2 \circ 0 = \{2\} \not\subseteq I$.

(ii) Let $H = \{0,1,2\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	1	2
0	$\{0,1,2\}$	$\{0,1,2\}$	$\{0,1,2\}$
1	$\{1\}$	$\{0,1,2\}$	$\{1,2\}$
2	$\{1,2\}$	$\{0,1\}$	$\{0,1,2\}$

Clearly that $I_2 = \{0,2\}$ is a positive implicative hyper$K$-ideal of type 2 but it is not of type 4, since $(1 \circ 2) \circ 0 = \{1,2\} < I_2$ and $2 \circ 0 = \{0,1\} < I_2$ but $1 \circ 0 = \{1\} \not< I_2$. Therefore this example show that the right scalar element in Corollary 3.16 is necessary.

Problem 1. Let H satisfies the simple condition. Are $I_1 = \{0,1\}$ and $I_2 = \{0,2\}$ are positive implicative hyper$K$-ideal of type 1?.

Problem 2. Give a characterization of hyper$K$-algebras of order 3 which have at least one proper positive implicative hyper$K$-ideals of type 2.

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Department of Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran, e-mail: zahedi@arg3.uk.ac.ir

Department of Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran, e-mail: borzoei@hamoon.usb.ac.ir

Department of Mathematics, Sistan and Baluchestan University, Zahedan, Iran, e-mail: hrezaei@hamoon.usb.ac.ir