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ON LOEB AND WEAKLY LOEB HAUSDORFF SPACES

KYRIAKOS KEREMEDIS AND ELEFTHERIOS TACHTSIS

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It is shown that the statements: *Compact T_2 spaces are weakly Loeb* and *the product of weakly Loeb T_2 spaces is weakly Loeb* are not provable in ZF^0, Zermelo-Fraenkel set theory without the axiom of regularity.

1. Introduction and definitions

Let (X,T) be a topological space. Then X is a *Loeb* (*weakly Loeb*) space if there is a choice (multiple choice) function on the family of its non-empty closed subsets. Examples of Loeb spaces are:

1. Compact linearly ordered spaces,
2. spaces with a finite number of open sets,
3. spaces (X,T) with X a well ordered set.

Remark. For any set X, the *Alexandroff one point compactification*, $X(a)$, of the discrete space X is an example of a weakly Loeb space. Indeed, for any non-empty closed set A of $X(a)$ either $a \in A$ and we may choose a from A, or A is a finite set and we may choose A itself.

P. Howard and J. Rubin ([5], p. 345) ask whether,

- **Form 115** : *The product of weakly Loeb* T_2 *spaces is weakly Loeb*

and,

- **Form 116** : *Compact* T_2 *spaces are weakly Loeb*

are provable in ZF^0. In view of the remark above, it follows that showing Form 116 is not provable in ZF^0 is not easy. Without AC there are not many compact T_2 topologies one can define on an infinite set X. The interested reader is invited to define such topologies other than $X(a)$.

The purpose of this paper is to show that neither Form 115 nor Form 116 are provable in ZF^0. Before we do this let us list some known results in this area and state the choice principles we are going to use in the sequel.

Theorem 1.1. [1] *In the basic Fraenkel permutation model (model $\mathcal{N}1$ in* [5]*) the following statements are valid.*
(i) Form 115.
(ii) Form 116.
(iii) A compact T_2 space (X,T) is weakly Loeb if and only if X is expressible as a well ordered union of compact sets.

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1. The *axiom of choice* **AC**, **Form 1 in** [5] : *For every family* $\mathcal{A} = \{A_i : i \in k\}$ *of disjoint non-empty sets there exists a set* $C = \{c_i : i \in k\}$ *such that* $c_i \in A_i$ *for all* $i \in k$.
2. The *axiom of multiple choice* **MC**, **Form 67 in** [5] : *For every family* $\mathcal{A} = \{A_i : i \in k\}$ *of disjoint non-empty sets there exists a set* $\mathcal{F} = \{F_i : i \in k\}$ *of finite non-empty sets such that* $F_i \subseteq A_i$ *for all* $i \in k$.
3. $\mathbf{AC_{fin}}$, **Form 62 in** [5] : AC *restricted to families of non-empty finite sets.*
4. **LN**, **Form 118 in** [5] : *Every linearly orderable space is normal.*

AC and MC are related to the notions of Loeb and weakly Loeb spaces respectively via the following

Theorem 1.2. *(i) AC if and only if every topological space is Loeb.*
(ii) MC if and only if every topological space is weakly Loeb.

Proof. (i). It suffices to show $(\leftarrow)$ as the other direction is evident. Let $\mathcal{A} = \{A_i : i \in k\}$ be a disjoint family of non-empty sets and let T be the discrete topology on $X = \bigcup \mathcal{A}$. By our hypothesis (X, T) is a Loeb space, and so the family $\mathcal{A}$ of closed subsets of X has a choice set.

(ii). This can be proved as in (i). ∎

Remark. U. Felgner and T. Jech in their joint paper [3] proved that in ZF (Zermelo-Fraenkel set theory minus AC), AC and MC are equivalent. Therefore, in conjuction with the latter theorem we deduce that in ZF the notions of Loeb and weakly Loeb spaces coincide. However, this is not the case in ZF^0. In the Second Fraenkel Model, see [4] and Model $\mathcal{N}2$ of [5], MC is true ([6], p. 135), and so every topological space in $\mathcal{N}2$ is a weakly Loeb space. But, AC fails in $\mathcal{N}2$. The family $\mathcal{A} = \{\{a_n, b_n\} : n \in \omega\}$, where $A = \bigcup \mathcal{A}$ is the set of atoms, has no choice set in $\mathcal{N}2$ (see [6] or [5]). Thus the space $A(a)$ is not Loeb since the family $\mathcal{A}$ of closed sets of $A(a)$ has no choice set in $\mathcal{N}2$. Hence, in ZF^0 weakly Loeb spaces need not be Loeb.

2. Main results

In what follows $\mathrm{CL}(\mathrm{T}_2)$ will denote the statement *compact* T_2 *spaces are Loeb.*

Proposition 1. [1]. *(i) If* (X, T) *is a Loeb (weakly Loeb) space and* G *a closed subset of* X*, then* G *is a Loeb (weakly Loeb) space.*
(ii) [1]. *If* (X, T) *is a Loeb (weakly Loeb) space,* (Y, Q) *a topological space and* $f : X \to Y$ *a continuous onto function, then* Y *is a Loeb (weakly Loeb) space.*
(iii) [8]. *A product of a well orderable family of compact spaces is compact if it is Loeb.*
(iv) If $(L, \leq)$ *is a conditionally complete linear order (i.e. each non-empty subset with an upper bound has a least upper bound), then* L *with the order topology is a Loeb space.*

Proof. (iv). Fix $\ell \in L$. Let $G \neq \emptyset$ be a closed subset of L. If $\ell \in G$ choose ℓ.
If $\ell \notin G$ and $G \cap [\ell, \infty) \neq \emptyset$ choose the inf of $G \cap [\ell, \infty)$.
If $\ell \notin G$ and $G \cap [\ell, \infty) = \emptyset$ choose the sup of $G \cap (-\infty, \ell]$. ∎

We would like to point out here that Proposition 1(i) is not provable in ZF if we require G to be an open subset of X. Indeed we have:

Theorem 2.1. *(i) MC if and only if open subspaces of weakly Loeb spaces are weakly Loeb.*
(ii) AC if and only if open subspaces of weakly Loeb spaces are weakly Loeb + AC_{fin}.

Proof. (i) $(\rightarrow)$. This is straightforward.

($\leftarrow$). Fix $\mathcal{A} = \{A_i : i \in k\}$ a disjoint family of non-empty sets. As we have remarked in the introduction $X(a), X = \bigcup \mathcal{A}$, is a weakly Loeb T_2 space. By our hypothesis the open subspace $\bigcup \mathcal{A}$ of $X(a)$ is weakly Loeb and consequently the family $\mathcal{A}$ of closed sets in $\bigcup \mathcal{A}$ admits a multiple choice function f. This completes the proof ($\leftarrow$) and of (i).

(ii). This follows from (i) and the fact that AC = MC + AC_{fin}. ∎

Next we give some easy topological characterizations of AC and MC.

Theorem 2.2. *(i) The following are equivalent:*
(a) AC.
(b) Every compact T_2 space has a well ordered dense set.
(c) Every compact T_2 space has a well ordered base.
(ii) The following are equivalent:
(a) MC.
(b) Every compact T_2 space has a dense set $\mathcal{D}$ which is the union of a well ordered set of finite sets.
(c) every compact T_2 space has a base $\mathcal{B}$ which is the union of a well ordered set of finite sets.
(d) Every point in a compact T_2 space has a well ordered neighborhood base.

Proof. (i), (ii). The conclusion follows from the fact that for any set X, $X(a)$ is a compact T_2 space and the well known results:
(A) [11] *AC if and only if every set can be well ordered*, and
(B) **Levy's Lemma** [7], *MC if and only if every set can be written as a well ordered union of finite sets.* ∎

Theorem 2.3. *(i) CL(T_2) implies AC_{fin}.*
(ii) AC if and only if MC+CL(T_2).
(iii) CL(T_2) if and only if every family of non-empty compact T_2 spaces has a choice set. (See [9] *and Form 343 in* [5]*).*

Proof. (i). Fix a family $\mathcal{A} = \{A_i : i \in k\}$ of disjoint non-empty finite sets. Then $\mathcal{A}$ is a family of closed sets in the compact T_2 space $X(a)$, $X = \bigcup \mathcal{A}$. Let f be a choice function for the family of all closed sets of X. Then the restriction of f to $\mathcal{A}$ is a choice function of $\mathcal{A}$.

(ii). This follows from (i) and the fact that AC = MC + AC_{fin}.

(iii) ($\rightarrow$). Let $\mathcal{A} = \{(X_i, T_i) : i \in k\}$ be a family of non-empty compact T_2 spaces. Without loss of generality we may assume that $\mathcal{A}$ is pairwise disjoint. Let $(X = \bigcup_{i\in k} X_i, T)$ be the disjoint topological union of the spaces X_i. That is, $O \in T$ if and only if $O \cap X_i \in T_i$ for each $i \in k$. Let $Y = X \cup \{\infty\}$, $\infty \notin X$, be the one-point compactification of X. Then Y is a compact T_2 space and we may finish the proof as in (i).

($\leftarrow$). Assume 343 and let (X, T) be a compact T_2 space. As each closed subset of X is a compact T_2 space (with the subspace topology), 343 implies that the family $\{F : F$ is closed in $X\}\backslash\{\emptyset\}$ has a choice set. This completes the proof of (iii) and of the theorem. ∎

In [2] it has been established that:

Van Douwen's Lemma. LN *if and only if every family $\mathcal{A} = \{A_i : i \in k\}$ of non-empty conditionally complete linear orders has a choice set.*

Theorem 2.4. *Form 115 is not provable in ZF.*

Proof. We show that Form 115 implies LN. As LN fails in the Pincus model II (see, [10] and Model $\mathcal{M}29$ of [5]) it will follow that Form 115 fails in $\mathcal{M}29$ and consequently Form

115 is not provable in ZF. To this end, it suffices, in view of van Douwen's lemma, to show that 115 implies *every family* $\mathcal{A} = \{A_i : i \in k\}$ *of non-empty conditionally complete linear orders has a choice set.* Fix such a family $\mathcal{A} = \{A_i : i \in k\}$ and let $X = \prod_{i \in k} X_i$, where $X_i = A_i \cup \{*\}$ is the disjoint topological union of A_i taken with the order topology and the discrete space $\{*\}$. By proposition 1(iv), it follows that each X_i is a Loeb (hence weakly Loeb) space. Thus, by Form 115 X is weakly Loeb and consequently the family $\{\pi_i^{-1}(A_i) : i \in k\}$ of non-empty closed sets in X has a multiple choice set $\mathcal{F} = \{F_i : i \in k\}$. It can be readily verified that

$$c = \{c_i = \max(\pi_i(F_i)) : i \in k\}$$

is a choice set of the family $\mathcal{A}$ finishing the proof of the theorem. ▮

Theorem 2.5. *Form 116 is not provable in ZF*^{0}.

Proof. We first give the description of a permutation model $\mathcal{N}$. The set of atoms $A = \cup\{A_n : n \in \omega^+ = \omega \backslash 1\}$, where $A_n = \{a_{nx} : x \in B(0, 1/n)\}$ and $B(0, 1/n)$ is the set of points on the circle of radius $1/n$ centered at 0. The group of permutations $\mathcal{G}$ is the group of all permutations on A which rotate the A_n's by an angle $\theta_n \in \mathbb{R}$ and supports are finite.

Claim 1. The family $\{A_n : n \in \omega^+\}$ does not have a multiple choice function in $\mathcal{N}$.

Proof of claim 1. Assume the contrary. Let f be a multiple choice function for the family $\{A_n : n \in \omega^+\}$ and let E be a support of f. Since E is a finite set there are only finitely many $n \in \omega^+$ such that $A_n \cap E \neq \emptyset$. Fix an n such that $E \cap A_m = \emptyset$ for all $m > n$. Let $m > n$ and let π be a permutation on A such that π is the identity map on each A_k, $k \neq m$ and $\pi|_{A_m}$ is a rotation of A_m by an angle $\theta \neq 0$ such that $\pi(f(A_m)) \neq f(A_m)$. Clearly π fixes E pointwise, therefore $\pi(f) = f$. Thus, $\pi(f(A_m)) = f(A_m)$, a contradiction finishing the proof of claim 1. ▮ (Claim 1)

Let $d : A \times A \to \mathbb{R}$ be the function given by:

$$d(a_{nx}, a_{my}) = \begin{cases} 1 \text{ if } n \neq m \\ \rho(x, y) \text{ if } n = m \end{cases},$$

where ρ is the Euclidean metric.

Claim 2. d is a metric on A and for every $n \in \omega^+$, (A_n, d) is compact.

Proof of claim 2. It can be readily verified that d is a metric on A.

To see that A_n is compact fix $n \in \omega^+$ and let $\mathcal{U}$ be an open cover of A_n in $\mathcal{N}$. As each $U \in \mathcal{U}$ is expressible as a union of open discs we may assume without loss of generality that each member of $\mathcal{U}$ is an open disc. Let f be the bijection $a_{nx} \mapsto x$ used to define A_n. f is in $\mathcal{N}$ since every singleton of A_n supports f. Then $f(\mathcal{U})$ is an open cover of $B(0, 1/n)$ and since $B(0, 1/n)$ is compact, $f(\mathcal{U})$ has a finite subcover $\mathcal{V}$. Clearly $f^{-1}(\mathcal{V})$ is an open cover of A_n and $f^{-1}(\mathcal{V}) \in \mathcal{N}$ (every singleton of A_n is a support for $f^{-1}(\mathcal{V})$). ▮ (Claim 2)

Let $A(*)$ be the Alexandroff one point compactification of A. Clearly $A(*)$ is a compact T_2 space. We claim that $A(*)$ is not weakly Loeb. Assume the contrary and let f be a multiple choice function on the set of all closed subsets of $A(*)$. Since each A_n is clearly a closed set, it follows that the restriction of f to the family $\{A_n : n \in \omega^+\}$ is a multiple choice function. This contradicts claim 1 and completes the proof of the theorem. ▮

3. Summary

The following diagram summarizes some of the results of the paper.

$$\begin{tikzcd} \boxed{\mathrm{AC}} \arrow[r] \arrow[d] & \boxed{\mathrm{CL}(\mathrm{T}_2) \equiv 343} \arrow[r] \arrow[d] & \boxed{\mathrm{AC}_{fin}} \\ \boxed{\mathrm{MC}} \arrow[r] \arrow[d] & \boxed{116} & \\ \boxed{115} \arrow[d] & & \\ \boxed{\mathrm{LN}} & & \end{tikzcd}$$

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Email-address; kker,ltah@aegean.gr