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CALDERÓN–ZYGMUND OPERATORS ON $H^p(R^n)$

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Dedicated to Professor Kôzô Yabuta on his sixtieth birthday

Abstract. We consider $H^p \to H^p$ and $H^p \to h^p$ (local Hardy space) boundedness of Calderón–Zygmund operators and give a counter example at critical index. We show $H^p \to h^p$ boundedness of Calderón's commutator.

1. Introduction

Consider the operator defined by

$$Tf(x) = p.v. \int_{R^n} K(x,y)f(y)dy,$$

where K is a Calderón–Zygmund kernel (see Sect.2).

Alvarez and Milman [1],[2] proved that if kernel $K(x,y)$ has some regularity then T is a bounded operator from H^p to L^p, and if $T^*1 = 0$ then T is a bounded operator from H^p to H^p.

In this paper we show that if T^*1 belongs to Lipschitz class then T is bounded operator from H^p to h^p (local Hardy space defined by Goldberg [4]).

2. Definitions and Notations

The following notation is used: For a set $E \subset R^n$ we denote the Lebesgue measure of E by $|E|$ and χ_E is a characteristic function of E.

We denote a ball of radius r centered at x_0 by $B(x_0, r) = \{x; |x - x_0| < r\}$.

We define two maximal functions.

Let $\varphi \in \mathcal{S}$ be a fixed function such that $\int \varphi(x)dx \neq 0$, then we define

$f^{++}(x) = \sup_{t>0} |\int f(y)\varphi_t(x-y)dy|$, $\quad f^{+}(x) = \sup_{1>t>0} |\int f(y)\varphi_t(x-y)dy|$,
where $\varphi_t(x) = t^{-n}\varphi(x/t)$.

Definition 2.1. (Fefferman–Stein's Hardy space [3])

$$H^p(R^n) = \{f \in \mathcal{S}'; \|f\|_{H^p} = \|f^{++}\|_{L^p} < \infty\}.$$

Definition 2.2. (local Hardy space [4])

$$h^p(R^n) = \{f \in \mathcal{S}';\ \|f\|_{h^p} = \|f^{+}\|_{L^p} < \infty\}.$$

Remark . $\|f\|_{h^p} \leq \|f\|_{H^p}$.

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Definition 2.3. (Lipschitz space)

$$\mathrm{Lip}_\epsilon(R^n) = \{f; \|f\|_{\mathrm{Lip}_\epsilon} = \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\epsilon} < \infty\} \quad \text{for} \quad 0<\epsilon<1.$$

Remark . $(H^p)^* = \mathrm{Lip}_{n(1/p-1)}$ where $n/(n+1) < p < 1$ (duality, see [3]).

Definition 2.4. Let T be a bounded linear operator from $\mathcal{S}$ to $\mathcal{S}'$. T is called a standard operator if T satisfies the following conditions.

(i) T extends to a continuous operator on L^2.

(ii) There exists a function $K(x,y)$ defined on $\{(x,y)\in R^n\times R^n; x\neq y\}$ which satisfies $|K(x,y)| \le \dfrac{C}{|x-y|^n}$.

(iii) $(Tf,g) = \displaystyle\int\int K(x,y)f(y)g(x)dydx$ for $f,g\in\mathcal{S}$ with disjoint supports.

Definition 2.5. A standard operator T is called a δ-Calderón–Zygmund operator if $K(x,y)$ satisfies

$$|K(x,y)-K(x,z)| + |K(y,x)-K(z,x)| \le C\frac{|y-z|^\delta}{|x-z|^{n+\delta}}$$

if $2|y-z| < |x-z|$, for some $0<\delta\le 1$.

Examples . Let T be a classical singular integral operator defined by

$$Tf(x) = p.v.\int_{R^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy,$$

where Ω satisfies the following conditions.

(iv) $\Omega(rx) = \Omega(x)$ for $r>0, x\neq 0$.

(v) $\int_{S^{n-1}} \Omega(x)d\sigma = 0$ where $d\sigma$ is the induced Euclidean measure on S^{n-1}.

(vi) $\Omega \in \mathrm{Lip}_\delta$.

Then T is a δ-Calderón–Zygmund operator.

The Hilbert transform and the Riesz transforms are 1-Calderón–Zygmund operators ($\delta = 1$).

Definition 2.6. A standard operator T is called a weak-δ-Calderón–Zygmund operator if $K(x,y)$ satisfies

$$\sup_{r>0}\sup_{|y-z|<r}\int_{2^j r\le|x-z|<2^{j+1}r} (|K(x,y)-K(x,z)| + |K(y,x)-K(z,x)|)\ dx \le C2^{-j\delta}$$

for some $0<\delta\le 1, j=1,2,3,\dots$.

Remark . If a standard operator T is δ-Calderón–Zygmund operator then it is weak-δ-Calderón–Zygmund operator.

Examples . Let $I_j = (2^j, 2^{j+1}]$ where $j\in\mathbb{Z}$. For $x>0$, we define $K(x) = 2^{-j}$ if $x\in I_j$. And for $x<0$, let $K(x) = -K(-x)$.

We define $Tf(x) = p.v.\int_{R^1} K(x-y)f(y)dy$. Then T is a weak-1-Calderón–Zygmund operator ($\delta = 1$).

The truncated Riesz transforms $(R_j)_a^b f(x) = \int_{a<|y|<b} y_j/|y|^n \cdot f(x-y)dy$ $(0<a<b)$ are weak-1-Calderón–Zygmund operators.

3. Theorems

Alvarez and Milman [1], [2] obtained next results.

Theorem A . *If T is a weak-δ-Calderón–Zygmund operator then T is a bounded operator from H^p to L^p where $\frac{n}{n+\delta} < p \le 1$.*

Theorem B . *If T is a δ-Calderón–Zygmund operator such that $T^*1 = 0$ then T is a bounded operator from H^p to H^p where $\frac{n}{n+\delta} < p \le 1$.*

Remark . T^* is an adjoint operator of T. T and T^* are simultaneously δ- or weak-δ-Calderón–Zygmund operators. For the definition of T^*1, see [6], p.412.

We have the following:

Theorem 1. *If T is a weak-δ-Calderón–Zygmund operator such that $T^*1 = 0$ then T is a bounded operator from H^p to H^p where $\frac{n}{n+\delta} < p \le 1$.*

Theorem 2. *If T is a weak-δ-Calderón–Zygmund operator such that $T^*1 \in Lip_\epsilon$ then T is a bounded operator from H^p to h^p where $\frac{n}{n+\delta} < p \le 1$ and $\frac{n}{n+\epsilon} \le p$.*

Remark . The conditions $\frac{n}{n+\delta} < p$ and $\frac{n}{n+\epsilon} \le p$ are the best possible (see Sect.6).

4. Lemmas

We shall show some properties about Hardy space. Let $\frac{n}{n+1} < p < 1$.

Definition 4.1. A function $a(x)$ is a (H^p, ∞)-atom centered at x_0 if there exists a ball $B(x_0, r)$ such that the following conditions are satisfied

(1) supp $a \subset B(x_0, r)$,
(2) $\|a\|_{L^\infty} \le r^{-n/p}$,
(3) $\int a(x)dx = 0$.

Definition 4.2. A function $a(x)$ is a $(H^p, 1)$-atom centered at x_0 if there exists a ball $B(x_0, r)$ such that the following conditions are satisfied (1), (3) and

(2') $\|a\|_{L^1} \le r^{n(1-1/p)}$.

Lemma 1 ([5], p.34)**.** *If a function $a(x)$ is a (H^p, ∞)-atom or $(H^p, 1)$-atom then we have $\|a\|_{H^p} \le C_{p,n}$ where $C_{p,n}$ is a constant depending only p and n.*

Remark . Note that $p < 1$.

Definition 4.3. A function $a(x)$ is a $(h^p, 1)$-atom centered at x_0 if there exists a ball $B(x_0, r)$ of radius $r \ge 1$ such that the following conditions are satisfied (1) and (2').

Lemma 2 ([4])**.** *If a function $a(x)$ is a $(h^p, 1)$-atom then we have $\|a\|_{h^p} \le C_{p,n}$.*

Lemma 3. *We assume a function $a(x)$ satisfies next conditions. There exists $0 < r < 1$ and $x_0 \in R^n$ such that (1), (2) and*

(3') $|\int a(x)dx| \le 1$.

Then we have $\|a\|_{h^p} \le C_{p,n}$.

Proof. We write

$$a(x) = (a(x) - a_B)\chi_B(x) + a_B\chi_B(x) = a_1(x) + a_2(x),$$

where $B = B(x_0, r)$ and $a_B = \frac{1}{|B|}\int_B a(y)dy$.

$a_1(x)/2$ is a (H^p, ∞)-atom, so by Lemma 1 we have $\|a_1\|_{H^p} \le C_{p,n}$.

supp $a_2 \subset B(x_0, 1)$ and $\int |a_2(x)|dx \le |a_B||B| = |\int_B a(y)dy| \le 1$. So $a_2(x)$ is a $(h^p, 1)$-atom. By Lemma 2 we have $\|a_2\|_{h^p} \le C_{p,n}$.

Definition 4.4. Suppose $\alpha > n(1/p-1)$. A function $M(x)$ is a $(h^p, 1, \alpha)$-molecule centered at x_0 if there exists $r > 0$ such that the following conditions are satisfied

$$(\mathrm{M}_1) \quad \int_{|x-x_0|<2r} |M(x)|dx \le r^{n(1-1/p)},$$

$$(\mathrm{M}_2) \quad \int_{|x-x_0|\ge 2r} |M(x)||x-x_0|^\alpha dx \le r^{\alpha+n(1-1/p)},$$

$$(\mathrm{M}_3) \quad \left|\int M(x)dx\right| \le 1.$$

Remark . For the definition of H^p-molecule, see [2] and [5].

Lemma 4. *If a function $M(x)$ is a $(h^p, 1, \alpha)$-molecule then we have $\|M\|_{h^p} \le C_{p,\alpha,n}$.*

Proof. Let $E_0 = \{x; |x-x_0| < 2r\}$ and $E_i = \{x; 2^i r \le |x-x_0| < 2^{i+1} r\}, i = 1, 2, 3, \dots,$ and let $\chi_i(x) = \chi_{E_i}(x)$, $\tilde{\chi}_i(x) = \frac{1}{|E_i|}\chi_{E_i}(x)$, $m_i = \frac{1}{|E_i|}\int_{E_i} M(y)dy$, $\tilde{m}_i = \int_{E_i} M(y)dy$ and $M_i(x) = (M(x) - m_i)\chi_i(x)$.

We write

$$M(x) = \sum_{i=0}^\infty M_i(x) + \sum_{i=0}^\infty m_i\chi_i(x) = \sum_{i=0}^\infty M_i(x) + \sum_{i=0}^\infty \tilde{m}_i\tilde{\chi}_i(x).$$

Let $N_j = \sum_{k=j}^\infty \tilde{m}_k$ and we write

$$\begin{aligned} M(x) &= \sum_{i=0}^\infty M_i(x) + \sum_{i=1}^\infty N_i(\tilde{\chi}_i(x) - \tilde{\chi}_{i-1}(x)) + N_0\tilde{\chi}_0(x) \\ &= I + II + III. \end{aligned}$$

We shall show $\|I\|_{H^p} \le C_{p,\alpha,n}, \|II\|_{H^p} \le C_{p,\alpha,n}$ and $\|III\|_{h^p} \le C_{p,n}$.

First we estimate I.

It is clear that supp $M_i \subset B(x_0, 2^{i+1}r), \int M_i(x)dx = 0$.

Furthermore $\int |M_0(x)|dx \le 2r^{n(1-1/p)}$ by the condition (M_1). So by Lemma 1 we have $\|M_0\|_{H^p} \le C_{p,n}$.

Using the condition (M_2), we have

$$\begin{aligned} \int |M_i(x)|dx &\le 2(2^i r)^{-\alpha} \int_{E_i} |M(x)||x-x_0|^\alpha dx \\ &\le 2(2^i r)^{-\alpha} r^{\alpha+n(1-1/p)} \le 2\cdot 2^{-\alpha i} r^{n(1-1/p)}. \end{aligned}$$

By Lemma 1 we have

$$\|M\|_{H^p} \le C_{p,n} 2^{-\alpha i} r^{n(1-1/p)} (2^{i+1}r)^{n(1/p-1)} = C_{p,n} 2^{(-\alpha+n(1/p-1))i}.$$

Since $\alpha > n(1/p-1)$, we obtain $\sum_{i=1}^\infty \|M_i\|_{H^p}^p \le C_{p,\alpha,n}$ and $\|I\|_{H^p} \le C_{p,\alpha,n}$.

Next we estimate II.

Let $A_i(x) = N_i(\tilde{\chi}_i(x) - \tilde{\chi}_{i-1}(x))$.

It is clear that supp $A_i \subset B(x_0, 2^{i+1}r), \int A_i(x)dx = 0$.

Using the condition (M_2), we have

$$\begin{aligned} \|A_i\|_{L^\infty} &\le C_n(2^i r)^{-n} \int_{|x-x_0|\ge 2^i r} |M(x)|dx \\ &\le C_n(2^i r)^{-n}(2^i r)^{-\alpha} \int_{|x-x_0|\ge 2^i r} |M(x)||x-x_0|^\alpha dx \\ &\le C_n 2^{i(-n-\alpha)} r^{-n-\alpha} r^{\alpha+n(1-1/p)} = C_n 2^{i(-n-\alpha)} r^{-n/p}. \end{aligned}$$

By Lemma 1 we have

$$\|A_i\|_{H^p} \le C_{p,n} 2^{i(-n-\alpha)} r^{-n/p} (2^{i+1} r)^{n/p} \le C_{p,n} 2^{i(-\alpha+n(1/p-1))}.$$

Since $\alpha > n(1/p-1)$, we obtain $\sum_{i=1}^{\infty} \|A_i\|_{H^p}^p \le C_{p,\alpha,n}$ and $\|II\|_{H^p} \le C_{p,\alpha,n}$.
Finally we estimate III.
It is clear that supp $N_0\tilde{\chi}_0 \subset B(x_0, 2r)$.
Using the conditions (M_1) and (M_2), we have

$$\begin{aligned}
&\|N_0\tilde{\chi}_0\|_{L^1} \le \int |M(x)|dx \\
&\le \int_{|x-x_0|<2r} |M(x)|dx + (2r)^{-\alpha} \int_{|x-x_0|\ge 2r} |M(x)||x-x_0|^{\alpha} dx \\
&\le r^{n(1-1/p)} + (2r)^{-\alpha} r^{\alpha+n(1-1/p)} \le 2r^{n(1-1/p)}.
\end{aligned}$$

Similarly we have

$$\|N_0\tilde{\chi}_0\|_{L^\infty} \le C_n r^{-n} \int |M(x)|dx \le C_n r^{-n/p}.$$

If $r \ge 1$, by Lemma 2 we have $\|N_0\tilde{\chi}_0\|_{h^p} \le C_{p,n}$.
If $r < 1$, using the condition (M_3), we have

$$\left|\int N_0\tilde{\chi}_0(x)dx\right| = \left|\int M(x)dx\right| \le 1.$$

By Lemma 3 we have $\|N_0\tilde{\chi}_0\|_{h^p} \le C_{p,n}$.
So we obtain $\|III\|_{h^p} \le C_{p,n}$.

5. Proof of Theorems

The proofs of two theorems are similar, so we prove only Theorem 2.

By the atomic decomposition, it suffices to show that there exists $C_{p,\epsilon,\delta,n} > 0$ such that $\|Ta\|_{h^p} \le C_{p,\epsilon,\delta,n}$, for every (H^p, ∞)-atom a.

By using the interpolation theorem between L^2 and H^p or h^p, we may assume $p < 1$.

We have to check that if an atom $a(x)$ is supported in $B(x_0, r)$ then $Ta(x)$ satisfies the conditions of Definition 4.4.

Since T is bounded on L^2, we have

$$\begin{aligned}
(4)\quad \int_{|x-x_0|\le 2r} |Ta(x)|dx &\le C_n r^{n/2} \|Ta\|_{L^2} \\
&\le C_n r^{n/2} \|a\|_{L^2} \le C_n r^{n/2} \|a\|_{L^\infty} r^{n/2} = C_n r^{n(1-1/p)}.
\end{aligned}$$

By the condition of Definition 2.6 and the cancellation property of atom we have

$$\begin{aligned}
&\int_{|x-x_0|\geq 2r} |Ta(x)||x-x_0|^\alpha dx = \sum_{j=1}^{\infty} \int_{2^j r\leq |x-x_0|<2^{j+1}r} |Ta(x)||x-x_0|^\alpha dx \\
&\leq \sum_{j=1}^{\infty} (2^{j+1}r)^\alpha \int_{2^j r\leq |x-x_0|<2^{j+1}r} \left| \int_{|y-x_0|<r} [K(x,y)-K(x,x_0)]a(y)dy \right| dx \\
&\leq \sum_{j=1}^{\infty} (2^{j+1}r)^\alpha r^{-n/p} \int_{|y-x_0|<r} \int_{2^j r\leq |x-x_0|<2^{j+1}r} |K(x,y)-K(x,x_0)| dx dy \\
&\leq \sum_{j=1}^{\infty} C_n 2^\alpha (2^j r)^\alpha r^{-n/p} r^n 2^{-j\delta} = \sum_{j=1}^{\infty} C_n 2^\alpha 2^{j(\alpha-\delta)} r^{\alpha+n(1-1/p)}.
\end{aligned}$$

Since $p > \frac{n}{n+\delta}$ we can choose α such that $n(1/p-1) < \alpha < \delta$.
So we have

$$\int_{|x-x_0|\geq 2r} |Ta(x)||x-x_0|^\alpha dx \leq C_{\delta,n} r^{\alpha+n(1-1/p)}. \tag{5}$$

If $r \geq 1$, by (1) and (2), we have

$$\left| \int Ta(x)dx \right| \leq \|Ta\|_{L^1} \leq C_{\delta,n} r^{n(1-1/p)} \leq C_{\delta,n}. \tag{6}$$

If $r < 1$, by the duality of H^p and Lip_ϵ, we have

$$\begin{aligned}
\left| \int Ta(x)dx \right| &= |(Ta,1)| = |(a,T^*1)| \leq C_n \|a\|_{H^{\frac{n}{n+\epsilon}}} \|T^*1\|_{\mathrm{Lip}_\epsilon} \\
&\leq C_n \|T^*1\|_{\mathrm{Lip}_\epsilon} r^{n+\epsilon-n/p}.
\end{aligned}$$

Since $p \geq \frac{n}{n+\epsilon}$ we have

$$\left| \int Ta(x)dx \right| \leq C_n \|T^*1\|_{\mathrm{Lip}_\epsilon}. \tag{7}$$

By (4)–(7) we obtain the desired result.

6. Example and Counterexamples

Definition 6.1. Calderón's commutator is defined as

$$T_b f(x) = p.v. \int_{R^1} \frac{b(x)-b(y)}{(x-y)^2} f(y)dy.$$

Theorem 3. *If $b' \in L^\infty \cap Lip_\epsilon$, then T_b is a bounded operator from H^p to h^p where $\frac{1}{1+\epsilon} \leq p \leq 1$.*

Proof. If $b' \in L^\infty$ then T_b is bounded on L^2 (see [6], p.408) and a 1-Calderón–Zygmund operator ($\delta = 1$).

We can write $T_b^*1(x) = -H(b')(x)$ where H is the Hilbert transform. Since H is bounded on Lip_ϵ (see [6], p.214), we have $T_b^*1(x) \in \mathrm{Lip}_\epsilon$.

By Theorem 2 we obtain the desired result.

Theorem 4. *The conclusion of Theorem A is not true in general for $p \leq \frac{n}{n+\delta}$.*

Proof. Let

$$\phi(x) = \begin{cases} x^\delta, & 0 \le x \le 1/2 \\ (1-x)^\delta, & 1/2 < x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

And let $I_j^k = [2^j + 2k, 2^j + 2k + 1]$ where $j = 1, 2, 3, \dots$, and k is an integer such that $0 \le k \le 2^{j-1} - 1$.

For $x \ge 0$, we define $K(x)$ as

$$K(x) = \begin{cases} 2^{-j(1+\delta)}\phi(x - 2^j - 2k), & \text{if } x \in I_j^k \text{ for some } j, k \\ 0, & \text{otherwise.} \end{cases}$$

And for $x \le 0$, let $K(x) = -K(-x)$.

We define $Tf(x) = \int_{R^1} K(x-y)f(y)dy$.

It is clear that T is a δ-Calderón–Zygmund operator.

We shall show that Ta does not belong to $L^p(R^1)$ for some $a(x) \in H^p$ where $p \le \frac{1}{1+\delta}$.

Let

$$a(x) = \begin{cases} 1, & 0 \le x < 1/2 \\ -1, & 1/2 \le x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

And let $I_j^{k*} = [2^j + 2k, 2^j + 2k + 1/2]$.

For $x \in I_j^{k*}$ we have

$$\begin{aligned} Ta(x) &= 2^{-j(1+\delta)} \int_{2^j+2k}^{x} (y - 2^j - 2k)^\delta dy \\ &= 2^{-j(1+\delta)}(x - 2^j - 2k)^{\delta+1}/(\delta+1). \end{aligned}$$

So we have

$$\begin{aligned} \int_{I_j^{k*}} |Ta(x)|^p dx &= C_{p,\delta} 2^{-j(1+\delta)p} \int_{I_j^{k*}} (x - 2^j - 2k)^{(\delta+1)p} dx \\ &= C_{p,\delta} 2^{-j(1+\delta)p} \int_0^{1/2} x^{(\delta+1)p} dx \\ &= C_{p,\delta} 2^{-j(1+\delta)p}, \end{aligned}$$

and

$$\begin{aligned} \int_{|x|\ge 2} |Ta(x)|^p dx &\ge \sum_{j=1}^{\infty} \sum_k \int_{I_j^{k*}} |Ta(x)|^p dx \\ &= C_{p,\delta} \sum_{j=1}^{\infty} 2^{-j(1+\delta)p} 2^{j-1} \\ &= C_{p,\delta} \sum_{j=1}^{\infty} 2^{j(1-(1+\delta)p)}. \end{aligned}$$

This series diverges if $p \le \frac{1}{1+\delta}$.

Remark . Similarly we can give counterexamples for $n \ge 2$.

Theorem 5. *The conclusion of Theorem 2 is not true in general for* $p < \frac{1}{1+\epsilon}$.

Proof. We consider Calderón's commutator $T_b f(x) = p.v. \int_{R^1} \frac{b(x)-b(y)}{(x-y)^2} f(y)dy$, where

$$b(x) = \begin{cases} \frac{1}{1+\epsilon} x^{1+\epsilon}, & 0 \le x < 1 \\ x - \frac{\epsilon}{1+\epsilon}, & 1 \le x \\ 0, & \text{otherwise.} \end{cases}$$

Then T_b is a 1-Calderón–Zygmund operator and $T_b^* 1 \in \mathrm{Lip}_\epsilon$, but we shall show $\lim_{r \to 0} \|T_b(a_r)\|_{h^p} = \infty$ for some (H^p, ∞)–atoms $\{a_r(x)\}$.

Let

$$a_r(x) = \begin{cases} -r^{-1/p}, & -r \le x < -r/2 \\ r^{-1/p}, & -r/2 \le x < 0 \\ 0, & \text{otherwise,} \end{cases}$$

where $r > 0$.

By the same argument used in the proof of Lemma 4 (see the estimate of III), it suffices to show

$$\lim_{r \to 0} \left| \int_{R^1} T_b(a_r)(x) dx \right| = \infty.$$

By calculations we have

$$\begin{aligned} T_b(a_r)(x) &= r^{-1/p}\ b(x) \left\{ - \int_{-r}^{-r/2} \frac{1}{(x-y)^2} dy + \int_{-r/2}^{0} \frac{1}{(x-y)^2} dy \right\} \\ &= \frac{r^{2-1/p}}{2(1+\epsilon)} \cdot \frac{x^\epsilon}{(x+r)(x+r/2)} \end{aligned}$$

for $0 < x < 1$.

Since $T_b(a_r)(x) \ge 0$, we have

$$\begin{aligned} \int_{R^1} T_b(a_r)(x) dx &\ge \frac{r^{2-1/p}}{2(1+\epsilon)} \int_0^r \frac{x^\epsilon}{(x+r)(x+r/2)}\ dx \\ &\ge \frac{r^{2-1/p}}{2(1+\epsilon)} \frac{1}{3r^2} \int_0^r x^\epsilon\ dx \\ &= \frac{r^{-1/p+1+\epsilon}}{6(1+\epsilon)^2}. \end{aligned}$$

If $p < \frac{1}{1+\epsilon}$, we have

$$\lim_{r \to 0} \int_{R^1} T_b(a_r)(x) dx = \infty.$$

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