STRUCTURE OF GROUP C^*–ALGEBRAS OF THE GENERALIZED DISCONNECTED DIXMIER GROUPS

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Abstract. In this paper we consider the structure of the group C^*-algebras of the generalized disconnected Dixmier groups. As an application we estimate the stable rank and connected stable rank of these C^*-algebras.

§0. Introduction

We first recall that the generalized discrete Heisenberg group $H_{2n+1}^{\mathbb{Z}}$ of rank $2n+1$ consists of all the $(n+2)\times(n+2)$ matrices:

$$(w,v,u) = \begin{pmatrix} 1 & u & w \\ & 1_n & v^t \\ 0 & & 1 \end{pmatrix} \quad w \in \mathbb{Z}, u = (u_j), v = (v_j) \in \mathbb{Z}^n,$$

which is isomorphic to the semi-direct product $\mathbb{Z}^{n+1} \rtimes \mathbb{Z}^n$ by the above identification. Then we define the generalized disconnected Dixmier group D_{2n}^d to be the semi-direct product $\mathbb{C}^{2n} \rtimes_\alpha H_{2n+1}^{\mathbb{Z}}$ with the action α defined by

$$\alpha_g(z_1, \cdots, z_n, w_1, \cdots, w_n) = (e^{iu_1}z_1, \cdots, e^{iu_n}z_n, e^{iv_1}w_1, \cdots, e^{iv_n}w_n)$$

(cf.[Sd5] for the Dixmier group). Then D_{2n}^d is a complex $(2n)$-dimensional, disconnected solvable (Lie) group. This definition is analogous with that of the discrete Mautner group (cf.[Bg], [Sd7]). We call D_2^d the disconnected Dixmier group.

The structure of the group C^*-algebra of $H_3^{\mathbb{Z}}$ was investigated in terms of continuous fields of C^*-algebras (cf.[AP], [Dv]). The stable rank and connected stable rank of this group C^*-algebra were estimated by the author [Sd6]. Refer to the reference for some other works about these ranks.

In this paper, we investigate the structure of the group C^*-algebras of the generalized disconnected Dixmier groups, and construct their finite composition series such that their subquotients are C^*-algebras of continuous fields with fibers noncommutative tori. This result would be useful to analyze structure of group C^*-algebras of the more general groups. As an application, we estimate the stable rank and connected stable rank of these group C^*-algebras using some results of [Rf1] and [Sd6] mainly.

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Notation. For a locally compact group G, we denote by $C^*(G)$ the (full) group C^*-algebra. For a locally compact Hausdorff space X, we let $C_0(X)$ the C^*-algebra of all continuous functions on X vanishing at infinity. If X is compact, we denote it by $C(X)$. For a C^*-algebra A and a locally compact group G, we denote by $\mathrm{A}\rtimes_\alpha G$ the (full) C^*-crossed product with the action α of G on A. By $\Gamma(\mathbb{T},\{\mathrm{A}_t\}_{t\in\mathbb{T}})$, we mean a C^*-algebra of continuous fields on the torus $\mathbb{T}$ with A_t fibers (cf.[Pd], [Dx]).

For a C^*-algebra A, we denote by sr(A), csr(A) the stable rank and the connected stable rank respectively (cf.[Rf1]).

§1. Generalized disconnected Dixmier groups

Let $D_{2n}^d = \mathbb{C}^{2n}\rtimes_\alpha H_{2n+1}^{\mathbf{Z}}$ be the generalized, disconnected Dixmier group defined in the introduction. Then the group C^*-algebra $C^*(D_{2n}^d)$ is isomorphic to the crossed product $C_0(\mathbb{C}^{2n})\rtimes_{\hat\alpha} H_{2n+1}^{\mathbf{Z}}$ via the Fourier transform, where $\hat\alpha$ is defined by

$$\hat\alpha_g(z_1,\cdots,z_n,w_1,\cdots,w_n) = (e^{-iu_1}z_1,\cdots,e^{-iu_n}z_n,e^{-iv_1}w_1,\cdots,e^{-iv_n}w_n).$$

Note that each restriction of $\hat\alpha$ to each direct summand $\mathbb{C}$ of $\mathbb{C}^{2n}$ is a rotation. Considering the restrictions of $\hat\alpha$ to the direct sums $(\mathbb{C}\setminus\{0\})^k$ of $\mathbb{C}\setminus\{0\}$ or $\{0\}$ with $0\le k\le 2n$, we have a composition series $\{\mathrm{I}_j\}_{j=1}^{2n+1}$ of $C^*(D_{2n}^d)$ such that

$$\begin{aligned}\mathrm{I}_{2n+1}/\mathrm{I}_{2n} &= C^*(D_{2n}^d)/\mathrm{I}_{2n}\cong C^*(H_{2n+1}^{\mathbf{Z}}),\\ \mathrm{I}_{2n+1-j}/\mathrm{I}_{2n-j} &\cong \oplus^{\binom{2n}{j}} C_0((\mathbb{C}\setminus\{0\})^j)\rtimes_{\hat\alpha} H_{2n+1}^{\mathbf{Z}}\end{aligned}$$

where $\oplus^{\binom{2n}{j}}$ means $\binom{2n}{j}$-direct sum, and $\binom{2n}{j}$ means the combination. Moreover, we have that

$$C_0((\mathbb{C}\setminus\{0\})^j)\rtimes_{\hat\alpha} H_{2n+1}^{\mathbf{Z}}\cong C_0(\mathbb{R}^j)\otimes(C(\mathbb{T}^j)\rtimes_{\hat\alpha} H_{2n+1}^{\mathbf{Z}}).$$

We now set the generators of $C^*(H_{2n+1}^{\mathbf{Z}})$ corresponding to those of $H_{2n+1}^{\mathbf{Z}}$:

$$\begin{cases} U_i\leftrightarrow(0,\cdots,0,u_i=1,0,\cdots,0), & 1\le i\le n,\\ V_i\leftrightarrow(0,\cdots,0,v_i=1,0,\cdots,0), & 1\le i\le n,\\ W\leftrightarrow(w=1,0,\cdots,0).\end{cases}$$

Then

$$C^*(H_{2n+1}^{\mathbf{Z}})\cong C(\mathbb{T}^{n+1})\rtimes\mathbb{Z}^n\cong C^*(C^*(W,V_1,\cdots,V_n),U_1,\cdots,U_n)$$

where $C(\mathbb{T}^{n+1}) = C^*(W,V_1,\cdots,V_n)$. Moreover, we set that

$$C(\mathbb{T}^{2n})\cong C^*(Z_1,\cdots,Z_n,W_1,\cdots,W_n)$$

where Z_i, W_i $(1\le i\le n)$ mean the coordinate functions of $\mathbb{C}^{2n}$. Then for $j=k+l$, we let that for $1\le i_1<\cdots<i_k\le n$ and $1\le j_1<\cdots<j_l\le n$,

$$C(\mathbb{T}^j)\cong C^*(Z_{i_1},\cdots,Z_{i_k},W_{j_1},\cdots,W_{j_l}).$$

We first assume that $C(\mathbb{T}^j)\cong C^*(Z_{i_1},\cdots,Z_{i_j})$. Then

$$\begin{aligned}C(\mathbb{T}^j)\rtimes_{\hat\alpha} H_{2n+1}^{\mathbf{Z}} &\cong C^*(C^*(Z_{i_1},\cdots,Z_{i_j},W,V_1,\cdots,V_n),U_1,\cdots,U_n)\\ &\cong C(\mathbb{T}^{j+n+1})\rtimes_u\mathbb{Z}^n\end{aligned}$$

where the action u is defined by

$$u_{(u_1,\cdots,u_n)}(z_{i_1},\cdots,z_{i_j},w,v_1,\cdots,v_n) = (e^{iu_{i_1}}z_{i_1},\cdots,e^{iu_{i_j}}z_{i_j},w,w^{u_1}v_1,\cdots,w^{u_n}v_n)$$

Therefore, the above crossed product is regarded as a C^*-algebra of continuous fields:

$$C(\mathbb{T}^{j+n+1}) \rtimes_u \mathbb{Z}^n \cong \Gamma(\mathbb{T}, \{C(\mathbb{T}^{j+n}) \rtimes_{w,u} \mathbb{Z}^n\}_{w\in\mathbb{T}})$$

where $C(\mathbb{T}^{j+n}) \rtimes_{w,u} \mathbb{Z}^n = C(\mathbb{T}^j \times \{w\} \times \mathbb{T}^n) \rtimes_u \mathbb{Z}^n$. Then the fiber decomposes into the tensor product:

$$\begin{aligned} C(\mathbb{T}^{j+n}) \rtimes_{w,u} \mathbb{Z}^n &\cong C(\mathbb{T}^{2j} \times \mathbb{T}^{n-j}) \rtimes_{w,u} \mathbb{Z}^n \\ &\cong (C(\mathbb{T}^{2j}) \rtimes \mathbb{Z}^j) \otimes (C(\mathbb{T}^{n-j}) \rtimes \mathbb{Z}^{n-j}) \\ &\cong (\otimes^j C(\mathbb{T}^2) \rtimes_{\theta\otimes w} \mathbb{Z})) \otimes (\otimes^{n-j} \mathrm{A}_w). \end{aligned}$$

where the action $\theta \otimes w$ means the product type action by the multi-rotation by the multiplication by $(e^{2\pi i\theta}, w)$ with $\theta = 1/2\pi$, and A_w is the rotation algebra by the multiplication by w. If the rotation by w on $\mathbb{T}$ is irrational, the fiber $C(\mathbb{T}^2) \rtimes_{\theta\otimes w} \mathbb{Z}$ is a simple noncommutative 3-torus (cf.[BKR]). By [EL1, 2] it is an inductive limit of direct sums of matrix algebras over $C(\mathbb{T})$, that is, an $\mathrm{A}\mathbb{T}$-algebra. If the rotation by w is rational, the fiber is non-simple and non-rational. By [Ln, Corollary 2] it is an inductive limit of direct sums of matrix algebras over a rational rotation algebra. We note that rational rotation algebras are homogeneous C^*-algebras (cf.[Dx], [Dv]).

We next assume that $C(\mathbb{T}^j) \cong C^*(W_{i_1},\cdots,W_{i_j})$. Then

$$\begin{aligned} C(\mathbb{T}^j) \rtimes_{\hat\alpha} H_{2n+1}^{\mathbb{Z}} &\cong C^*(C^*(W_{i_1},\cdots,W_{i_j},W,V_1,\cdots,V_n),U_1,\cdots,U_n) \\ &\cong ((\otimes_{s=1}^j C^*(W_{i_s},V_{i_s})) \otimes C(\mathbb{T}^{n-j+1})) \rtimes_u \mathbb{Z}^n \\ &\cong ((\otimes_{s=1}^j \mathrm{A}_\theta) \otimes C(\mathbb{T}^{n-j+1})) \rtimes_u \mathbb{Z}^n \end{aligned}$$

where the action u is defined by

$$u_{(u_1,\cdots,u_n)}(w_{i_1},\cdots,w_{i_j},w,v_1,\cdots,v_n) = (w_{i_1},\cdots,w_{i_j},w,w^{u_1}v_1,\cdots,w^{u_n}v_n)$$

and $\mathrm{A}_\theta \cong C^*(W_{i_s},V_{i_s})$ $(1 \le k \le n)$ is the irrational rotation algebra with $\theta = 1/2\pi$. Therefore, the above crossed product is regarded as a C^*-algebra of continuous fields:

$$((\otimes_{s=1}^j \mathrm{A}_\theta) \otimes C(\mathbb{T}^{n-j+1})) \rtimes_u \mathbb{Z}^n \cong \Gamma(\mathbb{T}, \{((\otimes_{s=1}^j \mathrm{A}_\theta) \otimes C(\mathbb{T}^{n-j})) \rtimes_{w,u} \mathbb{Z}^n\}_{w\in\mathbb{T}}).$$

Then the fiber decomposes into the tensor product:

$$\begin{aligned} ((\otimes_{s=1}^j \mathrm{A}_\theta) \otimes C(\mathbb{T}^{n-j})) \rtimes_{w,u} \mathbb{Z}^n &\cong (\otimes_{s=1}^j (\mathrm{A}_\theta \rtimes_w \mathbb{Z})) \otimes C(\mathbb{T}^{n-j} \rtimes \mathbb{Z}^{n-j}) \\ &\cong (\otimes_{s=1}^j (\mathrm{A}_\theta \rtimes_w \mathbb{Z})) \otimes (\otimes^{n-j} \mathrm{A}_w). \end{aligned}$$

We note that for $1 \le k \le n$,

$$\mathrm{A}_\theta \rtimes_w \mathbb{Z} \cong C^*(W_{i_s},V_{i_s}) \rtimes_w \mathbb{Z} \cong C^*(W_{i_s},U_{i_s}) \rtimes_{\theta\otimes\bar w} \mathbb{Z} \cong C(\mathbb{T}^2) \rtimes_{\theta\otimes\bar w} \mathbb{Z}.$$

Finally, we assume that $C(\mathbb{T}^j) \cong C^*(Z_{i_1},\cdots,Z_{i_k},W_{j_1},\cdots,W_{j_l})$ for $j = k + l$ with $k, l \ge 1$. Then

$$\begin{aligned} C(\mathbb{T}^j) \rtimes_{\hat\alpha} H_{2n+1}^{\mathbb{Z}} &\cong C^*(C^*(Z_{i_1},\cdots,Z_{i_k},W_{j_1},\cdots,W_{j_l},W,V_1,\cdots,V_n),U_1,\cdots,U_n) \\ &\cong (C(\mathbb{T}^k) \otimes (\otimes_{s=1}^l C^*(W_{j_s},V_{j_s})) \otimes C(\mathbb{T}^{n-l+1})) \rtimes_u \mathbb{Z}^n \\ &\cong (C(\mathbb{T}^{k_1}) \otimes (\otimes^{k_1} \mathrm{A}_\theta) \otimes (\otimes^{l-k_1} \mathrm{A}_\theta) \otimes C(\mathbb{T}^{n-l+1+k-k_1})) \rtimes_u \mathbb{Z}^n \end{aligned}$$

where $0 \leq k_1 \leq k, l$ and

$$u_{(u_1,\cdots,u_n)}(w_{i_1},\cdots,w_{i_j},w,v_1,\cdots,v_n) = (w_{i_1},\cdots,w_{i_j},w,w^{u_1}v_1,\cdots,w^{u_n}v_n).$$

Therefore, putting $p = n - l + k - k_1$,

$$\begin{aligned}
&(C(\mathbb{T}^{k_1}) \otimes (\otimes^{k_1} \mathrm{A}_\theta) \otimes (\otimes^{l-k_1} \mathrm{A}_\theta) \otimes C(\mathbb{T}^{p+1})) \rtimes_u \mathbb{Z}^n \\
&\cong \Gamma(\mathbb{T}, \{(\otimes^{k_1}(C(\mathbb{T}) \otimes \mathrm{A}_\theta) \rtimes_{\theta\otimes w} \mathbb{Z}) \otimes (\otimes^{l-k_1}(\mathrm{A}_\theta \rtimes_w \mathbb{Z})) \otimes (C(\mathbb{T}^p) \rtimes \mathbb{Z}^{n-l})\}_{w\in\mathbb{T}}).
\end{aligned}$$

Then noting $n - l \geq k - k_1$,

$$C(\mathbb{T}^p) \rtimes \mathbb{Z}^{n-l} \cong (\otimes^{k-k_1}(C(\mathbb{T}^2) \rtimes_{\theta\otimes w} \mathbb{Z})) \otimes (\otimes^{n-l-k+k_1} \mathrm{A}_w).$$

Moreover, $\mathrm{B}_{w,\theta} = (C(\mathbb{T}) \otimes \mathrm{A}_\theta) \rtimes_{\theta\otimes w} \mathbb{Z}$ is a simple, noncommutative torus. We note that it contains $(\mathbb{C} \otimes \mathrm{A}_\theta) \rtimes_{\theta\otimes w} \mathbb{Z} \cong \mathrm{A}_\theta \rtimes_w \mathbb{Z}$ as a C^*-subalgebra, and the Rieffel projections of A_θ (cf.[Wo]) commute with $C(\mathbb{T}) \otimes \mathbb{C}$. Therefore, the cut-down method for $\mathrm{A}_\theta \rtimes_w \mathbb{Z}$ ([EL1], [Ln]) is extended to the case of the fiber $\mathrm{B}_{w,\theta}$. In fact, if w is the irrational rotation, any element of $\mathrm{A}_\theta \rtimes_w \mathbb{Z}$ is approximated by matrix algebras of noncommutative 2-tori, so that any element of $\mathrm{B}_{w,\theta}$ is approximated by matrix algebras of noncommutative 3-tori. If w is rational, then $\mathrm{A}_\theta \rtimes_w \mathbb{Z}$ is an inductive limit of direct sums of matrix algebras of a rational rotation algebra. Therefore, in both cases we deduce that any element of $\mathrm{B}_{w,\theta}$ is approximated by A$\mathbb{T}$-algebras or approximately homogeneous C^*-algebras with slow dimension growth (cf.[BDR]).

Summing up we obtain that

Theorem 1.1. *Let $D_{2n}^d = \mathbb{C}^{2n} \rtimes_\alpha H_{2n+1}^{\mathbb{Z}}$ be the generalized disconnected Dixmier group. Then $C^*(D_{2n}^d)$ has a finite composition series $\{\mathrm{I}_j\}_{j=1}^{2n+1}$ such that*

$$\begin{aligned}
&\mathrm{I}_{2n+1}/\mathrm{I}_{2n} = C^*(D_{2n}^d)/\mathrm{I}_{2n} \cong C^*(H_{2n+1}^{\mathbb{Z}}), \\
&\mathrm{I}_{2n}/\mathrm{I}_{2n-1} \cong \oplus_{k=1}^{\binom{2n}{1}} \mathrm{K}_{1,k} \\
&\mathrm{K}_{1,k} \cong \begin{cases} C_0(\mathbb{R}) \otimes \Gamma(\mathbb{T}, \{(C(\mathbb{T}^2) \rtimes_{\theta\otimes w} \mathbb{Z}) \otimes (\otimes^{n-1} \mathrm{A}_w)\}_{w\in\mathbb{T}}), & \text{or} \\ C_0(\mathbb{R}) \otimes \Gamma(\mathbb{T}, \{(\mathrm{A}_\theta \rtimes_w \mathbb{Z}) \otimes (\otimes^{n-1} \mathrm{A}_w)\}_{w\in\mathbb{T}}), \end{cases} \\
&\mathrm{I}_{2n+1-j}/\mathrm{I}_{2n-j} \cong \oplus_{k=1}^{\binom{2n}{j}} \mathrm{K}_{j,k} \\
&\mathrm{K}_{j,k} \cong \begin{cases} C_0(\mathbb{R}^j) \otimes \Gamma(\mathbb{T}, \{(\otimes^j (C(\mathbb{T}^2) \rtimes_{\theta\otimes w} \mathbb{Z})) \otimes (\otimes^{n-j} \mathrm{A}_w)\}_{w\in\mathbb{T}}), & \text{or} \\ C_0(\mathbb{R}^j) \otimes \Gamma(\mathbb{T}, \{(\otimes^j (\mathrm{A}_\theta \rtimes_w \mathbb{Z})) \otimes (\otimes^{n-j} \mathrm{A}_w)\}_{w\in\mathbb{T}}), & \text{or} \\ C_0(\mathbb{R}^j) \otimes \Gamma(\mathbb{T}, \{(\otimes^{k_1}((C(\mathbb{T}) \otimes \mathrm{A}_\theta) \rtimes_{\theta\otimes w} \mathbb{Z})) \otimes (\otimes^{l-k_1}(\mathrm{A}_\theta \rtimes_w \mathbb{Z})) \\ \qquad \otimes (\otimes^{k-k_1}(C(\mathbb{T}^2) \rtimes_{\theta\otimes w} \mathbb{Z})) \otimes (\otimes^{n-l-k+k_1} \mathrm{A}_w)\}_{w\in\mathbb{T}}) \end{cases}
\end{aligned}$$

for $2 \leq j \leq 2n$, $k + l = j$, $0 \leq k_1 \leq k, l$, where the fibers of the C^*-algebras of continuous fields on $\mathbb{T}$ are tensor products of the noncommutative 2-tori A_w or 3-tori $C(\mathbb{T}^2) \rtimes_{\theta\otimes w} \mathbb{Z}$, $\mathrm{A}_\theta \rtimes_w \mathbb{Z}$ which are simple when w is a irrational rotation or the simple noncommutative 4-tori $(C(\mathbb{T}) \otimes \mathrm{A}_\theta) \rtimes_{\theta\otimes w} \mathbb{Z}$.

Remark. We note that

$$\begin{aligned}
&C^*(H_{2n+1}^{\mathbb{Z}}) \cong \Gamma(\mathbb{T}, \{\otimes^n \mathrm{A}_w\}_{w\in\mathbb{T}}), \quad \mathrm{A}_\theta \rtimes_w \mathbb{Z} \cong C(\mathbb{T}^2) \rtimes_{\theta\otimes\bar{w}} \mathbb{Z}, \\
&(C(\mathbb{T}) \otimes \mathrm{A}_\theta) \rtimes_{\theta\otimes w} \mathbb{Z} \cong (C(\mathbb{T}) \otimes C(\mathbb{T})) \rtimes_{(\theta\otimes w, 1\otimes\theta^{-1})} \mathbb{Z}^2.
\end{aligned}$$

Compare with the case of the generalized Dixmier groups [Sd5]. In particular, $C^*(D_2^d)$ has the finite composition series $\{\mathrm{I}_k\}_{k=1}^3$ such that

$$\begin{aligned} \mathrm{I}_3/\mathrm{I}_2 &= C^*(D_2^d)/\mathrm{I}_3 \cong C^*(H_3^{\mathbb{Z}}), \\ \mathrm{I}_2/\mathrm{I}_1 &\cong (C_0(\mathbb{R}) \otimes \Gamma(\mathbb{T}, \{C(\mathbb{T}^2) \rtimes_{\theta\otimes w} \mathbb{Z}\}_{w\in\mathbb{T}})) \oplus (C_0(\mathbb{R}) \otimes \Gamma(\mathbb{T}, \{\mathrm{A}_\theta \rtimes_w \mathbb{Z}\}_{w\in\mathbb{T}})), \\ \mathrm{I}_1 &\cong C_0(\mathbb{R}^2) \otimes \Gamma(\mathbb{T}, \{(C(\mathbb{T}) \otimes \mathrm{A}_\theta) \rtimes_{\theta\otimes w} \mathbb{Z}\}_{w\in\mathbb{T}}) \end{aligned}$$

As an application, we get that

Theorem 1.2. *Let D_{2n}^d be the generalized disconnected Dixmier group. Then*

$$\begin{cases} \mathrm{sr}(C^*(D_{2n}^d)) = n + 1 = \dim_{\mathbb{C}}(D_{2n}^d)_1^\wedge \equiv [\dim(D_{2n}^d)_1^\wedge/2] + 1, \\ 2 \le \mathrm{csr}(C^*(D_{2n}^d)) \le n + 1. \end{cases}$$

where $[x]$ means the maximal integer $\le x$.

Proof. By [Sd6], we have that

$$\mathrm{sr}(C^*(H_{2n+1}^{\mathbb{Z}})) = n + 1 \ge \mathrm{csr}(C^*(H_{2n+1}^{\mathbb{Z}})) \ge 2.$$

¿From Theorem 1.1 and the structure of $C^*(H_{2n+1}^{\mathbb{Z}})$ we have that $(D_{2n}^d)_1^\wedge$ is homeomorphic to $\mathbb{T}^{2n}$. Moreover, we have that

$$\mathrm{sr}(\mathrm{K}_{j,k}) \le 2, \quad \mathrm{csr}(\mathrm{K}_{j,k}) \le 2$$

where each inequality follows from [Sd6] and the structure of each subquotient, and combining the rank estimations for inductive limits and exact sequences in [Rf1] and [Sh] with the structure of each fiber given above. by [EL1,2], Combining these obtained estimations with the rank estimations for exact sequences inductively, we get that

$$\mathrm{sr}(C^*(D_{2n}^d)) = n + 1, \quad \mathrm{csr}(C^*(D_{2n}^d)) \le n + 1.$$

On the other hand, by [Eh] we get $\mathrm{csr}(C^*(D_{2n}^d)) \ge 2$. □

Remark. Using some methods in [Sd5] and this paper, it is possible to generalize the results of [Sd7] to the cases of the semi-direct products of $\mathbb{C}^n$ by $\mathbb{Z}^n$ with the multi-actions or the diagonal actions, and the semi-direct products of $\mathbb{C}^n \times \mathbb{R}^m$ by discrete groups with the actions induced from those of the quotient groups by their commutators.

As for the above remark, we give an example which is different from the generalized disconnected Dixmier groups.

Example. Let $G = \mathbb{R}^2 \rtimes_\alpha H_3^{\mathbb{Z}}$ with $\alpha_g = (e^u t, e^v s)$ for $t, s \in \mathbb{R}$. Then $C^*(G)$ has the following structure:

$$\begin{cases} 0 \to C_0(\mathbb{R}^2 \setminus \{0_2\}) \rtimes_\alpha H_3^{\mathbb{Z}} \to C^*(G) \to C^*(H_3^{\mathbb{Z}}) \to 0, \\ 0 \to C_0((\mathbb{R} \setminus \{0\})^2) \rtimes_\alpha H_3^{\mathbb{Z}} \to C_0(\mathbb{R}^2 \setminus \{0_2\}) \rtimes_\alpha H_3^{\mathbb{Z}} \to \oplus^2 C_0(\mathbb{R} \setminus \{0\}) \rtimes_\alpha H_3^{\mathbb{Z}} \to 0. \end{cases}$$

Taking the restrictions of α to $\mathbb{R} \setminus \{0\}, (\mathbb{R} \setminus \{0\})^2$,

$$\begin{cases} C_0(\mathbb{R} \setminus \{0\}) \rtimes_\alpha H_3^{\mathbb{Z}} \cong \oplus^2 C_0(\mathbb{R}_+) \rtimes_\alpha H_3^{\mathbb{Z}} \cong \oplus^2 (C(\mathbb{T}) \otimes (C_0(\mathbb{Z}) \rtimes H_3^{\mathbb{Z}})), \\ C_0((\mathbb{R} \setminus \{0\})^2) \rtimes_\alpha H_3^{\mathbb{Z}} \cong \oplus^4 C_0(\mathbb{R}_+^2) \rtimes_\alpha H_3^{\mathbb{Z}} \cong \oplus^4 (C_0(\mathbb{T} \times \mathbb{R}) \otimes (C_0(\mathbb{Z}) \rtimes H_3^{\mathbb{Z}})) \end{cases}$$

where the quotient spaces $\mathbb{R}_+/H_3^{\mathbb{Z}}, \mathbb{R}_+^2/H_3^{\mathbb{Z}}$ are homeomorphic to $\mathbb{T}$, $\mathbb{T} \times \mathbb{R}$ respectively. Moreover, we have that

$$C_0(\mathbb{Z}) \rtimes H_3^{\mathbb{Z}} \cong \Gamma(\mathbb{T}, \{C_0(\mathbb{Z}) \rtimes (\mathbb{Z} \rtimes_w \mathbb{Z})\}_{w\in\mathbb{T}}) \cong \Gamma(\mathbb{T}, \mathbb{K} \otimes C(\mathbb{T}))$$

where the last isomorphism follows from [Gr1] or [Gr2], so that no higher dimensional noncommutative tori appear in the fibers of the C^*-algebras of continuous fields on $\mathbb{T}$. On the other hand, the stable rank and connected stable rank of $C^*(G)$ are equal to 2.

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