ON A CLASS OF ULTRAMETRIC MEASURES

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Abstract. In this note we associate to any compact C in an ultrametric space (X, d) a real valued and a p-adic valued measure μ_C. We prove that any p-adic valued continuous function $f : C \to \mathbf{C}_p$ is μ_C-integrable. Using this measure we extend the definition of the trace function [2] to any $T \in \mathbf{C}_p$.

Let (X, d) be an ultrametric space and C be an infinite compact set in X. By $B[x, r] = \{y \in X \mid d(x, y) \leq r\}$ we denote the "closed" ball with centre in x and with radius r. From the topological point of view, this last one is a closed and an open (clopen) set in X. If $x \in C$, we denote by $D[x, r] = B[x, r] \cap C$. Moreover, if $B[x, r] \cap C \neq \emptyset$ we can choose x to be in C too. Let $\varepsilon_1 > 0$ be the diameter of C, i.e. the smallest $\varepsilon > 0$ such that $C \subset B[x, \varepsilon]$, for an $x \in C$. This means that $C = D[x, \varepsilon_1]$. For any $0 < \varepsilon < \varepsilon_1$, C can be covered with at least two balls. Let $\varepsilon_2 < \varepsilon_1$, be the smallest $\varepsilon < \varepsilon_1$ such that we can cover C with the smallest number of balls, $n_2 > 1$. By definition we put $n_1 = 1$. Suppose we have constructed ε_i and n_i. For any $0 < \varepsilon < \varepsilon_i$ let us denote by $n(\varepsilon)$ the number of distinct balls (they are uniquely determined by C and ε!) of radius $\varepsilon > 0$ which cover effectively the compact C. By the construction of ε_i and n_i we have that $n(\varepsilon) > n_i$. We choose now n_{i+1} to be the smallest $n(\varepsilon) > n_i$.Let $M_{i+1} = \left\{\varepsilon' \in \mathbf{R} \mid 0 < \varepsilon' < \varepsilon_i , n(\varepsilon') = n_{i+1}\right\}$ and denote by $\varepsilon_{i+1} = \inf M_{i+1}$. Since C is infinite $1 = n_1, n_2, ...$ is a strictly increasing sequence of natural numbers and $\varepsilon_1 > \varepsilon_2 > \cdots$ is a strictly decreasing sequence of positive numbers.

In the following, by a ball of C, we mean an intersection of the type $D[x, \varepsilon] = B[x, \varepsilon] \cap C = \{y \in C \mid d(x, y) \leq \varepsilon\}$, where $x \in C$. Let us denote by $\mathcal{S}_i$ the (finite) set of all distinct balls $D[x_j^{(i)}, \varepsilon_i]$, $j = 1, 2, ..., n_i$, which cover the compact C. For any $i = 1, 2, ...$ let k_{ij} be the number of balls $D[x_t^{(i)}, \varepsilon_i] \in \mathcal{S}_i$ which are contained in the fixed ball $D[x_j^{(i-1)}, \varepsilon_{i-1}]$ from $\mathcal{S}_{i-1}$. For instance, $k_{11} = 1$ and $k_{i1} + k_{i2} + \cdots k_{in_i} = n_i$, for every $i = 2, 3, ...$ The sequence $\{\varepsilon_1 > \varepsilon_2 > \cdots\}$ and the infinite matrix (k_{ij}), $i = 1, 2, ...$, $j = 1, 2, ..., n_i$ are called *the configuration of* C.

It is not difficult to see that any sequence of positive real numbers $\{\varepsilon_1 > \varepsilon_2 > \cdots\}$ and any infinite matrix of positive integers (k_{ij}), where $i = 1, 2, ...$, and $j = 1, 2, ..., n_i$, are the configuration of an infinite number of distinct compacts in $\mathbf{C}_p$, the complex p-adic numbers, i.e. the completion of an algebraic closure of the field of p-adic numbers $\mathbf{Q}_p$ relative to the usual p-adic distance.

Definition 0.1 *Let C be an infinite compact set in an ultrametric space (X, d). For any ball $D[x_{j_i}^{(i)}, \varepsilon_i] \in \mathcal{S}_i$, $j_i \in \{1, 2, ..., n_i\}$, let*

$D[x_{j_i}^{(i)}, \varepsilon_i] \subset D[x_{j_{i-1}}^{(i-1)}, \varepsilon_{i-1}] \subset \cdots \subset D[x_{j_1}^{(1)}, \varepsilon_1]$ be its saturated tower of balls (i.e. $D[x_{j_k}^{(k)}, \varepsilon_k] \in \mathcal{S}_k$, for any $k = 1, 2, ..., i$). For any $l = 1, 2, ..., i$ let k_{lj_l} be the number of balls of radius ε_l which are contained in the ball $D[x_{j_{l-1}}^{(l-1)}, \varepsilon_{l-1}]$. By definition, the measure of $D[x_{j_i}^{(i)}, \varepsilon_i]$, $\mu_C(D[x_{j_i}^{(i)}, \varepsilon_i]) = \frac{1}{N_{ij_i}}$, where $N_{ij_i} = k_{1j_1} \cdot k_{2j_2} \cdot \cdots \cdot k_{ij_i}$. We call this last number the (standard) ultrametric measure of $D[x_{j_i}^{(i)}, \varepsilon_i]$.

It is easy to prove the following result.

Lemma 0.1 *The real valued function μ_C can be uniquely extended to a σ-additive measure (also denoted by μ_C) on the Borel field of all the closed subset of C.*

For any $i = 1, 2, ...$ we consider the cannonical covering of C with all the disjoint balls from $\mathcal{S}_i$. In any ball $D[x_j^{(i)}, \varepsilon_i]$ we choose an element $x_j^{(i)}$, $j \in \{1, 2, ..., n_i\}$ (it can be thought to be the "centre" of the ball). Let now $f : C \to \mathbf{C}$ be a function defined on C with complex values.

Definition 0.2 *A function $f : C \to \mathbf{C}$ is said to be integrable on C if the set of complex numbers $S_i[f; (x_j^{(i)})_j] = \sum_{j=1}^{n_i} f(x_j^{(i)})\mu_C(D[x_j^{(i)}, \varepsilon_i])$ has a unique limit point in $\mathbf{C}$.*

Remark 0.1 *It is not difficult to see that any integrable function on C is a bounded function on C. Moreover, all the classical theory of the Riemann's and Darboux's sums (including Darboux's criteria) works well in this situation.*

Let $\omega(f; D[x_j^{(i)}, \varepsilon_i]) = \sup\left\{|f(x) - f(y)|, x, y \in D[x_j^{(i)}, \varepsilon_i]\right\}$
and $\omega(f; i) = \max\left\{\omega(f; D[x_j^{(i)}, \varepsilon_i]) \mid j = 1, 2, ..., n_i\right\}$.

Theorem 0.2 *Let C be an infinite compact in the ultrametric space (X, d) and $f : C \to \mathbf{C}$, be a continuous complex valued function defined on C,such that the series $\sum_{i=1}^{\infty} \omega(f; i)$ is convergent in $\mathbf{C}$. Then f is integrable on C.*

Proof. Let us remark that if *one* of the sequence of sums (for a fixed $(x_j^{(i)})_j$), $\left\{S_i[f; (x_j^{(i)})_j\right\}_i$ is convergent to a complex number I, then any other sequence of sums tends to the same number I. Hence, in the following we shall fix, for any $i = 1, 2, ...$, the set of elements $(x_j^{(i)})_j$, $j = 1, 2, ..., n_i$ and consider them to be the centers of their corresponding balls. We want now to estimate the difference $S_i - S_{i-1}$ (here $S_i = S_i[f; (x_j^{(i)})_j]$). Let us fix a term $f(x_{j_0}^{(i-1)})\mu_C(D[x_{j_0}^{(i-1)}, \varepsilon_{i-1}])$ from the sum S_{i-1} and denote by $D[x_{j_1}^{(i)}, \varepsilon_i], ..., D[x_{j_t}^{(i)}, \varepsilon_i]$, $t = k_{ij_0}$, all the balls from $\mathcal{S}_i$ which are contained in $D[x_{j_0}^{(i-1)}, \varepsilon_{i-1}]$. Using Definitions 1 and 2, $S_i - S_{i-1}$ can be grouped into sums of the following type: $\frac{1}{N_{ij_0}} \sum_{u=1}^{t} \left[f(x_{j_u}^{(i)}) - f(x_{j_0}^{(i-1)})\right]$. But $|S_i - S_{i-1}| \leq \omega(f; i-1)$, because $\sum_{j_0=1}^{n_{i-1}} \mu_C(D[x_{j_0}^{(i-1)}, \varepsilon_{i-1}]) = 1$.

Therefore $|S_{m+n} - S_n| \leq \sum_{i=n}^{m+n-1} \omega(f; i)$ and, using the covergence of the series $\sum_{i=1}^{\infty} \omega(f; i)$, we obtain that the sequence $\{S_n\}_n$ is uniformly convergent relative to the choice of $\{x_j^{(i)}\}_{i,j}$, $i = 1, 2, ...$, $j = 1, 2, ..., n_i$.

Remark 0.2 *If $f : C \to \mathbf{C}$ is a continuous function then there exists subsequences $\{S_{i_n}\}_n$ of $\{S_n\}_n$ such that the series $\sum_{n=1}^{\infty} \omega(f; i_n)$ is convergent. In this case we say that the function f is integrable relative to the subsequence $\{i_n\}_n$ of $\{1, 2, 3, ...\}$. Even in the particular case $f(x) = x$ and $\omega(f; i) = \varepsilon_i$ the series $\sum_{i=1}^{\infty} \varepsilon_i$ may be divergent. So that, generally speaking, we can say nothing about the set of limit points of the sums $\{S_i\}_i$.*

Let us now suppose that the measure μ_C has values in the p-adic complex number field $\mathbf{C}_p$. A function $f : C \to \mathbf{C}_p$ is called *p-adic integrable* if it is integrable (Definition 0.2 with $\mathbf{C}_p$ instead of $\mathbf{C}$!) relative to this p-adic valued measure μ_C. We denote by $\int f d\mu_C$ its p-adic measure.

Theorem 0.3 *Let C be an infinite compact in the ultrametric space (X, d) and $f : C \to \mathbf{C}_p$ be a continuous function defined on C with p-adic values. Then f is p-adic integrable on C.*

Proof. We follow the same reasoning as in the proof of Theorem 0.2. Since f is continuous the sequence $\omega(f; n) \to 0$ when $n \to \infty$. Hence $|S_{n+1} - S_n|_p \to 0$, when $n \to \infty$ and this one is enough to assure the uniform convergence of $\{S_n\}_n$ in $\mathbf{C}_p$.

Remark 0.3 *Let T be a transcendental element in $\mathbf{C}_p$ (relative to $\mathbf{Q}_p$) and $C(T)$ be the orbit of T with respect to the Galois group $G = Gal_{cont}(\mathbf{C}_p/\mathbf{Q}_p) \simeq Gal(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. We know [1], [2], [5] that $C(T)$ is an infinite compact set in $\mathbf{C}_p$. For any $k = 0, 1, 2, \ldots$ we define the k-th moment of T (it depends only of $C(T)$!) by $M_k^{(T)} = \int x^k d\mu_{C(T)}$. For instance, M_1 is the trace of T. The generating series (the trace function in [2]) $F(T, X) = 1 + M_1(T)X + M_2(T)X^2 + \cdots$ can now be defined for every $T \in \mathbf{C}_p$, not only for $(*)$-elements [2]. All the properties of the trace function on $\mathbf{C}_p$ [2] can be extended for every $T \in \mathbf{C}_p$ with our definition. These series and the above integral are fundamental tools for studing arithmetical properties of different infinite towers of algebraic extensions of $\mathbf{Q}_p$ ([2], Section 9).*

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