REMARKS ON THE SPACES OF RIEMANNIAN METRICS ASSOCIATED WITH CONTACT FORMS ON 3-MANIFOLDS

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Abstract. We recall and improve the correspondence theorem of Etnyre-Ghrist [1] between a positively rescaled Reeb field for a contact 1-form and a rotational Beltrami field for a Riemannian metric on a closed oriented 3-manifold. Given a contact form, we associate it with the space of Riemannian metrics for which the Reeb field is a Beltrami field with certain additional properties. We obtain a product structure on this space of metrics and then, by applying it, we characterize certain geometric structures on 3-manifolds.

1 Introduction. Recently, J.Etnyre and R.Ghrist [1] found a correspondence between a Reeb-like field in contact topology and a rotational Beltrami field in topological hydrodynamics. Let M^3 be a closed oriented 3-manifold. A *Reeb-like field* on M^3 is a positively rescaled Reeb field for a given positive contact form on M^3. On the other hand, a *rotational Beltrami field* on M^3 is a non-singular vector field X satisfying $\nabla \times X = fX$ for some function $f > 0$. Here $\nabla \times X$ denotes the curl of X with respect to a given Riemannian metric g and a fixed positive volume form ν on M^3. It is easy to see that the definition of a rotational Beltrami field is independent of the choice of ν. The correspondence theorem says that the set of all Reeb-like fields on M^3 can be regarded as the set of all rotational Beltrami fields on M^3 if we don't fix a contact form nor a metric.

We consider the case where the above f can be taken as $f \equiv 1$ with respect to g and the g-induced volume form ν_g. Then we call X a *normal Beltrami field* for g if moreover $g(X,X) \equiv 1$ holds. Fix a positive volume form ν on M^3. Then we can improve the above correspondence theorem as follows.

(1) A Reeb field X_α for some contact form α with $\alpha \wedge d\alpha = \nu$ corresponds to a normal Beltrami field for some ν-inducing metric g (Theorem 5).

(2) A rotational Beltrami field for a Riemannian metric g is a positively rescaled normal Beltrami field for some Riemannian metric g' (Remark 6).

Every positive contact form α on M^3, therefore, can be associated with the subspace $\mathcal{F}_\alpha$ of the space $\mathcal{R}(M^3)$ of all Riemannian metrics on M^3, where $\mathcal{F}_\alpha$ consists of any element g inducing the volume form $\alpha \wedge d\alpha$ and satisfying $g(X_\alpha, \cdot) = \alpha$. Set $\mathcal{B}_\alpha = \{\beta |\ X_\beta = X_\alpha\} (\subset \Gamma \mathrm{T}^* M^3)$ and $\mathcal{C}_\alpha = \bigcup_{\beta \in \mathcal{B}_\alpha} \mathcal{F}_\beta (\subset \mathcal{R}(M^3))$. In this paper, we study these spaces. Our results are the following theorems.

Theorem A(Theorem 10). *For any positive contact form α on a closed oriented 3-manifold M^3, the above $\mathcal{C}_\alpha$, $\mathcal{F}_\alpha$ and $\mathcal{B}_\alpha$ are connected and contractible. Moreover, $\mathcal{C}_\alpha$ is fibred trivially by $\{\mathcal{F}_\beta\}_{\beta \in \mathcal{B}_\alpha}$ over $\mathcal{B}_\alpha$.*

We obtain the following theorem by using Theorem A and a result of Geiges and Gonzalo [2] on the characterization of closed 3-manifolds admitting Cartan structures.

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Theorem B(Theorem 15). *Let M^3 be a closed orientable 3-manifold. Then there are two contact forms α and β on M^3 satisfying*

1) $\alpha \wedge d\alpha = h\beta \wedge d\beta$ *for some function* $h > 0$ *and*

2) $X_\alpha \perp X_\beta$ *with respect to some* $g \in \mathcal{C}_\alpha \cap \mathcal{C}_\beta$

if and only if M^3 is diffeomorphic to a $SU(2)$-, $\widetilde{SL_2}$- or $\widetilde{E_2}$-manifold.

The class of closed oriented 3-manifolds admitting positive contact forms α and β with $X_\beta \neq \pm X_\alpha$ and $\mathcal{C}_\alpha \cap \mathcal{C}_\beta \neq \emptyset$ seems very small since such a manifold has to admit another normal Beltrami field than $\pm X_\alpha$ for the same metric in $\mathcal{C}_\alpha$. It is even likely that this class coincides with one stated in Theorem B.

2 The correspondence theorem. We will work in the smooth category throughout this paper. First we prepare some definitions.

Definition 1. A vector field X on a Riemannian 3-manifold (M^3, g) is called a *Beltrami field* if it is everywhere colinear with its curl, that is, $\nabla \times X = fX$ for some function f on M^3. Here the curl $\nabla \times X$ is the vector field determined by $\iota_{\nabla\times X}\nu = d(g(X,\cdot))$ for a fixed volume form ν (ι denotes the interior product). A non-singular Beltrami field is called a *rotational Beltrami field* if the above f satisfies $f > 0$. A rotational Beltrami field for the particularly g-induced volume form ν_g is called a *normal Beltrami field* if $f \equiv 1$ and $g(X,X) \equiv 1$ hold.

Remark 2. Beltrami fields form an important and still mysterious class of steady (i.e., time-independent) solutions of the following Euler's equation for a perfect incompressible fluid with a volume form ν (see [1]).

Let $\{X_t\}_{t\in\mathbf{R}}$ be a family of vector fields. X_t can be considered as the velocity field of an perfect incompressible fluid if it satisfies the Euler's equation

$$\dot{X}_t + \nabla_{X_t} X_t = -\nabla p_t, \quad \mathcal{L}_{X_t}\nu = 0$$

for some family p_t of functions, called the pressure term. Here ∇_{X_t} denotes the covariant derivative with respect to g along X_t. Then the curl of X_t with respect to ν and g satisfies

$$\begin{aligned}(\iota_{X_t}\iota_{\nabla\times X_t}\mu)(Y) &= \iota_{X_t} d(g(X_t,\cdot))(Y)\\ &= X_t g(X_t, Y) - Yg(X_t, X_t) - g(X_t, \nabla_{X_t} Y - \nabla_Y X_t)\\ &= g(\nabla_{X_t} X_t, Y) - g(\nabla_Y X_t, X_t)\\ &= g(\nabla_{X_t} X_t, Y) - \tfrac{1}{2} Y g(X_t, X_t).\end{aligned}$$

From now on, we assume that X_t is time-independent, that is, $X_t = X$ holds for any $t \in \mathbf{R}$. Put $p = p_t$ and $P = p + \frac{1}{2}g(X,X)$. Then the first portion of the Euler's equation yields

$$\iota_X \iota_{\nabla\times X}\mu = -dP.$$

Then we see that a Beltrami field corresponds to a P-free steady fluid while a normal Beltrami field corresponds to a special kind of pressure-free fluid. Note also that a normal Beltrami field generates a divergence-free geodesical flow.

Definition 3. A 1-form α on an oriented 3-manifold M^3 is called a *positive contact form* if $\alpha \wedge d\alpha$ is a positive volume form. Then a vector field X on M^3 is called a *Reeb field* for α if $\iota_X d\alpha = 0$ and $\iota_X \alpha = 1$ hold. Such an X always exists and is determined uniquely by

α, so we denote it by X_α. A *Reeb-like field* Y is a positively rescaled Reeb-field, that is, $Y = fX_\alpha$ for some function $f > 0$.

The following is the Etnyre-Ghrist's correspondence theorem.

Theorem 4 ([1]). *Let M^3 be an oriented 3-manifold. Given a Riemannian metric g on M^3, any rotational Beltrami field for g, if it exists, is a Reeb-like field for some positive contact form on M^3. Conversely, given a positive contact form α on M^3, any Reeb-like field for α is a rotational Beltrami field for some Riemannian metric on M^3.*

Note that the definition of rotational Beltrami fields are independent of the choice of the fixed volume form ν in Definition 1. The above theorem says that a vector field X is a Reeb-like field for some positive contact form if and only if X is a rotational Beltrami field for some Riemannian metric. It may be difficult, however, to associate the set of all such contact forms with the set of all such metrics in a general way. So we state a more detailed correspondence theorem as follows.

Theorem 5. *Let M^3 be an oriented 3-manifold equipped with a positive volume form ν. Given a ν-inducing Riemannian metric g, any normal Beltrami field with respect to g, if it exists, is a Reeb field for some contact form α with $\alpha \wedge d\alpha = \nu$ on M^3. Conversely, for any contact form α on M^3 with $\alpha \wedge d\alpha = \nu$, the Reeb field X_α is a normal Beltrami field for some ν-inducing metric. Thus a vector field X is a Reeb field X_α for some contact form α with $\alpha \wedge d\alpha = \nu$ if and only if X is a normal Beltrami field for some ν-inducing metric.*

Proof. Suppose that $\nabla \times X = X$ with respect to ν and a ν-inducing metric g and $g(X, X) = 1$ hold. Putting $\alpha = g(X, \cdot)$, we have

$$\alpha \wedge d\alpha = g(X, \cdot) \wedge \iota_{\nabla \times X}\nu.$$

Since g is ν-inducing, we have

$$\nu = g(X, \cdot) \wedge g(e_2, \cdot) \wedge g(e_3, \cdot) = g(X, \cdot) \wedge \iota_X \nu$$

for a local orthonormal framing (X, e_2, e_3). Thus the condition $\nabla \times X = X$ implies $\alpha \wedge d\alpha = \nu$. Then $X = X_\alpha$ holds since $\iota_X d\alpha = \iota_X \iota_X \nu = 0$ and $\iota_X \alpha = g(X, X) = 1$.

Conversely, suppose that $X = X_\alpha$ for α with $\alpha \wedge d\alpha = \nu$. Then choose a local frame (X, e_2, e_3) such that (e_2, e_3) forms a symplectic basis for $d\alpha$ on $\ker \alpha$. Since there is a global complex structure J on $\ker \alpha$ compatible with $d\alpha$, we may assume $e_3 = Je_2$. Let g be the metric for which (X, e_2, e_3) is orthonormal. Note that g is globally defined since each transformation map between charts preserves the orthonormality of (e_2, e_3) as an element of $SU(1)$ with respect to J. Then X is a normal Beltrami field for g since $g(X, X) = 1$ and $d(g(X, \cdot)) = d\alpha = \iota_X \nu$. This completes the proof.

Remark 6. Our normal condition may seem much too strong at a glance. Note that, however, a given rotational Beltrami field for a metric g can be rescaled to be a normal Beltrami field for another metric g'. This fact follows immediately from the theorems 4 and 5. Moreover we can see, from Moser's theorem, that even when we fix arbitrary volume form ν the Weinstein conjecture translates to whether any normal Beltrami field for any ν-inducing Riemannian metric generates a flow with a closed orbit.

3 The Spaces of metrics. Given a positive volume form ν on a closed oriented 3-manifold M^3, we set $Cont(M^3) = \{\text{all positive contact forms on } M\}(\subset \Gamma T^* M^3)$ and $Cont(M^3, \nu) = \{\alpha |\ \alpha \wedge d\alpha = \nu\}(\subset Cont(M^3))$. Let $\mathcal{R}(M^3)$ be the space of all Riemannian metrics on M^3 and $\mathcal{R}(M^3, \nu)$ its subset consisting of all ν-inducing metrics. For any $\alpha \in Cont(M^3)$, put

$$\mathcal{F}_\alpha = \{g \in \mathcal{R}(M^3, \alpha \wedge d\alpha) |\ g(X_\alpha, \cdot) = \alpha\},$$

$$\widetilde{\mathcal{F}}_\alpha = \{g \in \mathcal{R}(M^3) |\ g(X_\alpha, \cdot) = \alpha\},$$

$$\mathcal{B}_\alpha = \{\beta \in Cont(M^3) |\ X_\beta = X_\alpha\}$$

and

$$\mathcal{C}_\alpha = \bigcup_{\beta \in \mathcal{B}_\alpha} \mathcal{F}_\beta \ (\subset \mathcal{R}(M^3)).$$

Note that we can also define the above $\mathcal{C}_\alpha$ by $\mathcal{C}_\alpha = \{g \in \mathcal{R}(M^3) |\ \beta := g(X_\alpha, \cdot) \in \mathcal{B}_\alpha$ and $g \in \mathcal{R}(M^3, \beta \wedge d\beta)\}$. Then the following lemmas hold.

Lemma 7. *$\mathcal{R}(M^3)$ and $\widetilde{\mathcal{F}}_\alpha$ are connected and contractible.*

Proof. For fixed $g_0 \in \mathcal{R}(M^3)$ and any $g \in \mathcal{R}(M^3)$, the family $\{tg_0 + (1-t)g\}_{t \in [0,1]}$ defines a contraction from $\mathcal{R}(M^3)$ to $\{g_0\}$. If g and g_0 are in $\widetilde{\mathcal{F}}_\alpha$ then so are $tg_0 + (1-t)g$ since $(tg_0 + (1-t)g)(X_\alpha, \cdot) = t\alpha + (1-t)\alpha = \alpha$. This ends the proof.

Lemma 8. *$\mathcal{F}_\alpha$ is connected and contractible.*

Proof. Any metric $g (\in \mathcal{F}_\alpha)$ has the following form on each Darboux coordinate (x, y, z) with $\alpha = xdy + dz$.

$$g = \begin{pmatrix} a & b & 0 \\ b & c + x^2 & x \\ 0 & x & 1 \end{pmatrix} \quad (a > 0, b > 0, ac - b^2 \equiv 1)$$

where a, b and c are some local functions. Fix the orthonormal framing

$$(X_\alpha, e_2, e_3) = \left(\frac{\partial}{\partial z}, \quad \frac{1}{\sqrt{a}} \frac{\partial}{\partial x}, \quad \frac{1}{\sqrt{a}} \left(-b \frac{\partial}{\partial x} + a \frac{\partial}{\partial y} - ax \frac{\partial}{\partial z} \right) \right)$$

on each Darboux coordinate, whose dual is

$$\left(\alpha (= xdy + dz), \quad \sqrt{a}dx + \frac{b}{\sqrt{a}} dy, \quad \frac{1}{\sqrt{a}} dy \right).$$

Put $h_\alpha = \alpha \otimes \alpha$. Then we have

$$h_\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x^2 & x \\ 0 & x & 1 \end{pmatrix}$$

on each Darboux coordinate. Note that h_α satisfies $h_\alpha(X_\alpha, \cdot) = \alpha$ and $h_\alpha(e, \cdot) = 0$ for any $e \in \ker \alpha$. Then for any $g \in \widetilde{\mathcal{F}}_\alpha$ inducing a volume form $f\nu$ $(f > 0)$, the family

$$\left((1-t) + t\frac{1}{f} \right) (g - h_\alpha) + h_\alpha \quad (t \in [0,1])$$

defines a retraction from $\widetilde{\mathcal{F}}_\alpha$ to $\mathcal{F}_\alpha$. Note that this retraction fixes any element of $\mathcal{F}_\alpha$ since $f \equiv 1$ holds in this case. Thus the lemma follows from Lemma 7.

Lemma 9. *Put* $\alpha_t = (1-t)\alpha + t\beta$ *for* $\beta \in \mathcal{B}_\alpha$. *Let* $\{Y_t\}_{t\in[0,1]}$ *be the family of vector fields determined by* $\alpha \wedge \beta = \iota_{Y_t}(\alpha_t \wedge d\alpha_t)$ *and* $\{\phi_t\}_{t\in[0,1]}$ *the family of diffeomorphisms on* M^3 *obtained by integrating* Y_t *under the initial condition* $\phi_0 = Id_{M^3}$. *Then* $\mathcal{F}_{\alpha_t} = (\phi_t)_*(\mathcal{F}_\alpha)$ $(t \in [0,1])$ *holds.*

Proof. On each Darboux coordinate (x, y, z) for α with $\alpha = xdy + dz$, we have

$$\beta - \alpha = pdx + qdy$$

for some functions p and q with $p_z = q_z = 0$ and $q_x - p_y > -1$. This yields

$$Y_t = \frac{1}{1 + t(q_x - p_y)}\left(-q\frac{\partial}{\partial x} + p\frac{\partial}{\partial y} - xp\frac{\partial}{\partial z}\right).$$

Then we have

$$\mathcal{L}_{Y_t}\alpha_t + \frac{d}{dt}\alpha_t = -pdx - qdy + pdx + qdy = 0.$$

Thus $\mathcal{F}_{\alpha_t} = (\phi_t)_*(\mathcal{F}_\alpha)$ holds for M is closed. This ends the proof.

These lemmas imply the following theorem.

Theorem 10. *For any contact form* α *on a closed oriented* 3*-manifold* M^3, *the above* $\mathcal{C}_\alpha$, $\mathcal{F}_\alpha$ *and* $\mathcal{B}_\alpha$ *are connected and contractible. Moreover,* $\mathcal{C}_\alpha$ *is fibred trivially by* $\{\mathcal{F}_\beta\}_{\beta\in\mathcal{B}_\alpha}$ *over the space* $\mathcal{B}_\alpha$ *with projection* $g \mapsto g(X_\alpha, \cdot)$.

Remark 11. We see that $\beta(\in \mathcal{B}_\alpha)$ satisfies $\int_{M^3} \beta \wedge d\beta = \int_{M^3} \alpha \wedge d\alpha$ from Lemma 9. Note that, however, this does not mean $\beta \wedge d\beta = \alpha \wedge d\alpha$ in general. It may be interesting to compare Lemma 9 and Theorem 10 with the Gray's stability theorem [3]. Note also that if $\pm\beta$ is in $\mathcal{B}_\alpha$ then $\mathcal{B}_\beta = \mathcal{B}_\alpha$ holds.

$\mathcal{C}_\alpha \cap \mathcal{C}_\beta$ may be non-empty even when $\pm\beta \notin \mathcal{B}_\alpha$. Here is an example.

Example 12. Put $M^3 = T^3 = \mathbf{R}^3/(2\pi\mathbf{Z})^3$, $\nu = dx \wedge dy \wedge dz$, $\alpha = \sin z dx + \cos z dy$, $\beta = -\cos z dx + \sin z dy$ and $\gamma = \sin x dy + \cos x dz$. Then we can easily see that the Eucridean metric $dx^2 + dy^2 + dz^2$ is in the intersection $\mathcal{C}_\alpha \cap \mathcal{C}_\beta \cap \mathcal{C}_\gamma$. Note that all these contact forms are, then, equivalent up to isometries.

4 Cartan structures. We recall a definition and one of the results in Geiges and Gonzalo [2].

Definition 13. We say that a pair (α, β) of contact forms on a closed oriented 3-manifold M^3 is a *Cartan structure* on M^3 if $\alpha \wedge d\alpha = \beta \wedge d\beta$ and $\alpha \wedge d\beta = \beta \wedge d\alpha = 0$ hold.

Theorem 14 ([2]). *Let* M^3 *be a closed orientable* 3*-manifold. Then* M^3 *admits a Cartan structure if and only if* M^3 *is diffeomorphic to a quotient of the Lie group* G *under a discrete and cocompact subgroup* Γ *acting from the left, where* G *is one of the following:*

1) $SU(2)$,
2) $\widetilde{SL}_2$, *the universal cover of* $PSL(2;\mathbf{R})$ *or*
3) $\widetilde{E}_2$, *the universal cover of the orientation-preserving isometry group of the Euclidean* $\mathbf{R}^2$.

The manifolds satisfying the above condition are called $SU(2)$-manifolds, $\widetilde{SL}_2$-manifolds or $\widetilde{E}_2$-manifolds respectively. As an application of Theorem 10 and this characterization theorem, we can prove the following theorem.

Theorem 15. *Let M^3 be a closed orientable 3-manifold. Then there are two contact forms α and β on M^3 satisfying*

1) $\alpha \wedge d\alpha = h\beta \wedge d\beta$ *for some function* $h > 0$ *and*

2) $X_\alpha \perp X_\beta$ *with respect to some* $g \in \mathcal{C}_\alpha \cap \mathcal{C}_\beta$

if and only if M^3 is diffeomorphic to a $SU(2)$-, $\widetilde{SL_2}$- or $\widetilde{E_2}$-manifold.

Proof. First, we prove the 'if' part. The Lie algebra of $G = SU(2), \widetilde{SL_2}$ or $\widetilde{E_2}$ admits a basis (e_1, e_2, e_3) with

$$[e_1, e_2] = \delta e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2$$

where $\delta = +1, -1$ or 0 respectively (see [2]). We regard these as left-invariant vector fields on G. Then take the dual frame $(\theta^1, \theta^2, \theta^3)$ and set $\alpha = \theta^1$ and $\beta = \theta^2$. Then the pair (α, β) is a Cartan structure. Take the metric g and the orientation of M by means of the oriented orthonormal basis (e_1, e_3, e_2). Then we have for example $d\alpha(e_3, e_2) = \alpha([e_2, e_3]) = 1 = \theta^3 \wedge \theta^2(e_3, e_2)$ and conclude the 'if' part.

Next, we prove the 'only if' part. Theorem 10 implies that there is the (unique) fibre $\mathcal{F}_{\alpha'}(\alpha' \in \mathcal{B}_\alpha)$ containing the metric g. Thus by changing α and β if necessary, we may assume $g \in \mathcal{F}_\alpha \cap \mathcal{F}_\beta$. Then we have $\ker \alpha \perp X_\alpha$ and $\ker \beta \perp X_\beta$. Thus $d\alpha|_{\ker \beta} = d\beta|_{\ker \alpha} = 0$ holds. This yields $\beta \wedge d\alpha = \alpha \wedge d\beta = 0$. The pair (α, β), therefore, forms a Cartan structure on M^3. Thus the theorem follows from Theorem 14.

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