AFFINE-DERIVED 3-DESIGNS

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Abstract. For any affine(or Hadamard) 3-(v,k,λ) design, four 3-designs are constructed. Namely a 3-$(4\lambda+4,\lambda+1,\binom{\lambda}{2})$ design, a 3-$(4\lambda+4,2\lambda+2,\binom{2\lambda+1}{2})$ design, a 3-$(4\lambda+4,\lambda+1,\binom{\lambda}{2})$ design and a 3-$(4\lambda+4,3\lambda+3,3\binom{3\lambda+2}{2})$ design. Moreover necessary conditions for the existence of simple such designs, i.e., with no repeated blocks, are also given.

1. Preliminaries. Let t,v,k,λ be integers such that $t \geq 0, k,\lambda > 0$ and $v > k+1$. A t-(v,k,λ) design (or t-design) is an ordered pair of points and blocks (V,B) where
(1) $|V| = v$, (2) B is a family of k-subsets of V, (3) every t-subset of V is contained in λ members of B.
A design will be called nontrivial if $t < k \leq v-2$. Moreover a t-design is also an s-design for every $s \leq t$. We shall denote by λ_s the number of blocks of B containing each s-subset of V. Thus

$$(\Delta) \qquad \lambda_s = \lambda\binom{v-s}{t-s}/\binom{k-s}{t-s}$$

see [2,7]. Let $r = \lambda_1$ and $b = \lambda_0 = |B|$. Then $bk = vr$.

An affine 3-design is a 3-$(4\lambda+4,2\lambda+2,\lambda)$ design. It is also called a Hadamard 3-design since it is an extension of a Hadamard 2-$(4\lambda+3,2\lambda+1,\lambda)$ design, see [2,4,7]. Using this the following properties of an affine 3-design (V,B) follow easily, see [2,7]: (i) $r = 4\lambda+3$, (ii) $\lambda_2 = 2\lambda+1$, (iii) $b = 8\lambda+6$, (iv) for each block $a \in B$, its compliment $a' = V - a$ is also a block of B, (v) $|a \cap c| = \lambda+1$, for all $a, c \in B, c \neq a, a'$.

Finally let A_1, A_2, A_3 be any three subsets of a set A. By the Sieve Principle see [3]

$$(*) \quad |A_1 \cup A_2 \cup A_3| = |A_1|+|A_2|+|A_3|-|A_1\cap A_2|-|A_1\cap A_3|-|A_2\cap A_3|+|A_1\cap A_2\cap A_3|\ldots$$

2. Designs Constructions.

Theorem 1. *Let (V,B) be an affine 3-$(4\lambda+4,2\lambda+2,\lambda)$ design where $\lambda > 2$. If $B_1 = \{a-c : a,c \in B, c \neq a,a'\}$, then (V,B_1) is a 3-$(4\lambda+4,\lambda+1,2\binom{\lambda}{2})$ design.*

Proof. Let $a - c \in B_1$. Hence

$$|a-c| = |a - a\cap c| = |a| - |a \cap c| = 2\lambda+2-\lambda-1 = \lambda+1,$$

which is independent of a, c. Now let $\{P,Q,S\}$ be any set of three points of V contained in $a - c \in B_1$. Therefore $\{P,Q,S\} \subseteq a$ and $\{P,Q,S\} \cap c = \emptyset$. Moreover there are λ choices of a. To find number of choices of c, let $l(P), l(Q), l(S)$ be the set of blocks of B containing

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P, Q and S respectively. By the Sieve Principle (eqn.($*$) above) and properties (i,ii), the number of blocks of B which contains at least one of P, Q and S is

$$\begin{aligned}|l(P) \cup l(Q) \cup l(S)| &= |l(P)| + |l(Q)| + |l(S)| - |l(P) \cap l(Q)| \\ &- |l(P) \cap l(S)| - |l(Q) \cap l(S)| + |l(P) \cap l(Q) \cap l(S)| \\ &= 3r - 3\lambda_2 + \lambda.\end{aligned}$$

Using this and properties (i,ii,iii) therefore the number of blocks of B disjoint from $\{P, Q, S\}$ is

$$b - 3r + 3\lambda_2 - \lambda = 8\lambda + 6 - 12\lambda - 9 + 6\lambda + 3 - \lambda = \lambda.$$

Since $c \neq a'$, the number of choices of c is therefore $\lambda - 1$. Hence $\{P, Q, S\}$ is contained in $\lambda(\lambda - 1) = 2\binom{\lambda}{2}$ blocks of B_1. Therefore (V, B_1) is a 3-$(4\lambda + 4, \lambda + 1, 2\binom{\lambda}{2})$ design. Since $\lambda > 2$, then $3 < \lambda + 1 < 4\lambda + 4 - 2$ and hence design is not trivial.

Next we investigate repetition of blocks in (V, B_1)

Theorem 2. *Let (V, B) be an affine 3-$(4\lambda+4, 2\lambda+2, \lambda)$ design and a, a', c, c', d, d' be distinct blocks of B. Then*
1) $V = a \cup c \cup d$ iff $|a \cap c \cap d| = \lambda + 1$
2) If $|a \cap c \cap d| = \lambda + 1$ then λ is an odd integer

Proof. Suppose $V = a \cup c \cup d$. As before by Sieve Principle and property (v), we get

$$4\lambda + 4 = |V| = |a \cup c \cup d| = 3(2\lambda + 2) - 3(\lambda + 1) + |a \cap c \cap d|.$$

Simplifying it gives $|a \cup c \cup d| = \lambda + 1$. Conversely suppose $|a \cap c \cap d| = \lambda + 1$. By Sieve Principle and property (v) again

$$\begin{aligned}|a \cup c \cup d| &= 3(2\lambda + 2) - 3(\lambda + 1) + |a \cap c \cap d| = 3\lambda + 3 + \lambda + 1 \\ &= 4\lambda + 4 = |V|.\end{aligned}$$

Hence $V = a \cup c \cup d$.
2) Obviously $a \cap c, a \cap d, c \cap d \supseteq a \cap c \cap d$. By property (v), therefore

$$|a \cap c| = |a \cap d| = |c \cap d| = \lambda + 1 = |a \cap c \cap d|.$$

Using these relations we get $a \cap c = a \cap d = c \cap d = a \cap c \cap d$ and hence for any $f \in B - \{a, a', c, c', d, d'\}, f \cap a \cap c = f \cap a \cap d = f \cap c \cap d = f \cap a \cap c \cap d$. Let

$$|f \cap a \cap c| = |f \cap a \cap d| = |f \cap c \cap d| = |f \cap a \cap c \cap d| = i. \tag{1}$$

By part (1) of the theorem $V = a \cup c \cup d$ and therefore

$$\begin{aligned}2\lambda + 2 = |f| = |f \cap (a \cup c \cup d)| &= |(f \cap a) \cup (f \cap c) \cup (f \cap d)| \\ &= |f \cap a| + |f \cap c| + |f \cap d| - |f \cap a \cap c| - |f \cap a \cap d| - |f \cap c \cap d| \\ &+ |f \cap a \cap c \cap d| = 3(\lambda + 1) - 2i,\end{aligned}$$

see eqn.(1). Therefore $i = \frac{\lambda+1}{2}$ and hence λ must be an odd integer.

Theorem 3. *Let* (V, B) *be an affine 3-*$(4\lambda+4, 2\lambda+2, \lambda)$ *design. If* λ *is even, then* (V, B_1) *consists of two copies of a 3-*$(4\lambda+4, \lambda+1, \binom{\lambda}{2})$ *design which has no repeated blocks.*

Proof. Let $a-c, d-e \in B_1$ where $\{a,c\} \neq \{d,e\}$ and such that $a-c = d-e (\neq \emptyset)$. Therefore $e \cap a - e \cap c = e \cap (a-c) = e \cap (d-e) = \emptyset$, which implies that $e \cap a \subseteq e \cap c$. Similarly $c \cap d \subseteq e \cap c$. Therefore

$$e \cap a, c \cap d \subseteq c \cap e \tag{1}$$

Since $a - c = d - e \neq \emptyset$, therefore $e \neq a, c'$ and $c \neq d$. By property (v) this implies that $|e \cap a|, |c \cap d| = 0$ or $\lambda+1$ and $|c \cap e| = \lambda+1$ or $2\lambda+2$. Equivalently this implies that: (i) Either $e = a'$ or $|e \cap a| = \lambda+1$, (ii) Either $d = c'$ or $|c \cap d| = \lambda+1$ and (iii) Either $c = e$ or $|c \cap e| = \lambda+1$. Using the above assumption that $a-c = d-e \neq \emptyset$, we investigate the various possibilities of (i,ii,iii) in the following cases: (1) $e = a', d = c' \in B$. Since $c' \neq a'$ therefore $c' - a' \in B_1$. Moreover $c' - a' = a - c$ and hence $a - c \in B_1$ is a repeated block of B_1. (2) $e = a', |c \cap d| = \lambda+1$. Since $a - c \in B_1$ and $c' \in B$, therefore $|c \cap e| = |c \cap a'| = |c \cap a| = \lambda+1$, by property (v). This implies that $c \cap d = c \cap e$ since $c \cap d \subseteq c \cap e$, see eqn.(1) above. Therefore $c \cap d \cap e = c \cap e$ and consequently $|c \cap d \cap e| = |c \cap e| = \lambda+1$. Since $|c \cap e| = |c \cap d| = |d \cap e| = \lambda+1$, from above, therefore c, c', d, d', e, e' are distinct. By theorem 2 part 2 therefore λ is an odd integer. (3) $|e \cap a| = \lambda+1$ and $d = c'$. Since $d - e \in B_1$, therefore $|e \cap c'| = |e \cap d| = \lambda+1$ and hence $|e \cap c| = \lambda+1 = |e \cap a|$. By eqn.(1) $e \cap a \subseteq c \cap e$. Consequently $c \cap e = a \cap e$ which implies that $a \cap c \cap e = a \cap e$. Therefore $|a \cap c|, |c \cap e|, |a \cap e|, |a \cap c \cap e| = \lambda+1$ and hence a, a', c, c', e, e' are distinct. By Theorem 2 part 2 it follows that λ is an odd integer. (4) $|a \cap e| = |c \cap d| = \lambda+1$. By eqn.(1) and referring to (iii) of this proof this case splits into two parts as follows: I) $|c \cap e| = \lambda+1$. By eqn.(1) therefore $e \cap a = c \cap e = d \cap c$ which implies that $a \cap c \cap e = c \cap e$ and as in case (3) λ must be an odd integer.
II) $c = e$. Therefore $d - c = d - e = a - c \in B_1$. This implies that

$$\emptyset = a' \cap (a-c) = a' \cap (d-c) = a' \cap d - a' \cap c. \tag{2}$$

It also implies that $d \neq a, a', c$, since we are investigating distinct repeated blocks. By eqn.(2) therefore $a' \cap d = a' \cap c$ and hence $a' \cap c \cap d = a' \cap c$. Moreover $|a' \cap d| = |a \cap d| = \lambda+1$, and $|a' \cap c| = |a \cap c| = \lambda+1$. Similarly as in case (3) λ must be an odd integer. Since λ is even, it follows by case (1) above that each block of B_1 is repeated once and therefore (V, B_1) consists of two copies of a 3-$(4\lambda+4, \lambda+1, \binom{\lambda}{2})$ design which has no repeated blocks.

Corollary 1.*If there exists a Hadamard matrix of order* $4(\lambda+1)$ *for even* $\lambda \geq 4$, *then there exists a simple 3-*$(4\lambda+4, \lambda+1, \lambda(\lambda-1)/2)$ *design.*

Remark 1. Corollary 1 is a corollary of Theorems 3 and 6.

Remark 2. If there exists a Hadamard matrix of order $4n$ for a positive integer n, then there exists an affine 3-$(4n, 2n, n-1)$ design. It was conjectured by Hadamard [6] that there exists a Hadamard matrix of order $4n$ for any positive integer n. It is known (cf. Colbourn and Dinitz [5]) that (1) there exists a Hadamard matrix of order $4n$ for any positive integer n less than or equal to 107 and (2) there is no positive integer n such that the nonexistence of Hadamard matrices of order $4n$ is known and (3) there exists a Hadamard matrix of order $4n$ for infinitely many even integers n. Hence Corollary 1 gives a new series of simple 3-designs.

Theorem 4. *Let* (V, B) *be an affine 3-*$(4\lambda+4, 2\lambda+2, \lambda)$ *design, If* $B_2 = \{(a-c) \cup (c-a) : a, c \in B, c \neq a, a'\}$, *then* (V, B_2) *is a 3-*$(4\lambda+4, 2\lambda+2, 2\binom{2\lambda+1}{2})$ *design.*

Proof. Let $(a-c)\cup(c-a)\in B_2$. By Sieve Principle and property (v)

$$\begin{aligned}|(a-c)\cup(c-a)| &= |a-c|+|c-a| = |a|-|a\cap c|+|c|-|a\cap c|\\ &= 2(2\lambda+2)-2(\lambda+1) = 2\lambda+2,\end{aligned}$$

which is independent of a and c. Next let $\{P,Q,S\}$ be any set of three points of V contained in $(a-c)\cup(c-a)\in B_2$. There are two types of such block:(1) $\{P,Q,S\}\subseteq a$, $\{P,Q,S\}\cap c=\emptyset$. Hence $\{P,Q,S\}\subseteq a-c\in B_1$. By theorem 1 therefore $\{P,Q,S\}$ is contained in $\lambda(\lambda-1)$ blocks of this type. (2) One block, a say, contains two points only, $\{P,Q\}$ say, and the other block c contains S only. Since the number of blocks of B which contain $\{P,Q,S\}$ is λ and the number of blocks which contain $\{P,Q\}$ is $\lambda_2=2\lambda+1$ by property (ii), therefore there are $\lambda_2-\lambda=\lambda+1$ choices of a. Since each of $\{P,S\},\{Q,S\}$ is on λ_2 blocks of B and $\{P,Q,S\}$ is on λ blocks, this implies that S is on

$$r-2\lambda_2+\lambda = 4\lambda+3-2(2\lambda+1)+\lambda = \lambda+1,$$

blocks of B which do not contain $\{P,Q\}$; see properties above. Since $c\neq a'$ therefore the number of choices of c is $\lambda+1-1=\lambda$. Consequently the number of blocks $(a-c)\cup(c-a)$ such that a contains $\{P,Q\}$ only and c contains S only is $\lambda(\lambda+1)$. Since $\{P,Q\}$ is one of three ways of choosing two points from $\{P,Q,S\}$, therefore the number of blocks of this type is $3\lambda(\lambda+1)$. Combining the two counts in the two types implies that $\{P,Q,S\}$ is contained in

$$\lambda(\lambda-1)+3\lambda(\lambda+1) = 4\lambda^2+2\lambda = 2\binom{2\lambda+1}{2}$$

blocks of B_2. Therefore (V,B_2) is a 3-$(4\lambda+4, 2\lambda+2, 2\binom{2\lambda+1}{2})$ design which is nontrivial since $3<2\lambda+2<4\lambda+4-2$.

Next we investigate repetition of blocks in B_2.

Theorem 5. *Let (V,B) be an affine 3-$(4\lambda+4,2\lambda+2,\lambda)$ design. If λ is even, then (V,B_2) consists of two copies of a 3-$(4\lambda+4,2\lambda+2,\binom{2\lambda+1}{2})$ design which has no repeated blocks.*

Proof. Let $(a-c)\cup(c-a),(d-e)\cup(e-d)$ be any two blocks of B_2, where $\{a,c\}\neq\{d,e\}$ and such that

$$(a-c)\cup(c-a)=(d-e)\cup(e-d) \tag{1}$$

$$\begin{aligned}a-c &= a\cap((a-c)\cup(c-a)) = a\cap((d-e)\cup(e-d)) = (a\cap(d-e))\cup(a\cap(e-d))\\ &= (a\cap d-a\cap e)\cup(a\cap e-a\cap d)) = (a\cap d-a\cap d\cap e)\cup(a\cap e-a\cap d\cap e).\end{aligned}$$

Using this we get

$$\begin{aligned}\lambda+1 &= |a\cap c'| = |a-c| = |(a\cap d-a\cap d\cap e)\cup(a\cap e-a\cap d\cap e)|\\ &= |a\cap d|-|a\cap d\cap e|+|a\cap e|-|a\cap d\cap e| = |a\cap d|+|a\cap e|-2|a\cap d\cap e|.\end{aligned} \tag{2}$$

Since $\{a,c\}\neq\{d,e\}$ we can assume that $a\neq d,e$ and hence $|a\cap d|,|a\cap e|=0$, or $\lambda+1$, see property (v). If $|a\cap e|=|a\cap d|=\lambda+1$, then substituting in eqn.(2) we get $|a\cap d\cap e|=\frac{1}{2}(\lambda+1)$, contradiction since λ is even. Therefore exactly one of $|a \cap e|$ and $|a \cap d|$ is equal to 0. If $|a \cap d| = 0$, then $d = a'$. Substituting in eqn.(1) we get $(a - c) \cup (c - a) = (a' - e) \cup (e - a')$. Using this equation as we did with eqn.(1) above we get

$$\begin{aligned} c - a &= c \cap ((a - c) \cup (c - a)) = c \cap ((a' - e) \cup (e - a')) \\ &= (c \cap a' - c \cap a' \cap e) \cup (c \cap e - c \cap a' \cap e). \end{aligned}$$

Similarly and as in eqn.(2) this gives

$$\lambda + 1 = |c - a| = |c \cap a'| + |c \cap e| - 2|c \cap a' \cap e| = \lambda + 1 + |c \cap e| - 2|c \cap a' \cap e|$$

since $c \neq a', a$. Therefore $|c \cap a' \cap e| = \frac{1}{2}|c \cap e|$. Since $|c \cap e| = 0$ or $\lambda + 1$ or $2\lambda + 2$, then $c = e'$ or $c \neq e, e'$ or $c = e$ respectively. Therefore we have three cases to consider: (i) If $c = e'$ then $e = c'$. Moreover $d = a'$, from above. Using this we get

$$(a - c) \cup (c - a) = (a \cap c') \cup (c \cap a') = (c' - a') \cup (a' - c') = (d - e) \cup (e - d).$$

Since $(c' - a') \cup (a' - c') \in B_2$, therefore $(a - c) \cup (c - a)$ is a repeated block.(ii) If $c \neq e, e'$, therefore $|c \cap e| = \lambda + 1$. Since $|c \cap a' \cap e| = \frac{1}{2}|c \cap e|$ from above, therefore $|c \cap a' \cap e| = \frac{1}{2}(\lambda + 1)$. Contradiction since λ is even. (iii) If $c = e$ then by substituting in eqn.(1) we get $(a - c) \cup (c - a) = (a' - c) \cup (c - a')$, since $d = a'$ from above. Contradiction, since the two sides of this equation are non empty and have no common points. Therefore by case (i) above each block of B_2 is repeated once and therefore (V, B_2) consists of two copies of a 3-$(4\lambda + 4, 2\lambda + 2, \binom{2\lambda+1}{2})$ design which has no repeated blocks.

Corollary 2. *If there exists a Hadamard matrix of order $4(\lambda + 1)$ for even $\lambda \geq 4$, then there exists a simple 3-$(4\lambda + 4, 2\lambda + 2, \lambda(2\lambda + 1))$ design.*

Remark 3. Corollary 2 gives a new series of simple 3-designs.

Theorem 6.*Let (V, B) be an affine 3-$(4\lambda + 4, 2\lambda + 2, \lambda)$ design. If $B_3 = \{a \cap c : a, c \in B, c \neq a, a'\}$ and $\lambda > 2$, then (V, B_3) is a 3-$(4\lambda + 4, \lambda + 1, \binom{\lambda}{2})$ design which has no repeated blocks if λ is even.*

Proof. Let $a \cap c \in B_3$, then by property (v) $|a \cap c| = \lambda + 1$ which is independent of a and c. Let $\{P, Q, S\}$ be any set of three points of V. Since there are λ blocks of B containing it, it follows that the number of blocks $a \cap c \in B_3$ which contain it is $\binom{\lambda}{2}$. Therefore (V, B_3) is a 3-$(4\lambda + 4, \lambda + 1, \binom{\lambda}{2})$ design which is not trivial since $3 < \lambda + 1 < 4\lambda + 4 - 2$. Let $a \cap c, d \cap e \in B_3$ where $\{a, c\} \neq \{d, e\}$ and such that $a \cap c = d \cap e$. Therefore $a \cap c \cap d = d \cap e$ and $|a \cap c \cap d| = |d \cap e| = \lambda + 1$. Since $\{a, c\} \neq \{d, e\}$ assume that $d \neq a, c$. Therefore a, a', c, c', d, d' are distinct. By theorem 2, λ must be odd. Contradiction since λ is even. Therefore (V, B_3) has no repeated blocks.

Remark 4. By Corollary 1 and Remarks 1 and 2 above theorem 6 gives a new series of simple 3- designs.

Theorem 7. *Let (V, B) be an affine 3-$(4\lambda + 4, 2\lambda + 2, \lambda)$ design. If $B_4 = \{a \cup c : a, c \in B, c \neq a, a'\}$, then (V, B_4) is a 3-$(4\lambda + 4, 3\lambda + 3, 3\binom{3\lambda+2}{2})$ design which has no repeated blocks if λ is even.*

Proof. Let $a \cup c \in B_4$. Hence $|a \cup c| = |a| + |c| - |a \cap c| = 4\lambda + 4 - \lambda - 1 = 3\lambda + 3$ which is independent of a and c. For any $a, c \in B, (a \cup c)' = a' \cap c' \in B_3$, see theorem 6 and property (iv) above. Conversely for any $a \cap c \in B_3, (a \cap c)' = a' \cup c' \in B_4$. This implies that $B_4 = \{a' : a \in B_3\}$. Let $\{P, Q, S\}$ be any set of three points of V and $l(P), l(Q)$ and $l(S)$ be the set of blocks of B_3 containing P, Q and S respectively. By the Sieve Principle the number of blocks of B_3 which contains at least one of P, Q and S is equal to

$$|l(P) \cup l(Q) \cup l(S)| = 3r - 3\lambda_2 + \binom{\lambda}{2}, \tag{1}$$

where r and λ_2 here are the blocks of B_3 which contain each point and each two points of V respectively. By eqn.(Δ) above $r = \frac{1}{2}(4\lambda+3)(4\lambda+2)$ and $\lambda_2 = \frac{1}{2}\lambda(4\lambda+2)$. Using the relation $bk = vr$ for the design (V, B_3) we get $b(\lambda+1) = (4\lambda+4)r$ which implies that $b = 4r = 2(4\lambda+3)(4\lambda+2)$. Using eqn.(1) above the number of blocks of B_3 disjoint from $\{P, Q, S\}$ is therefore equal to $b - 3r + 3\lambda_2 - \binom{\lambda}{2}$. Since $B_4 = \{a'; a \in B_3\}$ therefore $\{P, Q, S\}$ is contained in $b - 3r + 3\lambda_2 - \binom{\lambda}{2}$ blocks of B_4. Substituting for b, r and λ_2 this number is equal to

$$\begin{aligned} b - 3r + 3\lambda_2 - \binom{\lambda}{2} &= 2(4\lambda+3)(4\lambda+2) - \frac{3}{2}(4\lambda+3)(4\lambda+2) \\ &\quad + \frac{3}{2}\lambda(4\lambda+2) - \frac{1}{2}\lambda(\lambda-1) \\ &= 3\binom{3\lambda+2}{2}. \end{aligned}$$

Hence (V, B_4) is a 3-$(4\lambda+4, 3\lambda+3, 3\binom{3\lambda+2}{2})$ design. Since λ is even, therefore B_3 has no repeated blocks by theorem 6 and hence B_4 has not repeated blocks as $B_4 = \{a' : a \in B_3\}$.

Remark 5. Using the points and blocks of an affine 3-$(4\lambda+4, 2\lambda+2, \lambda)$ Cameron has constructed a 2-design see [4].

Remark 6. Although we have focused on affine 3-designs only, yet our methods may apply for any affine design. In fact it applies to any symmetric design.

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