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RELATIONS BETWEEN SOME CLASSES OF FUNCTIONS

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ABSTRACT. In this paper the necessary and sufficient conditions for the inclusion of classes H^{ω} and $V[\upsilon(n)]$ in the class $BV(p(n)\uparrow\infty)$ is found.

It is well-known that the notion of variation of a function was introduced by C. Jordan in 1881 in the paper [6], devoted to the convergence of Fourier series. In 1924 N. Wiener [11] generalized this notion and introduced the notion of p-variation. L. Young [12] introduced the notion of Φ-variation of a function.

Definition 1 (see [12]) *Let Φ be a strictly increasing continuous function on $[0,+\infty)$ and $\Phi(0) = 0$. f will be said to have bounded Φ-variation on $[0,1]$, or $f \in V_{\Phi}$ if*

$$\upsilon_{\Phi}(f) = \sup_{\Pi} \sum_{k=1}^{n} \Phi(|f(x_k) - f(x_{k-1})|) < \infty,$$

where $\Pi = \{0 \le x_0 < x_1 < \cdots < x_n \le 1\}$ is an arbitrary partition.

If $\Phi(u) = u$ the V_{Φ} coincides with the Jordan class V and when $\Phi(u) = u^p$, $p > 1$ it coincides with the Wiener class V_p.

$C(0,1)$ and $B(0,1)$ are, respectively, spaces of continuous and bounded functions given on $[0,1]$.

In 1974 Z.A. Chanturia [3] introduced the notion of the modulus of variation of a function.

Definition 2 (see [3]) *The modulus of variation of function $f \in B(0,1)$ is said to be the function $\upsilon(n,f)$ defined as: $\upsilon(0,f) = 0$ and for $n \ge 1$*

$$\upsilon(n,f) = \sup_{\Pi_n} \sum_{k=0}^{n-1} |f(t_{2k+1}) - f(t_{2k})|,$$

where Π_n is an arbitrary partition of $[0,1]$ into n disjoint intervals (t_{2k}, t_{2k+1}), $k = 0,1,...,n-1$.

If $\upsilon(n)$ is a non-decreasing and convex upwards function and $\upsilon(0) = 0$ then $\upsilon(n)$ will be called the modulus of variation [3].

Let the modulus of variation $\upsilon(n)$ is given, then the class of functions f, given on $[0,1]$, for which $\upsilon(n,f) = O(\upsilon(n))$ when $n \to \infty$, will be denoted by $V[\upsilon(n)]$ [3].

In 1990 H. Kita and K. Yoneda [7] introduced the notion of the generalized Wiener's class $BV(p(n)\uparrow p)$.

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Let f be a function defined on $(-\infty,+\infty)$ with period 1. Δ is said to be a partition with period 1, if

$$\Delta : \cdots < t_{-1} < t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} < \cdots \tag{1}$$

satisfies $t_{k+m} = t_k + 1$ for $k = 0, \pm 1, \pm 2, ...$, where m is a positive integer.

Definition 3 (see [7]) *When $1 \le p(n) \uparrow p$ as $n \to +\infty$, where $1 \le p \le +\infty$, f is said be a function of $BV(p(n)\uparrow p)$ if and only if*

$$V(f; p(n)\uparrow p) =$$

$$= \sup_{n\ge 1} \sup_{\Delta} \left\{ \left(\sum_{k=1}^{m} |f(t_k) - f(t_{k-1})|^{p(n)} \right)^{1/p(n)} : \rho(\Delta) \ge \frac{1}{2^n} \right\} < +\infty,$$

where $\rho(\Delta) = \inf_k |t_k - t_{k-1}|$.

When $p(n) = p$ for all n, $BV(p(n)\uparrow p)$ coincides with V_p which is the Wiener's classes of bounded p-variation.

If $f \in C(0,1)$, then the function

$$\omega(\delta, f) = \max\{|f(x) - f(y)| : |x - y| \le \delta,\ x, y \in [0,1]\}$$

is called the modulus of continuity of the function f.

The modulus of continuity of an arbitrary function $f \in C(0,1)$ has the following properties:

1. $\omega(0) = 0$;
2. $\omega(\delta)$ is nondecreasing;
3. $\omega(\delta)$ is continuous on $[0,1]$;
4. $\omega(\delta_1 + \delta_2) \le \omega(\delta_1) + \omega(\delta_2)$ for $0 \le \delta_1 \le \delta_2 \le \delta_1 + \delta_2 \le 1$.

An arbitrary function $\omega(\delta)$ which is defined on $[0,1]$ and has the properties 1-4 is called the modulus of continuity.

If the modulus of continuity $\omega(\delta)$ is given then H^ω denoted the class of function $f \in C(0,1)$ for which $\omega(\delta, f) = O(\omega(\delta))$ as $\delta \to 0$.

The relation between different classes of generalized bounded variation was taken into account in the works of Avdaspahic [1],Kovocik [8], Belov [2], Chanturia [4] and Medvedieva [9].

H.Kita and K.Yoneda [7] proved some sufficient conditions for the inclusion of classes H^ω and $V[\upsilon(n)]$ in the class $BV(p(n)\uparrow\infty)$. In this paper the necessary and sufficient conditions for this inclusion is found.In particular,we prove the followings

Theorem 1 $H^\omega \subset BV(p(n)\uparrow\infty)$ *if and only if*

$$\omega(t) = O\left(t^{1/p([\log_2 1/t])}\right) \text{ as } t \to 0+. \tag{2}$$

Theorem 2 $V[\upsilon(n)] \subset BV(p(n)\uparrow\infty)$ *if and only if*

$$\overline{\lim_{n\to\infty}} \left(\sum_{k=1}^{2^n} (\upsilon(k) - \upsilon(k-1))^{p(n)} \right)^{1/p(n)} < +\infty. \tag{3}$$

For the proof of this theorems two lemmas are needed:

Lemma 1 *(Oskolkov [10]) Let there be given disjoint intervals* $\Delta_k \subset [0,1]$, $k = 1, 2, ...,$ *and* $\{g_k : k \geq 1\}$ *be a sequence of periodic functions with period 1 such that* $g_k(x) = 0$ *for* $x \in [0,1] \setminus \Delta_k$, *if* $\omega(\delta, g_k) \leq \omega(\delta)$ *and the functions* g *is defined by*

$$g(x) = \sum_{k=1}^{\infty} g_k(x)$$

then

$$\omega(\delta, g) \leq 2\omega(\delta).$$

Lemma 2 *(see [5], p. 111) Let* $0 \leq a_n \downarrow$, $0 \leq b_n \downarrow$, *and let the relations*

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i$$

be true for $k = 1, 2, ..., m$. *Then for convex functions* Φ *the inequality*

$$\sum_{i=1}^{m} \Phi(a_i) \leq \sum_{i=1}^{m} \Phi(b_i)$$

holds.

Proof of Theorem 1. Let $f \in H^{\omega}$ and Δ be a partition defined by (1) such that $\rho(\Delta) \geq \frac{1}{2^n}$. Then from the condition of the theorem we get

$$\left(\sum_{j=1}^{m} \left| f(t_j) - f(t_{j-1}) \right|^{p(n)} \right)^{1/p(n)}$$

$$\leq \left(\sum_{j=1}^{m} \left(\omega(t_j - t_{j-1}, f) \right)^{p(n)} \right)^{1/p(n)}$$

$$= O\left(\left(\sum_{j=1}^{m} \left(\omega(t_j - t_{j-1}) \right)^{p(n)} \right)^{1/p(n)} \right)$$

$$= O\left(\left(\sum_{j=1}^{m} (t_j - t_{j-1})^{\frac{p(n)}{p\left(\left[\log \frac{1}{t_j - t_{j-1}}\right]\right)}} \right)^{1/p(n)} \right)$$

$$= O\left(\left(\sum_{j=1}^{m} (t_j - t_{j-1})^{\frac{p(n)}{p\left(\left[\log \frac{1}{\rho(\Delta)}\right]\right)}} \right)^{1/p(n)} \right)$$

$$= O\left(\left(\sum_{j=1}^{m}(t_j - t_{j-1})\right)^{1/p(n)}\right) = O(1) \quad \text{as } n \to \infty.$$

Therefore $f \in BV(p(n) \uparrow \infty)$ holds.

Next we suppose that $\{p(n) : n \geq 1\}$ and $\omega(\delta)$ does not satisfy (2). As an example we construct function from H^ω which is not in $BV(p(n) \uparrow \infty)$.

Since $\omega(t)\left(t^{1/p([\log_2 1/t])}\right)^{-1}$ is not bounded by hypothesis, there exists a sequence of positive numbers $\{u_k' \downarrow 0 : k \geq 1\}$ such that

$$\omega(u_k')(u_k')^{-1/p\left([\log_2 1/u_k']\right)} \to \infty \text{ as } k \to \infty.$$

Then it is evident that there exists a sequence $\{u_k : k \geq 1\} \subset \{u_k' : k \geq 1\}$ such that

$$4\sqrt{\frac{u_k}{\omega(u_k)^{p([\log 1/u_k])}}} + 5u_k \leq u_{k-1}. \tag{4}$$

Consider the function f_k defined by

$$f_k(x) = \begin{cases} \omega(u_k), \text{if } x = (4j+3)u_k,\ j = 0, 1, 2, ..., m_k; \\ 0 \text{ if } x \in [0, u_k] \cup [(4m_k+5)u_k, 1],\ x = (4j+1)u_k, \\ j = 1, 2, ..., m_k; \\ \text{is linear and continuous for other } x \in [0,1], \end{cases}$$

where

$$m_k = \left[\sqrt{\frac{1}{\omega(u_k)^{p([\log 1/u_k])}u_k}}\right].$$

Let

$$f_0(x) = \sum_{k=1}^{\infty} f_k(x),\ f_0(0) = 0$$

and

$$f_0(x+l) = f_0(x),\ \ l \in Z.$$

First we prove that $f_0 \in H^\omega$. Let $\delta \leq u_k$. Since $\frac{\omega(\delta_1)}{\delta_1} \leq 2\frac{\omega(\delta_2)}{\delta_2}$, $0 < \delta_2 < \delta_1$, it follows that

$$\omega(\delta, f_k) = O\left(\delta\frac{\omega(u_k)}{u_k}\right) = O(\omega(\delta)). \tag{5}$$

Let $\delta > u_k$. Since $\omega(\delta)$ is non-decreasing function we get

$$\omega(\delta, f_k) \leq 2\|f_k\|_C = 2\omega(u_k) \leq 2\omega(\delta). \tag{6}$$

From (5)and (6) we have

$$\omega(\delta, f_k) = O(\omega(\delta)). \tag{7}$$

From Lemma 1 and by (4), (7) we obtain

$$f_0 \in H^\omega.$$

Next we shall prove that $f_0 \notin BV\left(p\left(n\right)\uparrow\infty\right)$. From the construction of the function we get

$$\left(\sum_{j=1}^{m_k}\left|f_0\left(\left(4j+3\right)u_k\right)-f_0\left(\left(4j+1\right)u_k\right)\right|^{p([\log 1/u_k])}\right)^{1/p([\log 1/u_k])}$$

$$=\left(\sum_{j=1}^{m_k}\left|f_k\left(\left(4j+3\right)u_k\right)-f_k\left(\left(4j+1\right)u_k\right)\right|^{p([\log 1/u_k])}\right)^{1/p([\log 1/u_k])}$$

$$=\left(\sum_{j=1}^{m_k}\left|f_k\left(\left(4j+3\right)u_k\right)\right|^{p([\log 1/u_k])}\right)^{1/p([\log 1/u_k])}$$

$$=\left(\sum_{j=1}^{m_k}\omega\left(u_k\right)^{p([\log 1/u_k])}\right)^{1/p([\log 1/u_k])}$$

$$=\omega\left(u_k\right)m_k^{1/p([\log 1/u_k])}$$

$$\geq c\omega\left(u_k\right)\left(\sqrt{\frac{1}{\omega\left(u_k\right)^{p([\log 1/u_k])}u_k}}\right)^{1/p([\log 1/u_k])}$$

$$=c\sqrt{\omega\left(u_k\right)u_k^{-1/p([\log 1/u_k])}}\to\infty \text{ as } k\to\infty.$$

Therefore we get $f_0 \notin BV\left(p\left(n\right)\uparrow\infty\right)$ and the proof is complete.

Proof of Theorem 2. Let $f\in V\left[\upsilon\left(n\right)\right]$ and $\Delta:\cdots<t_{-1}<t_0<t_1<\cdots<t_m<t_{m+1}<\cdots$ be any partition with period 1 and $\rho\left(\Delta\right)\geq\frac{1}{2^n}$. Without loss of generality it may be assumed that

$$\left|f\left(t_j\right)-f\left(t_{j-1}\right)\right|\geq\left|f\left(t_{j+1}\right)-f\left(t_j\right)\right|,\ \ j=1,...,m-1.$$

It is evident that

$$\sum_{j=1}^{m}\left|f\left(t_j\right)-f\left(t_{j-1}\right)\right|\leq\upsilon\left(m\right)=\sum_{j=1}^{m}\left(\upsilon\left(j\right)-\upsilon\left(j-1\right)\right).$$

Since $\upsilon\left(n\right)$ is upwards convex,for any $n\geq 1$

$$\upsilon\left(n+1\right)-\upsilon\left(n\right)\leq\upsilon\left(n\right)-\upsilon\left(n-1\right), \tag{8}$$

if we take $a_k=\left|f\left(t_k\right)-f\left(t_{k-1}\right)\right|$, $b_k=\upsilon\left(k\right)-\upsilon\left(k-1\right)$ and $\Phi\left(u\right)=u^{p(n)}$, and apply Lemma 2, from the condition of the theorem we get

$$\sum_{j=1}^{m}\left|f\left(t_j\right)-f\left(t_{j-1}\right)\right|^{p(n)}$$

$$\leq\sum_{j=1}^{m}\left(\upsilon\left(j\right)-\upsilon\left(j-1\right)\right)^{p(n)}$$

$$\leq \sum_{j=1}^{2^n} (v(j) - v(j-1))^{p(n)},$$

$$\left(\sum_{j=1}^{m} |f(t_j) - f(t_{j-1})|^{p(n)} \right)^{1/p(n)}$$

$$\leq \left(\sum_{j=1}^{2^n} (v(j) - v(j-1))^{p(n)} \right)^{1/p(n)} \leq c < \infty, \text{ for } n = 1, 2, ...$$

Therefore we proved that $f \in BV(p(n) \uparrow \infty)$.

Next we suppose that the condition (3) does not satisfy. As an example we construct function from $V[v(n)]$ which is not in $BV(p(n) \uparrow \infty)$.

Since

$$\overline{\lim_{n\to\infty}} \left(\sum_{j=1}^{2^n} (v(j) - v(j-1))^{p(n)} \right)^{1/p(n)} = \infty,$$

there exists a sequence of integers $\{m_k : k \geq 1\}$ such that

$$\lim_{k\to\infty} \left(\sum_{j=1}^{2^{m_k}} (v(j) - v(j-1))^{p(m_k)} \right)^{1/p(m_k)} = \infty. \tag{9}$$

We choose a monotone increasing sequence of positive integers $\{n_k : k \geq 1\} \subset \{m_k : k \geq 1\}$ such that

$$p(n_k) \geq n_{k-1}. \tag{10}$$

$$n_k \geq 2n_{k-1} \tag{11}$$

From (10) it is evident that

$$\left(\sum_{j=1}^{2^{n_{k-1}-n_{k-2}-1}} (v(j) - v(j-1))^{p(n_k)} \right)^{1/p(n_k)} \tag{12}$$

$$\leq c 2^{\frac{n_{k-1}}{p(n_k)}} \leq c < \infty.$$

Applying the inequality

$$\left(\sum_{k=0}^{\infty} a_k \right)^p \leq \sum_{k=0}^{\infty} a_k^p \quad (0 < p \leq 1,\ a_k \geq 0,\ k = 1, 2, ...), \tag{13}$$

by (9) and (12) we get

$$\left(\sum_{j=2^{n_{k-1}-n_{k-2}-1}+1}^{2^{n_k}} (v(j) - v(j-1))^{p(n_k)} \right)^{1/p(n_k)} \to \infty \text{ as } k \to \infty. \tag{14}$$

From (13) we have

$$\left(\sum_{j=2^{n_{k-1}-n_{k-2}-1}+1}^{2^{n_k}} (v(j)-v(j-1))^{p(n_k)}\right)^{1/p(n_k)} \tag{15}$$

$$\leq \left(\sum_{j=2^{n_{k-1}-n_{k-2}-1}+1}^{2^{n_k-n_{k-1}-1}} (v(j)-v(j-1))^{p(n_k)}\right)^{1/p(n_k)}$$

$$+\left(\sum_{j=2^{n_k-n_{k-1}-1}+1}^{2^{n_k}} (v(j)-v(j-1))^{p(n_k)}\right)^{1/p(n_k)}.$$

First we prove that

$$\left(\sum_{j=2^{n_{k-1}-n_{k-2}-1}+1}^{2^{n_k-n_{k-1}-1}} (v(j)-v(j-1))^{p(n_k)}\right)^{1/p(n_k)} \to +\infty \quad \text{as} \quad k\to\infty. \tag{16}$$

We suppose that $\{n_k : k \geq 1\}$ does not satisfy (16).
From (8),(10) and (11) we obtain

$$\left(\sum_{j=2^{n_{k-1}-n_{k-2}-1}+1}^{2^{n_k-n_{k-1}-1}} (v(j)-v(j-1))^{p(n_k)}\right)^{1/p(n_k)}$$

$$\geq \Big[\left(v\left(2^{n_k-n_{k-1}-1}\right)-v\left(2^{n_k-n_{k-1}-1}-1\right)\right)^{p(n_k)}$$

$$\times\left(2^{n_k-n_{k-1}-1}-2^{n_{k-1}-n_{k-2}-1}\right)\Big]^{1/p(n_k)}$$

$$\geq c\left[v\left(2^{n_k-n_{k-1}-1}\right)-v\left(2^{n_k-n_{k-1}-1}-1\right)\right]\frac{2^{n_k/p(n_k)}}{2^{n_{k-1}/p(n_k)}}$$

$$\geq c\left[v\left(2^{n_k-n_{k-1}-1}\right)-v\left(2^{n_k-n_{k-1}-1}-1\right)\right]2^{n_k/p(n_k)},$$

then by hypothesis we get

$$v\left(2^{n_k-n_{k-1}-1}\right)-v\left(2^{n_k-n_{k-1}-1}-1\right)=O\left(2^{-n_k/p(n_k)}\right). \tag{17}$$

By (8) and (17) we get

$$\left(\sum_{j=2^{n_k-n_{k-1}-1}+1}^{2^{n_k}} (v(j)-v(j-1))^{p(n_k)}\right)^{1/p(n_k)} \tag{18}$$

$$\leq \left[v\left(2^{n_k-n_{k-1}-1}\right)-v\left(2^{n_k-n_{k-1}-1}-1\right)\right]2^{\frac{n_k}{p(n_k)}}$$

$$=O(1) \quad \text{as} \quad k\to\infty.$$

From (15),(18) and by hypothesis we obtain

$$(19)\qquad \left(\sum_{j=2^{n_{k-1}-n_{k-2}-1}+1}^{2^{n_k}} (v(j)-v(j-1))^{p(n_k)}\right)^{1/p(n_k)} \le c<\infty \quad \text{as} \quad k\to\infty.$$

We arrive at a contradiction (see (14)).

Therefore we get

$$(20)\qquad \left(\sum_{j=2^{n_{k-1}-n_{k-2}-1}+1}^{2^{n_k-n_{k-1}-1}} (v(j)-v(j-1))^{p(n_k)}\right)^{1/p(n_k)}$$

$$=\left(\sum_{j=1}^{2^{n_k-n_{k-1}-1}-2^{n_{k-1}-n_{k-2}-1}} \left(v\left(j+2^{n_{k-1}-n_{k-2}-1}\right)\right.\right.$$

$$\left.\left.-\,v\left(j+2^{n_{k-1}-n_{k-2}-1}-1\right)\right)^{p(n_k)}\right)^{1/p(n_k)} \to +\infty \text{ as } k\to\infty.$$

Consider the function $g_k(x)$ defined by

$$g_k(x)=\begin{cases} v\left(2^{n_{k-1}-n_{k-2}-1}+j\right)-v\left(2^{n_{k-1}-n_{k-2}-1}+j-1\right), \text{if } x=\frac{2j}{2^{n_k}}, \\ j=1,2,\ldots,2^{n_k-n_{k-1}-1}-2^{n_{k-1}-n_{k-2}-1}-1, \\ 0, \text{if } x\in\left[0,\frac{1}{2^{n_k}}\right]\cup\left[\frac{2^{n_k-n_{k-1}-1}-2^{n_{k-1}-n_{k-2}-1}-1}{2^{n_k}},1\right], \\ x=\frac{2j+1}{2^{n_k}},, j=0,1,2,\ldots,2^{n_k-n_{k-1}-1}-2^{n_{k-1}-n_{k-2}-1}-1, \\ \text{is linear and continuous for other } x\in[0,1]. \end{cases}$$

Let

$$g(x)=\sum_{k=3}^{\infty} g_k(x), \qquad g(0)=0$$

and

$$g(x+l)=g(x), \quad l\in Z.$$

First we prove that $g\in V[v(n)]$. For any positive integer $n\ge 2^{n_2-n_1-1}$ we choose an integer $k(n)$ such that

$$2^{n_{k(n)-1}-n_{k(n)-2}-1}\le n<2^{n_{k(n)}-n_{k(n)-1}-1}.$$

Denote

$$m(n)=2^{n_{k(n)-1}-n_{k(n)-2}-1}.$$

It is evident that

$$(21)\qquad v(n,g)\le$$

$$\le c\left(\sum_{k=3}^{k(n)-1} v\left(2^{n_k-n_{k-1}-1}-2^{n_{k-1}-n_{k-2}-1},g_k\right)+v\left(n-m(n),g_{k(n)}\right)\right).$$

From the construction of the function we obtain

$$\sum_{k=3}^{k(n)-1} \upsilon\left(2^{n_k-n_{k-1}-1}-2^{n_{k-1}-n_{k-2}-1}, g_k\right) \tag{22}$$

$$\le \sum_{k=3}^{k(n)-1} \sum_{j=1}^{2^{n_k-n_{k-1}-1}-2^{n_{k-1}-n_{k-2}-1}-1} \left[\upsilon\left(2^{n_{k-1}-n_{k-2}-1}+j\right)\right.$$
$$\left.-\upsilon\left(2^{n_{k-1}-n_{k-2}-1}+j-1\right)\right]$$
$$\le \sum_{k=3}^{k(n)-1} \left[\upsilon\left(2^{n_k-n_{k-1}-1}\right)-\upsilon\left(2^{n_{k-1}-n_{k-2}-1}\right)\right]$$
$$\le c\upsilon\left(2^{n_{k(n)-1}-n_{k(n)-2}-1}\right) \le c\upsilon(n).$$

Analogously,we get

$$\upsilon\left(n-m(n), g_{k(n)}\right) \tag{23}$$

$$\le \sum_{j=1}^{n-m(n)} \left[\upsilon\left(2^{n_{k(n)-1}-n_{k(n)-2}-1}+j\right)-\upsilon\left(2^{n_{k(n)-1}-n_{k(n)-2}-1}+j-1\right)\right]$$
$$=\upsilon\left(n-m(n)+2^{n_{k(n)-1}-n_{k(n)-2}-1}\right)-\upsilon\left(2^{n_{k(n)-1}-n_{k(n)-2}-1}\right) \le \upsilon(n).$$

Owing to (21), (22) and (23) we get $g \in V[\upsilon(n)]$.

Finally we prove that $g \notin BV(p(n)\uparrow\infty)$ .By (20) and from the construction of the function we get

$$\left(\sum_{j=1}^{2^{n_k-n_{k-1}-1}-2^{n_{k-1}-n_{k-2}-1}-1} \left|g\left(\frac{2j-1}{2^{n_k}}\right)-g\left(\frac{2j}{2^{n_k}}\right)\right|^{p(n_k)}\right)^{1/p(n_k)}$$
$$=\left(\sum_{j=1}^{2^{n_k-n_{k-1}-1}-2^{n_{k-1}-n_{k-2}-1}-1} \left|g_k\left(\frac{2j-1}{2^{n_k}}\right)-g_k\left(\frac{2j}{2^{n_k}}\right)\right|^{p(n_k)}\right)^{1/p(n_k)}$$
$$=\left(\sum_{j=1}^{2^{n_k-n_{k-1}-1}-2^{n_{k-1}-n_{k-2}-1}-1} \left|g_k\left(\frac{2j}{2^{n_k}}\right)\right|^{p(n_k)}\right)^{1/p(n_k)}$$
$$=\left(\sum_{j=1}^{2^{n_k-n_{k-1}-1}-2^{n_{k-1}-n_{k-2}-1}-1} \left(\upsilon\left(2^{n_{k-1}-n_{k-2}-1}+j\right)\right.\right.$$
$$\left.\left.-\upsilon\left(2^{n_{k-1}-n_{k-2}-1}+j-1\right)\right)^{p(n_k)}\right)^{1/p(n_k)} \to \infty \quad \text{as} \quad k\to\infty.$$

Therefore we get $g \notin BV(p(n)\uparrow\infty)$ and the proof of Theorem 2 is complete.

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