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ON ATOMS OF BCK-ALGEBRAS

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Abstract. Atoms in BCK-algebras are considered. The notions of the star BCK-algebras and the star part of BCK-algebras are introduced. The properties of some substructures which consist of atoms are investigated. Furthermore, by isomorphic view, there are and only $n+1$ BCK-algebras X with $|X| = n+1$ and $|S_t(X)| = n$.

1. Introduction By a BCK-algebra we mean an algebra $(X;*,0)$ of type (2, 0) satisfying the axioms:

(1) $((x*y)*(x*z))*(z*y) = 0$,
(2) $(x*(x*y))*y = 0$,
(3) $x*x = 0$,
(4) $x*y = y*x = 0$ impies $x = y$,
(5) $0*x = 0$

for any x, y and z in X. For any BCK-algebra X, the relation $\leq$ defined by $x \leq y$ if and only if $x*y = 0$ is a partial order on X (see [1]).

A BCK-algebra X has the following properties for any $x, y, z \in X$:

(6) $x*0 = x$,
(7) $(x*y)*z = (x*z)*y$,
(8) $x \leq y$ implies that $x*z \leq y*z$ and $z*y \leq z*x$.

In a BCK-algebra X, if an element a satisfying:

(a) $a \neq 0$,
(b) $x \in X \setminus \{0\}$ and $x \leq a$ imply $x = a$

then the element a is called an atom of X. Since 0 is the least element of X, it is obvious that an atom of X is a minimal element of X.

Let $(X;*,0)$ be a BCK-algebra. A non-empty subset S of X is called a subalgebra if $x, y \in S$ implies $x*y \in S$. By an ideal I of X we mean $0 \in I$ and $y, x*y \in I$ imply $x \in I$. If an ideal I of X is also a subalgebra of X, then I is called a close ideal of X. It has been known that an ideal of a BCK-algebra is a close ideal (see [2]).

2. Star subalgebras of BCK-algebras Let X be a BCK-algebra. We define

$$S_t(X) = \{a \in X; a = 0 \text{ or } a \text{ is an atom of } X\ .\}$$

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The subset $S_t(X)$ is called the star part of X.

Propsition 2.1. Let X be a BCK-algebra. If $a, b \in S_t(X)$ and $a \neq b$, then $a * b = a$.

Proof. In case $a = 0$ or $b = 0$, the proof is trivial. Assume $a \neq 0$ and $b \neq 0$, by $a * b \leq a$ and a is an atom of X, we get $a * b = 0$ or $a * b = a$. If $a * b = 0$, then $a = 0$ or $a = b$ since b is an atom of X. It is a contradiction, hence $a * b = a$. The proof is completed.

By Propsition 2.1 we can immediately get

Theorem 2.2. For any BCK-algebra X, $S_t(X)$ is a subalgebra of X.

Let X be a BCK-algebra and S be a subalgebra of X. S is called a star subalgebra of X if $S_t(S) = S$. Particularly X is called a star BCK-algebra if $X = S_t(X)$.

Remark. $S_t(X)$ may be not a maximal star subalgebra.

Example 1. Let $X = \{0, \cdots, -n-1, -n, -n+1, \cdots, -3, -2, -1\}$ and partial order $\leq$ as follows $0 \leq \cdots \leq -n-1 \leq -n \leq -n+1 \leq \cdots -3 \leq -2 \leq -1$. Define operation $*$ by

$$x * y = \begin{cases} 0, & x \leq y \\ x, & others \end{cases}$$

for any x, y in X. Then $(X; *, 0)$ is a BCK-algebra and $S_t(X) = \{0\}$. If take the subalgebra $S = \{0, 1\}$ of X, then $S_t(S) = S$. In this example, $S_t(X)$ is not a maximal star subalgebra of X.

Example 2. Let $X = \{0, 1, 2, 3\}$. Take the operation table of X as follows

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Then $(X; *, 0)$ is a BCK-algebra. $S_t(X) = \{0, 1, 3\}$ is a maximal star subalgebra of X but not the largest star subalgebra of X since $S = \{0, 2\}$ is a star subalgebra of X.

Theorem 2.3. Let X be a BCK-algebra and S be a subalgebra of X. Then S is a star subalgebra of X if and only if for any $a, b \in S$, $a \neq b$ implies $a * b = a$.

Proof. By Propsition 2.1, the necessity part is obvious. In the sufficiency part, for any $b \in S \backslash \{0\}$, if there exists $x_0 \in S \backslash \{0\}$ such that $x_0 \leq b$, then we have $x_0 = b$ or $x_0 \neq b$. If $x_0 \neq b$, then we get $x_0 * b = 0$ by $x_0 \leq b$ and $x_0 * b = x_0$ by the condition of the Theorem, hence $x_0 = 0$. It is contradictory that $x_0 \in S \backslash \{0\}$. Hence $x_0 = b$ and b is an atom of S. The proof is completed.

Theorem 2.4. Let X be a BCK-algebra. Then $S_t(X)$ is a maximal star subalgebra of X if and only if for any element x in $X \backslash S_t(X)$, there exists an element a in $S_t(X) \backslash \{0\}$ suh that $a \leq x$.

Proof. Assume S be a star subalgebra of X and $S_t(X) \subseteq S$. If there exists an element x_0 of X in $S \backslash S_t(X)$, then there exists an element a in $S_t(X) \backslash \{0\} \subseteq S$ such that $a \leq x_0$. Hence the element x_0 is not an atom of S. It is contrdictory that S is a star subalgebra of X. And the sufficient part is proved. On the other hand, if there exists x_0 in $X \backslash S_t(X)$ such that for all a in $S_t(X) \backslash \{0\}$, $a * x_0 \neq 0$, then we have $a * x_0 = a$ by $a * x_0 \leq a$ and $a \in S_t(X) \backslash \{0\}$. Assume $x_0 * a = b$, we get

$$(x_0 * b) * a = (x_0 * a) * b = b * b = 0,$$

that is $x_0 * b \leq a$, hence $x_0 * b = 0$ or $x_0 * b = a$. If $x_0 * b = a$, then

$$a * x_0 = (x_0 * b) * x_0 = (x_0 * x_0) * b = 0 * b = 0.$$

It is a contradition. Hence $x_0 * b = 0$. By $b * x_0 = (x_0 * a) * x_0 = (x_0 * x_0) * a = 0 * a = 0$, we have $x_0 = b = x_0 * a$. Then we get $x_0 * a = x_0$ and $a * x_0 = a$ for all $a \in S_t(X)$. Therefore $S = S_t(X) \bigcup \{x_0\}$ is a star subalgebra of X by Theorem 2.3. It is contradictory that $S_t(X)$ is a maximal star subalgebra. The proof is completed.

Corollary 2.5. For a finite BCK-algebra X, $S_t(X)$ is a maximal star subalgebra of X.

Theorem 2.6. Let X be a BCK-algebra. $S_t(X)$ is the largest star subalgebra of X if and only if $X = S_t(X)$.

Proof. The sufficiency part is obvious. Conversely, for any $x \in X \backslash \{0\}$, $S = \{0, x\}$ is a star subalgebra of X, hence $x \in S_t(X)$ since $S_t(X)$ is the largest star subalgebra. The proof is completed.

Let $(X; *_1, 0)$, $(Y; *_2, 0)$ be two BCK-algebras. The set $X \times Y = \{(x, y); x \in X, y \in Y\}$ about operation $*$: $(x_1, y_1) * (x_2, y_2) = (x_1 *_1 x_2, y_1 *_2 y_2)$ becomes a BCK-algebra, and (0,0) is the zero element of $X \times Y$.

Generally, $S_t(X \times Y) \neq S_t(X) \times S_t(Y)$, but we have

Theorem 2.7. Let X, Y be two BCK-algebras. Then

$$S_t(X \times Y) = (S_t(X) \times \{0\}) \bigcup (\{0\} \times S_t(Y))$$

Proof. It is obvious that $S_t(X \times Y) \supseteq (S_t(X) \times \{0\}) \bigcup (\{0\} \times S_t(Y))$. Furthermore, for any $(x_0, y_0) \in S_t(X \times Y)$, if $x_0 \neq 0$ and $y_0 \neq 0$, then we get $(x_0, 0) * (x_0, y_0) = (0, 0)$. It is contradictory that $(x_0, y_0) \in S_t(X \times Y)$. Hence we get $x_0 = 0$ or $y_0 = 0$. If $x_0 = 0$, then it is easy to prove that $y_0 \in S_t(Y)$. Similarly, if $y_0 = 0$, then $x_0 \in S_t(X)$. The proof is completed.

Corollary 2.8. For any finite BCK-algebra X, Y, we have $|S_t(X \times Y)| = |S_t(X)| + |S_t(Y)| - 1$.

Corollary 2.9. Let X, Y be two BCK-algebras. Then $S_t(X \times Y) = S_t(X) \times S_t(Y)$ if and only if $S_t(x) = \{0\}$ or $S_t(Y) = \{0\}$

Let X be a BCK-algebra. If an atom b of X satisfies that $b * x = b$ for any $x \in X \backslash \{b\}$, then we call b is a strong atom of X. Take the subset of X

$$D(X) = \{\ b \in S_t(X);\ b \text{ is a strong atom of } X \text{ or } b = 0\ \}$$

we have

Theorem 2.10. For any BCK-algebra X, $D(X)$ is a closed ideal of X.

Proof. We need to prove that $D(X)$ is an ideal of X only. Assume $y, x * y \in D(X)$, if $x * y = x$, then $x \in D(X)$. If $x * y \neq x$, then $(x * y) * x = x * y$ by the definition of $D(X)$. On the other hand,

$$(x * y) * x = (x * x) * y = 0 * y = 0,$$

hence we get $x * y = 0$. By $y \in D(X)$ and $x * y = 0$, we have $x = 0$ or $x = y$, hence $x \in D(X)$. The proof is completed.

3. On star BCK-algebras Suppose $(X; *, 0)$ be a BCK-algebra. For any $a \in X$, we use a^{-1} denote the selfmap of defined by $xa^{-1} = x * a$. Let $M(X)$ denote the set of all finite product $a^{-1}\cdots b^{-1}$ of selfmaps with $a,\cdots,b\in X$. It is clear that $M(X)$ becomes a commutative monoid under composition of maps and 0^{-1} is the identity. We difine a relation $\leq_1$ on $M(X)$ as follows:

$$u^{-1}\cdots v^{-1}\leq_1 a^{-1}\cdots b^{-1}\Longleftrightarrow(xu^{-1}\cdots v^{-1})*(xa^{-1}\cdots b^{-1})=0$$

for any $x\in X$. We call $M(X)$ the adjoint semigroup of X (see [3]). It is obvious that $M(S)=\{u^{-1}\cdots v^{-1};u,\cdots,v\in S\}$ becomes a subsemigroup of $M(X)$ for any non-empty subset S of X.

Lemma 3.1. Let X be a BCK-algebra and $\sigma=a_1{}^{-1}\cdots a_n{}^{-1}\in M(S_t(X))$. If $S_t(X)$ is an ideal of X, then $Ker\sigma=\{0,a_1,a_2,\cdots,a_n\}$

Proof. It is obvious that $\{0,a_1,a_2,\cdots,a_n\}\subseteq Ker\sigma$ by Section 1. Conversely, if $\sigma=a_1{}^{-1}$ and $b\in Ker\sigma$, then $ba_1{}^{-1}=b*a_1=0$. We get $b=0$ or $b=a_1$ by $a_1\in S_t(X)$, hence $b\in\{0,a_1\}$ and the Lemma holds for $n=1$. Now we assume the Lemma has already been proved for $n=k$, then we prove the case of $\sigma=a_1{}^{-1}a_2{}^{-1}\cdots a_k{}^{-1}a_{k+1}{}^{-1}$. If $b\in Ker\sigma$, $b\sigma=0$, then we have $b\in S_t(X)$ by $S_t(X)$ is an ideal of X and $a_1,\cdots,a_{k+1}\in S_t(X)$. Since $(b*a_{k+1})*b=0$ and $b\in S_t(X)$, we get $b*a_{k+1}=0$ or $b*a_{k+1}=b$. If $b*a_{k+1}=0$, then $b=a_{k+1}$ or $b=0$ hence $b\in\{0,a_1,\cdots,a_{k+1}\}$. If $b*a_{k+1}=b$, then

$$0=b\sigma=ba_1{}^{-1}\cdots a_k{}^{-1}a_{k+1}{}^{-1}=(ba_{k+1}{}^{-1})a_1{}^{-1}\cdots a_k{}^{-1}=ba_1{}^{-1}\cdots a_k{}^{-1},$$

we have $b\in\{0,a_1,\cdots,a_k\}\subseteq\{0,a_1,\cdots,a_k,a_{k+1}\}$ by our assumption. The proof is completed.

Theorem 3.2. Let X be a BCK-algebra. Then X is a star BCK-algebra if and only if for all $\sigma=a_1{}^{-1}\cdots a_n{}^{-1}\in M(X)$, $Ker\sigma=\{0,a_1,\cdots,a_n\}$.

Proof. By Lemma 3.1, the necessity part is obvious. In sufficiency part, for any element $a\in X\backslash\{0\}$, if there exists $x\in X$ such that $x\leq a$, then $x\in Kera^{-1}=\{0,a\}$, hence $x=0$ or $x=a$, and a is an atom of X. The proof is completed.

By a positive implicative BCK-algebra, we mean a BCK-algebra $(X;*,0)$ such that for all $x,y,z\in X$, $(x*y)*z=(x*z)*(y*z)$. If for all $x,y\in X$, $y*(y*x)=x*(x*y)$, then X is said to be a commutative BCK-algebra. It is alse noteworthy that X is a positive implicative BCK-algebra if and only if $(x*y)*y=x*y$ for all $x,y\in X$ (see [4]). By using these results, we have

Theorem 3.3. If X is a star BCK-algebra, then the following results hold:

(a) X is a positive implicative BCK-algebra;

(b) X is a commutative BCK-algebra.

Proof. (a) For any $x,y\in X$, if $x*y=0$, it is obvious that $(x*y)*y=x*y$. Assume $x*y\neq 0$, then we have $x*y=x$ by $x*y\leq x$ and x is an atom of X. Hence $(x*y)*y=x*y$.

(b) For any $x,y\in X$, if $x*y=0$ or $y*x=0$, then we get $x=0$ or $x=y$ or $y=0$, hence it is obvious that $y*(y*x)=x*(x*y)$. Assume $x*y\neq 0$ and $y*x\neq 0$, then we get $x*y=x$ and $y*x=y$ by $x*y\leq y$ and $y*x\leq y$, hence $x*(x*y)=0=y*(y*x)$. The proof is completed.

4. The count of a class finite BCK-algebras Let u be an element in BCK-algebra X. If for any $x\in X$, $u*x=0$ implies $u=x$, then u is called a maximal element of X.

Theorem 4.1. Let X be a BCK-algebra. If $u\in X\backslash D(X)$ is a maximal element of X, then for any $b\in D(X)$, $u*b=u$.

Proof. It is trivial to see the case $b=0$. If $b\neq 0$, then we have $(u*(u*b))*b=0$ by BCK axiom (2), hence $u*(u*b)=0$ or b, since b is an atom of X. If $u*(u*b)=b$, then we can get

$$b*u=(u*(u*b))*u=(u*u)*(u*b)=0*(u*b)=0,$$

it is contradictory that b is a strong atom of X. Hence $u*(u*b)=0$. Therefore $u*b=u$ by u is a maximal element of X. The proof is completed.

Lemma 4.2. If $X=\{0,a_1,\cdots,a_n\}$ is a BCK-algebra with $S_t(X)=\{0,a_1,\cdots,a_{n-1}\}$ and $D(X)=\{0,a_1,\cdots,a_i\}$ $(0\leq i\leq n-2)$, then for all $a_k\in S_t(X)\backslash D(X)$, $a_k*a_n=0$.

Proof. By $(a_k*a_n)*a_k=(a_k*a_k)*a_n=0*a_n=0$, we get $a_k*a_n=0$ or $a_k*a_n=a_k$ since a_k is an atom of X. If $a_k*a_n=a_k$, then we have $a_k\in D(X)$ by Propsition 2.1, it is contradictory that $a_k\notin D(X)$. Hence $a_k*a_n=0$. The proof is completed.

Corollary 4.3. In Lemma 4.2, the element a_n is a maximal element of X.

Let X be a BCK-algebra with $|X|=n+1$, $|S_t(X)|=n$ and $|D(X)|=i+1$ $(0\leq i\leq n-2)$. Assuming $X=\{0,a_1,a_2,\cdots,a_n\}$, $S_t(X)=\{0,a_1,\cdots,a_{n-1}\}$ and $D(X)=\{0,a_1,\cdots,a_i\}$, by above discussing, the operation table of X must be as table one.

In table one, $a_{nk}=a_n*a_k(i+1\leq k\leq n-1)$. After, we shall give the number of this class BCK-algebras by determining the value of a_{nk} in table one.

Lemma 4.4. Let BCK-algebra $X=\{0,a_1,a_2,\cdots,a_n\}$ with $S_t(X)=\{0,a_1,a_2,\cdots,a_{n-1}\}$ and $D(X)=\{0,a_1,a_2,\cdots,a_i\}$. If $|S_t(X)\backslash D(X)|\geq 2$, then the following conclusions hold:

(a) For any $a_k\in S_t(X)\backslash D(X)$, $a_n*a_k\neq a_k$;

(b) If there exists $a_k\in S_t(X)\backslash D(X)$ such that $a_n*a_k=a_l$ and $a_l\neq a_n$, then $a_n*a_l=a_k$;

(c) If there exists $a_k,a_l\in S_t(X)\backslash D(X)$ and $a_k\neq a_l$ such that $a_n*a_k=a_l$, $a_n*a_l=a_k$, then for all $a_p\in S_t(X)\backslash\{D(X)\bigcup\{a_k,a_l\}\}$, $a_n*a_p=a_n$.

Proof. (a) If there exists $a_k\in S_t(X)\backslash D(X)$, such that $a_n*a_k=a_k$, then take $a_l\in S_t(X)\backslash D(X)$, $a_l\neq a_k$, we have

$$\begin{aligned}
0 =\ & ((a_l*a_k)*(a_l*a_n))*(a_n*a_k) && (\textit{by axiom } (1)\ )\\
=\ & (a_l*(a_l*a_n))*(a_n*a_k) && (\textit{by Proposition } 2.1)\\
=\ & (a_l*0)*(a_n*a_k) && (\textit{by Lemma } 4.2)\\
=\ & a_l*(a_n*a_k) && \\
=\ & a_l*a_k && (\textit{by our assumption}\ )\\
=\ & a_l && (\textit{by Propsition } 2.1\ )
\end{aligned}$$

It is a contradiction. Hence (a) holds.

(b) By BCK axioms (2), we have $0=(a_n*(a_n*a_k))*a_k=(a_n*a_l)*a_k$. Hence $a_n*a_l=0$ or $a_n*a_l=a_k$. If $a_n*a_l=0$, then we get $a_n=a_l$ by Lemma 4.2. It is contradictory that $a_l\neq a_n$. Therefore $a_n*a_l=a_k$, and the proof of (b) is completed.

(c) If there exists $a_p\in S_t(X)\backslash\{D(X)\bigcup\{a_k,a_l\}\}$ such that $a_n*a_p=a_q$ and $a_q\neq a_n$, then by BCK axioms (1) we have $0=((a_n*a_p)*(a_n*a_k))*(a_k*a_p)=(a_q*a_l)*a_k$. If $a_q\neq a_l$, then $a_q*a_l=a_q$ by Propsition 2.1. Hence $0=a_q*a_k$ and $a_q=a_k$, therefore we get

$$0=(a_n*(a_n*a_p))*a_p=(a_n*a_q)*a_p=(a_n*a_k)*a_p=a_l*a_p=a_l.$$

It is a contradiction. If $a_q=a_l$, then it is contradictory that

$$0=(a_n*(a_n*a_p))*a_p=(a_n*a_q)*a_p=(a_n*a_l)*a_p=a_k*a_p=a_k.$$

Hence (c) holds. And the proof of the Lemma is completed.

Theorem 4.5. By isomorphic view, there are total $n+1$ BCK-algebras X with $|X| = n+1$ and $|S_t(X)| = n$.

Proof. Assuming $X = \{0, a_1, a_2, \cdots, a_n\}$ with $S_t(X) = \{0, a_1, a_2, \cdots, a_{n-1}\}$ and $D(X) = \{0, a_1, a_2, \cdots, a_i\}$ $(0 \leq i \leq n-2)$, we determine the operation tables of X according to the order of $D(X)$.

Case 1. $|D(X)| = n-1$, that is $D(X) = \{0, a_1, a_2, \cdots, a_{n-2}\}$. In this case, we only need to determine the value of $a_{n(n-1)} = a_n * a_{n-1}$ in table one. By $a_n * a_{n-1} \neq 0$ and $a_n * a_{n-1} \leq a_n$, we have $a_n * a_{n-1} = a_{n-1}$ or $a_n * a_{n-1} = a_n$. Taking $a_{n(n-1)} = a_{n-1}$ and $a_{n(n-1)} = a_n$ each, we get two different operation tables—table two and table three

By BCK-algebra axioms (1)—(5) we can verify that table two and table three indeed give two BCK-algebras. Hence, there are and only two BCK-algebras in Case 1.

Case 2. $|D(X)| = n-2$, that is $|S_t(X) \backslash D(X)| = 2$. In this case, if $a_n * a_{n-2} = a_n * a_{n-1} = a_n$, then by table one we get the operation table of X as table four

If the operation table of X is different from table four, then we have $a_n * a_{n-2} = a_{n-1}$ and $a_n * a_{n-1} = a_{n-2}$ by Lemma 4.4. Hence by table one the operation table must be as table five

By BCK-algebra axioms (1)—(5) we can verify X which are given by table four and table five are BCK-algebras. Hence, there are and only two BCK-algebras in Case 2.

Case 3. $|D(X)| < n-2$, that is $|S_t(X) \backslash D(X)| > 2$. In this case, if $a_n * a_k = a_n$, $k = i+1, \cdots, n-1$, then by table one we get the operation table of X as table six

By BCK-algebra axioms (1)—(5) we can verify X which is given by table six is BCK-algebra. If the operation table of X is different from table six, then by Lemma 4.4, there are two elements $a_k, a_l \in S_t(X) \backslash D(X)$ such that $a_n * a_k = a_l$ and $a_n * a_l = a_k$. Assume $a_k = a_{n-2}$ and $a_l = a_{n-1}$. We get $a_n * a_p = a_n, p = i+1, \cdots, n-3$ by Lemma 4.4. Hence by table one the operation table must be as follows

$*$	0	a_1	a_2	$\cdots$	a_i	a_{i+1}	$\cdots$	a_{n-3}	a_{n-2}	a_{n-1}	a_n
0	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	0
a_1	a_1	0	a_1	$\cdots$	a_1	a_1	$\cdots$	a_1	a_1	a_1	a_1
a_2	a_2	a_2	0	$\cdots$	a_2	a_2	$\cdots$	a_2	a_2	a_2	a_2
$\vdots$	.	.	.	$\cdots$	.	.	$\cdots$	.	.	.	.
a_i	a_i	a_i	a_i	$\cdots$	0	a_i	$\cdots$	a_i	a_i	a_i	a_i
a_{i+1}	a_{i+1}	a_{i+1}	a_{i+1}	$\cdots$	a_{i+1}	0	$\cdots$	a_{i+1}	a_{i+1}	a_{i+1}	0
$\vdots$	.	.	.	$\cdots$	.	.	$\cdots$	.	.	.	.
a_{n-3}	a_{n-3}	a_{n-3}	a_{n-3}	$\cdots$	a_{n-3}	a_{n-3}	$\cdots$	0	a_{n-3}	a_{n-3}	0
a_{n-2}	a_{n-2}	a_{n-2}	a_{n-2}	$\cdots$	a_{n-2}	a_{n-2}	$\cdots$	a_{n-2}	0	a_{n-2}	0
a_{n-1}	a_{n-1}	a_{n-1}	a_{n-1}	$\cdots$	a_{n-1}	a_{n-1}	$\cdots$	a_{n-1}	a_{n-1}	0	0
a_n	a_n	a_n	a_n	$\cdots$	a_n	a_n	$\cdots$	a_n	a_{n-1}	a_{n-2}	0

(table seven)

But the algebra defined by table seven is not a BCK-algebra, for, we have

$$((a_{n-3} * a_{n-1}) * (a_{n-3} * a_n)) * (a_n * a_{n-1}) = (a_{n-3} * 0) * a_{n-2} = a_{n-3} \neq 0,$$

namely, the BCK-algebra axiom (1) does not hold. Hence, there exists and only one BCK-algebra X with $|D(X)| = i+1 < n-2$ in Case 3 by table six. Since the order of $D(X)$

can take $1, 2, \cdots, n-3$, the proof is completed by combinig Case 1, Case 2, Case 3, and the operation tables are given by table two — table six.

$*$	0	a_1	a_2	$\cdots$	a_i	a_{i+1}	$\cdots$	a_{n-3}	a_{n-2}	a_{n-1}	a_n
0	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	0
a_1	a_1	0	a_1	$\cdots$	a_1	a_1	$\cdots$	a_1	a_1	a_1	a_1
a_2	a_2	a_2	0	$\cdots$	a_2	a_2	$\cdots$	a_2	a_2	a_2	a_2
$\vdots$	.	.	.	$\cdots$	.	.	$\cdots$	.	.	.	.
a_i	a_i	a_i	a_i	$\cdots$	0	a_i	$\cdots$	a_i	a_i	a_i	a_i
a_{i+1}	a_{i+1}	a_{i+1}	a_{i+1}	$\cdots$	a_{i+1}	0	$\cdots$	a_{i+1}	a_{i+1}	a_{i+1}	0
$\vdots$	.	.	.	$\cdots$	.	.	$\cdots$	.	.	.	.
a_{n-3}	a_{n-3}	a_{n-3}	a_{n-3}	$\cdots$	a_{n-3}	a_{n-3}	$\cdots$	0	a_{n-3}	a_{n-3}	0
a_{n-2}	a_{n-2}	a_{n-2}	a_{n-2}	$\cdots$	a_{n-2}	a_{n-2}	$\cdots$	a_{n-2}	0	a_{n-2}	0
a_{n-1}	a_{n-1}	a_{n-1}	a_{n-1}	$\cdots$	a_{n-1}	a_{n-1}	$\cdots$	a_{n-1}	a_{n-1}	0	0
a_n	a_n	a_n	a_n	$\cdots$	a_n	$a_{n(i+1)}$	$\cdots$	$a_{n(n-3)}$	$a_{n(n-2)}$	$a_{n(n-1)}$	0

(table one)

$*$	0	a_1	a_2	$\cdots$	a_i	a_{i+1}	$\cdots$	a_{n-3}	a_{n-2}	a_{n-1}	a_n
0	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	0
a_1	a_1	0	a_1	$\cdots$	a_1	a_1	$\cdots$	a_1	a_1	a_1	a_1
a_2	a_2	a_2	0	$\cdots$	a_2	a_2	$\cdots$	a_2	a_2	a_2	a_2
$\vdots$	.	.	.	$\cdots$	.	.	$\cdots$	.	.	.	.
a_i	a_i	a_i	a_i	$\cdots$	0	a_i	$\cdots$	a_i	a_i	a_i	a_i
a_{i+1}	a_{i+1}	a_{i+1}	a_{i+1}	$\cdots$	a_{i+1}	0	$\cdots$	a_{i+1}	a_{i+1}	a_{i+1}	a_{i+1}
$\vdots$	.	.	.	$\cdots$	.	.	$\cdots$	.	.	.	.
a_{n-3}	a_{n-3}	a_{n-3}	a_{n-3}	$\cdots$	a_{n-3}	a_{n-3}	$\cdots$	0	a_{n-3}	a_{n-3}	a_{n-3}
a_{n-2}	a_{n-2}	a_{n-2}	a_{n-2}	$\cdots$	a_{n-2}	a_{n-2}	$\cdots$	a_{n-2}	0	a_{n-2}	a_{n-2}
a_{n-1}	a_{n-1}	a_{n-1}	a_{n-1}	$\cdots$	a_{n-1}	a_{n-1}	$\cdots$	a_{n-1}	a_{n-1}	0	0
a_n	a_n	a_n	a_n	$\cdots$	a_n	a_n	$\cdots$	a_n	a_n	a_{n-1}	0

(table two)

$*$	0	a_1	a_2	$\cdots$	a_i	a_{i+1}	$\cdots$	a_{n-3}	a_{n-2}	a_{n-1}	a_n
0	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	0
a_1	a_1	0	a_1	$\cdots$	a_1	a_1	$\cdots$	a_1	a_1	a_1	a_1
a_2	a_2	a_2	0	$\cdots$	a_2	a_2	$\cdots$	a_2	a_2	a_2	a_2
$\vdots$	.	.	.	$\cdots$	.	.	$\cdots$	.	.	.	.
a_i	a_i	a_i	a_i	$\cdots$	0	a_i	$\cdots$	a_i	a_i	a_i	a_i
a_{i+1}	a_{i+1}	a_{i+1}	a_{i+1}	$\cdots$	a_{i+1}	0	$\cdots$	a_{i+1}	a_{i+1}	a_{i+1}	a_{i+1}
$\vdots$	.	.	.	$\cdots$	.	.	$\cdots$	.	.	.	.
a_{n-3}	a_{n-3}	a_{n-3}	a_{n-3}	$\cdots$	a_{n-3}	a_{n-3}	$\cdots$	0	a_{n-3}	a_{n-3}	a_{n-3}
a_{n-2}	a_{n-2}	a_{n-2}	a_{n-2}	$\cdots$	a_{n-2}	a_{n-2}	$\cdots$	a_{n-2}	0	a_{n-2}	a_{n-2}
a_{n-1}	a_{n-1}	a_{n-1}	a_{n-1}	$\cdots$	a_{n-1}	a_{n-1}	$\cdots$	a_{n-1}	a_{n-1}	0	0
a_n	a_n	a_n	a_n	$\cdots$	a_n	a_n	$\cdots$	a_n	a_n	a_n	0

(table three)

$*$	0	a_1	a_2	$\cdots$	a_i	a_{i+1}	$\cdots$	a_{n-3}	a_{n-2}	a_{n-1}	a_n
0	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	0
a_1	a_1	0	a_1	$\cdots$	a_1	a_1	$\cdots$	a_1	a_1	a_1	a_1
a_2	a_2	a_2	0	$\cdots$	a_2	a_2	$\cdots$	a_2	a_2	a_2	a_2
$\vdots$	.	.	.	$\cdots$	.	.	$\cdots$	.	.	.	.
a_i	a_i	a_i	a_i	$\cdots$	0	a_i	$\cdots$	a_i	a_i	a_i	a_i
a_{i+1}	a_{i+1}	a_{i+1}	a_{i+1}	$\cdots$	a_{i+1}	0	$\cdots$	a_{i+1}	a_{i+1}	a_{i+1}	a_{i+1}
$\vdots$	.	.	.	$\cdots$	.	.	$\cdots$	.	.	.	.
a_{n-3}	a_{n-3}	a_{n-3}	a_{n-3}	$\cdots$	a_{n-3}	a_{n-3}	$\cdots$	0	a_{n-3}	a_{n-3}	a_{n-3}
a_{n-2}	a_{n-2}	a_{n-2}	a_{n-2}	$\cdots$	a_{n-2}	a_{n-2}	$\cdots$	a_{n-2}	0	a_{n-2}	0
a_{n-1}	a_{n-1}	a_{n-1}	a_{n-1}	$\cdots$	a_{n-1}	a_{n-1}	$\cdots$	a_{n-1}	a_{n-1}	0	0
a_n	a_n	a_n	a_n	$\cdots$	a_n	a_n	$\cdots$	a_n	a_n	a_n	0

(table four)

$*$	0	a_1	a_2	$\cdots$	a_i	a_{i+1}	$\cdots$	a_{n-3}	a_{n-2}	a_{n-1}	a_n
0	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	0
a_1	a_1	0	a_1	$\cdots$	a_1	a_1	$\cdots$	a_1	a_1	a_1	a_1
a_2	a_2	a_2	0	$\cdots$	a_2	a_2	$\cdots$	a_2	a_2	a_2	a_2
$\vdots$	.	.	.	$\cdots$	.	.	$\cdots$	.	.	.	.
a_i	a_i	a_i	a_i	$\cdots$	0	a_i	$\cdots$	a_i	a_i	a_i	a_i
a_{i+1}	a_{i+1}	a_{i+1}	a_{i+1}	$\cdots$	a_{i+1}	0	$\cdots$	a_{i+1}	a_{i+1}	a_{i+1}	a_{i+1}
$\vdots$	.	.	.	$\cdots$	.	.	$\cdots$	.	.	.	.
a_{n-3}	a_{n-3}	a_{n-3}	a_{n-3}	$\cdots$	a_{n-3}	a_{n-3}	$\cdots$	0	a_{n-3}	a_{n-3}	a_{n-3}
a_{n-2}	a_{n-2}	a_{n-2}	a_{n-2}	$\cdots$	a_{n-2}	a_{n-2}	$\cdots$	a_{n-2}	0	a_{n-2}	0
a_{n-1}	a_{n-1}	a_{n-1}	a_{n-1}	$\cdots$	a_{n-1}	a_{n-1}	$\cdots$	a_{n-1}	a_{n-1}	0	0
a_n	a_n	a_n	a_n	$\cdots$	a_n	a_n	$\cdots$	a_n	a_{n-1}	a_{n-2}	0

(table five)

$*$	0	a_1	a_2	$\cdots$	a_i	a_{i+1}	$\cdots$	a_{n-3}	a_{n-2}	a_{n-1}	a_n
0	0	0	0	$\cdots$	0	0	$\cdots$	0	0	0	0
a_1	a_1	0	a_1	$\cdots$	a_1	a_1	$\cdots$	a_1	a_1	a_1	a_1
a_2	a_2	a_2	0	$\cdots$	a_2	a_2	$\cdots$	a_2	a_2	a_2	a_2
$\vdots$	.	.	.	$\cdots$	.	.	$\cdots$	.	.	.	.
a_i	a_i	a_i	a_i	$\cdots$	0	a_i	$\cdots$	a_i	a_i	a_i	a_i
a_{i+1}	a_{i+1}	a_{i+1}	a_{i+1}	$\cdots$	a_{i+1}	0	$\cdots$	a_{i+1}	a_{i+1}	a_{i+1}	0
$\vdots$	.	.	.	$\cdots$	.	.	$\cdots$	.	.	.	.
a_{n-3}	a_{n-3}	a_{n-3}	a_{n-3}	$\cdots$	a_{n-3}	a_{n-3}	$\cdots$	0	a_{n-3}	a_{n-3}	0
a_{n-2}	a_{n-2}	a_{n-2}	a_{n-2}	$\cdots$	a_{n-2}	a_{n-2}	$\cdots$	a_{n-2}	0	a_{n-2}	0
a_{n-1}	a_{n-1}	a_{n-1}	a_{n-1}	$\cdots$	a_{n-1}	a_{n-1}	$\cdots$	a_{n-1}	a_{n-1}	0	0
a_n	a_n	a_n	a_n	$\cdots$	a_n	a_n	$\cdots$	a_n	a_n	a_n	0

(table six)

References

[1] K.Iseki, *An algebra related with a propositional calculus*, Proc. Japan Acad., 42(1966), 26-29
[2] Wenping Huang, *Nil-radical in BCI-algebras*, Math. Japonica 37, No.2 (1992), 363-366
[3] Wenping Huang, *Adjoint semigroups of BCI-algebras,* SEA Bull. Math., Vol.19, No.3 (1995) 95-98
[4] K.Iseki and S.Tanaka, *An introduction to the theory of BCK-algebra,* Math. Japonica 23, No.1(1978), 1-26

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