ON CLOSED RANGE MULTIPLIERS ON TOPOLOGICAL ALGEBRAS

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ABSTRACT. In this paper, we investigate several conditions pertaining to closed range multipliers on topological algebras. We first obtain some general results which give several equivalent conditions for a continuous linear operator T on a Fréchet locally convex space to have a closed range. In particular, when we assume T to be a multiplier on a topological algebra without order, a number of other conditions also appear. For instance, if T is a multiplier on a semiprime Fréchet locally convex algebra A such that $T^2A = TA$, then the range TA is closed. Finally, as a converse result, it is shown that if A is a Fréchet locally C^*-algebra and T a multiplier on A, then TA is closed, if, and only if, $T^2A = TA$.

1. INTRODUCTION

The class of multipliers with closed range, in the context of semisimple commutative Banach algebras, has been studied by several authors (see e.g. [1], [6], [8], [13]). The most significant applications of such multipliers are to group algebras $L^1(G)$ and measure algebras $M(G)$. Host and Parreau [8,Théorém 1] gave a complete description of closed range multipliers on $L^1(G)$ and established that a multiplier T on $L^1(G)$ has closed range if and only if there exists a factorization $T = PB$, where P is an idempotent and B an invertible multiplier. This partially resolved a question raised by Glicksberg [6] whether the factorization $T = PB$ is necessary and sufficient to ensure the closedness of TA for any multiplier T on a semisimple Banach algebra A. Various equivalent conditions have been determined in [1] and [13] under which a multiplier T has closed range. The aim of this paper is to consider this problem for a more general situation in (non-normed) topological algebras. We first establish that for an arbitrary continuous linear operator T on a complete metrizable locally convex space X, the decomposition $X = TX \oplus \mathrm{Ker}T$ ensures a factorization $T = PB$, where B is invertible, P is an idempotent, and P, B commute. We also show that the decomposition $X = TX \oplus \mathrm{Ker}T$ implies that TX is necessarily closed, and this happens if and only if there exists a commuting generalized inverse S of T. When thses equivalent conditions are considered for multipliers on Fréchet locally convex algebras, a number of other conditions also appear. Moreover, it is proved (Corollary 3.4) that if A is a semiprime Fréchet locally convex algebra and $T \in M(A)$ such that $T^2A = TA$, then TA is closed; also, in this case, T is injective if and only if it is surjective. Finally, as a converse result, it is shown (Theorem 3.6) that if A is a Fréchet locally C^*-algebra and $T \in M(A)$, then TA is closed, if, and only if, $T^2A = TA$.

The concepts are introduced as needed. We refer to [14] for the general theory of topological algebras (see also [4, 5, 9]); [9, 10] for multipliers on topological algebras; and [12] for multipliers on Banach algebras.

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2. Closed Range Operators

Let X denote a complete metrizable locally convex space with a family $\{p_n\}_{n\in N}$ of seminorms, usually called a Fréchet locally convex space, and let $B(X)$ be the algebra of all continuous linear operators of X into itself. For $T \in B(X), TX$ and Ker$T$ will denote the range and kernel of T, respectively.

First we discuss the problem in somewhat greater generality and establish that for an operator $T \in B(X)$, there exists a factorization $T = PB$ if and only if the decomposition $X = TX \oplus \text{Ker}T$ holds, where P is an idempotent, B an invertible operator and P, B commute. Moreover, when X decomposes in this way, TX is necessarily closed.

We begin with the following result which is essentially a consequence of the open mapping theorem.

Theorem 2.1. *Assume that $TX \cap \text{Ker}T = \{0\}$ and that $TX + \text{Ker}T$ is closed, for any $T \in B(X)$. Then T^nX is closed for every $n \in \mathbb{N}$.*

Proof. First we show that TX is closed with respect to the given Fréchet topology. By hypothesis, $X_0 = TX \oplus \text{Ker}T$ is closed, therefore it is a Fréchet locally convex space. Moreover, it is easy to verify that TX is a Fréchet locally convex space when equipped with the family $\{q_n\}_{n\in\text{N}}$ of seminorms given by

$$q_n(y) = p_n(y) + \inf_{x\in X, y=Tx} P_n(x),$$

for every $n \in \mathbb{N}$. Further, since $p_n(y) \leq q_n(y)$ for every $y \in TX$ and every $n \in \mathbb{N}$, the injection $TX \to X_0$ is continuous.

Define $\psi : TX \times \text{Ker}T \to X_0$ by $\psi(y, x) = y + x$. Then ψ is a continuous bijection. Therefore, by virtue of the open mapping theorem [10, Corollary 3.4, p. 30], ψ is bicontinuous. Thus $TX = \psi(TX \times \{0\})$ is closed in X_0, and hence closed in X. Thus T has closed range.

Since $TX \cap \text{Ker}T = \{0\}$, $\text{Ker}T^2 = \text{Ker}T$, and also $\text{Ker}T^n = \text{Ker}T$ for every $n \in \mathbb{N}$, we can accomplish the proof by an inductive argument. To do this, assume that T^n has closed range for some $n \in \mathbb{N}$. Since $TX \oplus \text{Ker}T = TX \oplus \text{Ker}T^n$ is closed, $T^{n+1}X = T^n(TX \oplus \text{Ker}T) = T^n(TX \oplus \text{Ker}T^n)$ is closed. □

The preceding result has the following converse.

Theorem 2.2. *Let $T \in B(X)$. If T^2X is closed, then $TX + \text{Ker}T$ is closed (without an assumption of direct sum).*

Proof. Suppose that T^2X is closed, and let $Ta_n + b_n \to c$, where $b_n \in \text{Ker}T$. Then $T^2a_n \to Tc$, so by assumption $Tc \in T^2X$, i.e., there exists an element $x \in X$ for which $Tc = T^2x$. Since $z = c - Tx \in \text{Ker}T$, it follows that $c = Tx + z \in TX + \text{Ker}T$. Thus $TX + \text{Ker}T$ is closed. □

Now we collect this information to get the following result:

Corollary 2.3 *Let $T \in B(X)$ satisfy the property $TX \cap \text{Ker}T = \{0\}$. Then the following conditions are equivalent:*

(1) *$TX + \text{Ker}T$ is closed.*

(2) *T^nX is closed for all $n \in \mathbb{N}$.*

(3) T^nX *is closed.*

(4) *The induced map* $\tilde{T} : X/\text{Ker}T \to X/\text{Ker}T$, *defined by* $\tilde{T}(x + \text{Ker}T) = Tx + \text{Ker}T$, *has closed range.*

Proof. By Theorem 2.1 together with Theorem 2.2, it remains only to show the equivalence (1) $\Leftrightarrow$ (4). Let $\pi : X \to X/\text{Ker}T$ be the quotient map. Then $\tilde{T}(X/\text{Ker}T) = \pi(TX + \text{Ker}T)$ and hence $\pi^{-1}(\tilde{T}(X/\text{Ker}T)) = TX + \text{Ker}T$. Therefore $\tilde{T}(X/\text{Ker}T)$ is closed if and only if $TX + \text{Ker}T$ is closed. This completes the proof. □

We say that an operator $T \in B(X)$ has a *generalized inverse,* and write that T has a $g-$inverse, or that T is $g-$*invertible,* if there is an operator $S \in B(X)$ such that $T = TST$ and $S = STS$. The operator T is also called *relatively regular*[7]. We make a few observations about these operators for our subsequent discussion.

Remark 1. (i) There is no gain of generality in requiring only that $T = TST$. In fact, if $T = TST$, then $S' = STS$ will satisfy $T = TS'T$, as well as $S' = S'TS'$.

(ii) If $T = TST$ and $S = STS$, then TS and ST are idempotents and hence projections for which $TS(X) = T(X)$ and $\text{Ker}T = \text{Ker}ST$. Indeed, $(TS)^2 = TSTS = TS$ and $(ST)^2 = STST = ST$. Moreover, from $T(X) = TST(X) \subseteq TS(X) \subseteq T(X)$, $\text{Ker}T \subseteq \text{Ker}(ST) \subseteq \text{Ker}(TST) = \text{Ker}T$, we obtain $TS(X) = T(X)$ and $\text{Ker}(ST) = (I - ST)X = \text{Ker}T$, where I denotes the identity element in $B(X)$.

(iii) Generally speaking, a generalized inverse of T is rarely uniquely determined. For instance, if $T = TST$, then S can be anything on $\text{Ker}T$. But there is at most one generalized inverse which commutes with the given $T \in B(X)$. In fact, if S and S' are $g-$inverses of T, both commuting with T, then $TS' = TSTS' = ST$, and hence $S' = S'TS' = S'TS = STS = S$.

There is an intimate relationship between commuting $g-$invertible operators T and the factorization problem as given below:

Theorem 2.4. *For any* $T \in B(X)$ *the following conditions are equivalent:*

(1) T *has a generalized inverse* $S \in B(X)$ *such that* $ST = TS$

(2) $TX \oplus \text{Ker}T = X$.

(3) $T = PB$, *where* $B \in B(X)$ *is invertible and* $P \in B(X)$ *is an idempotent.*

(4) $T = TCT$, *where* $C \in B(X)$ *is invertible and* $TC = CT$.

Proof. Assume that (1) holds, and let S be a $g-$inverse of T such that $ST = TS$. Then the identity $I = ST + (I - ST) = TS + (I - ST)$, together with Remark 1(ii) yields (2). Suppose that (2) holds. Then by Theorem 2.1, TX is closed. Moreover, since $T^2X = T(TX) = T(TX \oplus \text{Ker}T) = TX$ and $TX \cap \text{Ker}T = \{0\}$, it follows that $T|TX$ is invertible. Now define $B = T|TX \oplus I_{\text{Ker}T}$. Then clearly B is invertible. Let $P : X \to X$ be the projection of X onto TX with $\text{Ker}P = \text{Ker}T$, then $T = PB = BP$, and hence (3) is established. The implication (3) $\Rightarrow$ (4) follows immediately by choosing $C = B^{-1}$. Finally, if (4) holds, then $S = C^2T$ is $g-$inverse of T satisfying $ST = TS$. This completes the proof. □

Remark 2. Condition (2) of Theorem 2.4 is equivalent to the condition

$$T^2X = TX \text{ and } \text{Ker}T^2 = \text{Ker}T \quad [7, \text{Proposition 38.4}].$$

This last condition is also described by saying that T has descent and ascent both equal to 1.

We recall that T is said to have *descent* (ascent) n if n is the smallest positive integer such that $T^nX = T^{n+1}X(\mathrm{Ker}T^n = \mathrm{Ker}T^{n+1})$.

3. Closed Range Multipliers

Before proceeding to the particular situation of multipliers on topological algebras, we recall some fundamental concepts for the sake of development of the results.

An algebra A is said to be *without order,* or *proper,* if zero is the only element that annihilates the whole algebra, i.e., if $aA = \{0\}$ or $Aa = \{0\}$, then $a = 0$. By *a Fréchet locally convex algebra A,* we mean a complete metrizable locally convex algebra A whose topology is generated by a family $\{p_n\}_{n\in \mathrm{N}}$of seminorms. In what follows, A denotes a Fréchet locally convex algebra without order, unless specified otherwise explicitly. Following [9], a mapping $T : A \to A$ is said to be a *multiplier* if $x(Ty) = (Tx)y$ for all $x, y \in A$. We denote the set of all multipliers on A by $M(A)$. Because A is without order, any multiplier $T \in M(A)$ turns out to be linear; the identities $x(Ty) = T(xy)$ and $(Ty)x = T(yx)$ hold for any $x, y \in A$. Using the closed graph theorem, the definition of multiplier and the properness of A one can show that all multipliers are necessarily continuous and hence bounded (see e.g. [9], Corollary 2.3). An application of the above identities implies that $M(A)$ may be described as the commutant in $B(A)$ of all operators of multiplication (on the right or on the left) by the elements of the algebra A. It is well known that $M(A)$ is a commutative closed subalgebra of $B(A)$ with respect to the strong operator topology ([9], Theorem 2.4). The commutativity of $M(A)$ is purely algebraic and can be proved as in ([12], Theorem 1.1.1). Since $x(Ty) = T(xy)$ and $(Ty)x = T(yx)$ for any $x, y \in A$, both TA, and $\mathrm{Ker}T$ are two-sided ideals of A.

Since $M(A)$ is commutative, it follows from Remark 1 (iii) that for any $T \in M(A)$ there is at most one $g-$inverse in $M(A)$. We shall see in Theorem 3.1 that if $T \in M(A)$ has a commuting $g-$inverse at all, then this will necessarily be a multiplier. This corresponds to the fact that if a multiplier has an inverse (as a linear operator), then this inverse is necessarily a multiplier ([12], Theorem 1.1.3).

The following result is an extension of ([13], Theorem 5) to the general framework of Fréchet locally convex algebras.

Theorem 3.1. *Let A be a Fréchet locally convex algebra without order and $T \in M(A)$. Then the following statements are equivalent.*

(1) *T has a $g-$inverse S such that $ST = TS$.*

(2) *T has a $g-$inverse $S \in B(A)$ such that $TS \in M(A)$.*

(3) *T has a $g-$inverse $S \in B(A)$ such that TS commutes with T.*

(4) *T has a $g-$inverse $S \in M(A)$.*

(5) *$TA \oplus \mathrm{Ker}T = A$.*

(6) *$T^2A = TA$ and $\mathrm{Ker}T^2 = \mathrm{Ker}T$.*

(7) *$T = PB = BP$, where $B \in M(A)$ is invertible and $P \in M(A)$ is an idempotent.*

(8) *T is decomposably regular in $M(A)$, i.e., $T = TCT$, where C is an invertible multiplier.*

Proof. (1) $\Rightarrow$ (2). Let S be a $g-$inverse of T such that $ST = TS$. Then by Remark 1(ii), $P = TS$ is an idempotent for which $PA = TA$ and $KerT = \mathrm{Ker}P$, i.e., both kernel and range of P are two-sided ideals. This implies that P is a multiplier. In fact, $x = Px + (I - P)x$ implies $xPy = PxPy + (I - P)xPy$, and since $(I - P)xPy \in \mathrm{Ker}P \cap PA = \{0\}$, it follows that $xPy = PxPy$. Similarly, $(Px)y = PxPy$, hence $(Px)y = xPy$ for all $x, y \in A$.

The implication (2) $\Rightarrow$ (3) is trivial since $M(A)$ is commutative algebra.

(3) $\Rightarrow$ (5). We have already seen that if $P = TS$ then $PA = TA$. Therefore, if $x \in A$, then $Tx = Pz$ for some $z \in A$. Hence, if $Px = 0$, then $Tx = P^2z = PTx = TPx = 0$, so $\mathrm{Ker}P \subseteq \mathrm{Ker}T$. Thus it follows that $A = TA + \mathrm{Ker}T$. It only remains to show that $TA \cap \mathrm{Ker}T = \{0\}$. Let $x \in TA \cap \mathrm{Ker}T$, then $x = 0$ provided we show that $xTA = x\mathrm{Ker}T = \{0\}$. But $xTA = TxA = \{0\}$, so only $x\mathrm{Ker}T = \{0\}$ remains to be verified. If $x = Tz \in TA$, while $y \in \mathrm{Ker}T$, then $xy = (Tz)y = z(Ty) = 0$. Thus $TA \cap \mathrm{Ker}T = \{0\}$, and hence $TA \oplus \mathrm{Ker}T = A$.

By virtue of Theorem 2.4, we have thus established the equivalence of $(1), (2), (3), (5)$ and (6).

(5) $\Rightarrow$ (7). Assume that condition (5) holds (and hence also (6)), then the projection $P : A \to A$ with $PA = TA$ and $\mathrm{Ker}P = \mathrm{Ker}T$ is a multiplier, by condition (2). Consequently, $B = T + (I - P) \in M(A)$. Note that B is the same operator as the operator B described in the proof of Theorem 2.4, and hence it is invertible. Since $T = BP = PB$ we get (7).

The implication (7) $\Rightarrow$ (6) follows immediately by taking $S = PB^{-1}$.

(4) $\Rightarrow$ (5). Since $M(A)$ is commutative, this follows from the implication (1) $\Rightarrow$ (2) of Theorem 2.4.

(7) $\Rightarrow$ (8). If $T = PB = BP$, where $B \in M(A)$ is invertible and $P \in M(A)$ is idempotent, then $TB^{-1}T = PBB^{-1}T = PT = T$.

(8) $\Rightarrow$ (1). If $T = TCT$, where C is an invertible multiplier, then $S = CTC \in M(A)$ is a $g-$inverse of T satisfying $ST = TS$. This complete the proof.

□

We recall that an algebra A is said to be *semiprime* if $\{0\}$ is the only two-sided ideal J such that $J^2 = \{0\}$ ([2], Definition IV. 30.3). In other words, A is semiprime if and only if $aAa = \{0\}$ implies $a = 0$. Clearly, a semiprime algebra is without order.

One fact about multipliers on semiprime algebras that we shall use below is that they have ascent ≤ 1, i.e., $\mathrm{Ker}T^2 = \mathrm{Ker}T$. In fact, if $T^2x = 0$, then $(Tx)a(Tx) = T(xT(ax)) = T^2(xax) = (T^2x)ax = 0$ for any $a \in A$. Hence $Tx = 0$, and so $\mathrm{Ker}T^2 \subseteq \mathrm{Ker}T$. Since the reverse inclusion is trivial, it follows that $\mathrm{Ker}T^2 = \mathrm{Ker}T$ for any $T \in M(A)$, when A is a semiprime algebra.

Theorem 3.2. *Let A be a semiprime Fréchet locally convex algebra and $T \in M(A)$. Then the following conditions are equivalent to those specified in Theorem 3.1:*

(9) $T^2A = TA$, *i.e., T has descent* ≤ 1.

(10) T *has finite descent.*

Proof. We have already seen that T has ascent ≤ 1, and so the equivalence of these two conditions is a general fact (see for instance [7], §38).

(5) $\Rightarrow$ (9). This follows immediately from Remark 2.

(9) $\Rightarrow$ (5). Assume that $T^2A = TA$. Since $\mathrm{Ker}T^2 = \mathrm{Ker}T$, it follows from ([7], Proposition 38.4) that $A = TA \oplus \mathrm{Ker}T$. □

Corollary 3.3. *Let A be a semiprime Fréchet locally convex algebra and $T \in M(A)$. Then any one of the conditions of Theorem 3.1 implies that* $\mathrm{dist}(0, \sigma(T)\backslash\{0\}) > 0$.

Proof. Clearly only the case $0 \in \sigma(T)$ concerns us. For the sake of definiteness, assume that condition (5) of Theorem 3.1 holds, i.e., $A = TA \oplus \mathrm{Ker}T$. Since the operator $(T - \lambda I)$ is invertible if and only if $(T - \lambda I)|TA$ and $(T - \lambda I)|\mathrm{Ker}T$ both are invertible, the result then follows because $T|TA$ is invertible, while $\sigma(T|\mathrm{Ker}T) = \{0\}$. □

Corollary 3.4. *Let A be a semiprime Fr'echet locally convex algebra and $T \in M(A)$. If $T^2A = TA$, then TA is closed.*

Proof. Assume that $T^2A = TA$. Since $\mathrm{Ker}T^2 = \mathrm{Ker}T$, as we have already seen, it follows from condition (5) of Theorem 3.1 that $A = TA \oplus \mathrm{Ker}T$. Hence by Theorem 2.1, TA is closed. □

Corollary 3.5. *Let A be a semiprime Fréchet locally convex algebra and $T \in M(A)$. If $T^2A = TA$, then T is injective if and only if it is surjective.*

Proof. Let T be surjective. Since $TA \cap \mathrm{Ker}T = \{0\}$ implies $\mathrm{Ker}T = \{0\}$, we see that T is injective. Conversely, suppose that $\mathrm{Ker}T = \{0\}$. Since $T^2A = TA$ by assumption, it follows from Theorem 3,2 that $A = TA \oplus \mathrm{Ker}T$, and so $TA = A$, i.e., T is surjective. □

Remark 3. The converse of Corollary 3.4 need not be in the case of general Banach algebras as shown in [13]. For instance, if $A = A(D)-$ the disc algebra of complex-valued continuous functions on the closed unit disc D which are analytic in the interior of D, and T_g is the multiplication operator, corresponding to the function $g(z) = z$ for every $z \in D$, defined by $(T_g f)(z) = zf(z)$ for every $f \in A(D)$, then $T_g \in M(A)$. Moreover, $T_gA = \{f \in A : f(0) = 0\}$ and $T_g^2A = \{f \in A : f(0) = f'(0) = 0\}$. Both T_gA and T_g^2A are closed, but clearly $T_gA \neq T_g^2A$. This also shows that condition (5) of Theorem 3.1 cannot be relaxed to that of Theorem 2.1, i.e., to the requirement that $TA \oplus \mathrm{Ker}T$ be closed; since $\mathrm{Ker}T_g = \{0\}$, $T_gA \oplus \mathrm{Ker}T_g$ is closed, but none of the conditions of Theorem 3.1 holds for T_g.

It is, however, shown in ([13], Theorem 13) that the converse of Corollary 3.4 dose hold if A is C^*- algebra and $T \in M(A)$. But, we observe below (Theorem 3.6) that it is true even when A is a Fréchet locally C^*-algebra. This provides a positive answer to a question raised by the referee. To do this, we recall some definitions.

Let A be a complete Hausdorff locally m-convex algebra whose topology is generated by a family $\{p_\gamma : \gamma \in J\}$ of submultiplicative seminorms. Following Inoue [11], A is called a *locally* $C^* - algebra$ if it has an involution $*$ and $p_\gamma(x^*x) = (p_\gamma(x))^2$ for all $\gamma \in J$ and $x \in A$. A net $\{e_\alpha : \alpha \in I\}$ in A is called a *bounded approximate identity* (abbreviated bai) if $p_\gamma(e_\alpha) \leq 1$ for all $\gamma \in J, \alpha \in I$ and $\lim_\alpha e_\alpha x = \lim_\alpha xe_\alpha = x$ for all $x \in A$. Every locally C^*-algebra has a bai (([11], Theorem 2.6),([4], Theorem 4.5)) and hence is also without order. Besides, Craw ([3], p. 610) has constructed a subalgebra of $L^1(\mathbb{R})$ which is a Fréchet locally m- convex algebra with bai.

Theorem 3.6. *Let A be Fréchet locally C^*-algebra and $T \in M(A)$. Then TA is closed if, and only if, $T^2A = TA$.*

Proof. Suppose that TA is closed. Then it is a closed two-sided ideal in a locally C^*-algebra and so has a bai. Since TA is also Fréchet, by a generalized version of the Cohen's factorizatiopn theorem ([3], p. 610), for each $x \in TA$, there exist $y, z \in TA$ such that $x = yz$; i.e., $TA = (TA)^2$. Clearly, $T^2A \subseteq TA = (TA)^2$. On the other hand, for any $x, y \in A$,

$$TxTy = T(xTy) = T^2(xy) \in T^2A,$$

and so $(TA)^2 \subseteq T^2A$. Thus $TA = T^2A$. Conversely, suppose that $T^2A = TA$. In view of Corollary 3.4, it suffices to show that A is semiprime. Using the terminology of M. Fragoulopoulou [4, 5], A is $*-$semisimple([4], Corollary 5.6), and hence semisimple([5], Lemma 8.14(ii)). Consequently, by ([2], p. 155, Proposition 30.5), A is semiprime. □

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