APPROXIMATING COMMON FIXED POINTS OF TWO ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

SAFEER HUSSAIN KHAN AND WATARU TAKAHASHI

Received May 12, 2000

Abstract. This paper deals with appproximating common fixed points of two asymptotically nonexpansive mappings through weak and strong convergence of an iterative sequence in a uniformly convex Banach space.

1. Introduction

Let C be a nonempty subset of a real Banach space E. A mapping S of C into itself is called asymptotically nonexpansive if for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, $\|S^n x - S^n y\| \le k_n \|x - y\|$ holds for all $x, y \in C$ and all $n = 1, 2, \ldots$. S is also called uniformly k-Lipschitzian if for some $k > 0$, $\|S^n x - S^n y\| \le k \|x - y\|$ is true for all $n = 1$, $2, \ldots$ and all $x, y \in C$. Moreover, S is termed as nonexpansive if $\|Sx - Sy\| \le \|x - y\|$ for all $x, y \in C$ and quasi-nonexpansive if $F(S)$, the set of fixed points of S, is nonempty and $\|Sx - y\| \le \|x - y\|$ for all $x \in C$ and $y \in F(S)$. Das and Debata [1] considered the following iteration scheme for two quasi-nonexpansive mappings S and T :

$$x_1 \in C, \;\; x_{n+1} = (1 - a_n)x_n + a_n S[(1 - b_n)x_n + b_n T x_n],$$

for all $n = 1, 2, \ldots$, where $\{a_n\}$ and $\{b_n\}$ are in $[0, 1]$. Takahashi and Tamura [8] studied the above scheme for two nonexpansive mappings. As is clear from definitions, the idea of asymptotic nonexpansiveness is more general than both nonexpansiveness and quasi-nonexpansiveness. Asymptotically nonexpansive mappings, since their introduction in 1972 by K.Goebel and W.A.Kirk [2], have remained under study by various authors. For example, see [4] and [6] besides [2].

In this paper, we take up the problem of approximating the common fixed points of two asymptotically nonexpansive mappings S and T through weak and strong convergence of the sequence defined by :

$$x_1 \in C, \;\; x_{n+1} = (1 - a_n)x_n + a_n S^n[(1 - b_n)x_n + b_n T^n x_n], \tag{1.1}$$

for all $n = 1, 2, \ldots$, where $\{a_n\}$ and $\{b_n\}$ in $[0, 1]$ satisfy certain conditions.

2. Preliminaries

Let E be a Banach space and let C be a nonempty bounded convex subset of E. We need the following lemma which can be found in [6].

Lemma 1. *Suppose that E is a uniformly convex Banach space and $0 < p \le t_n \le q < 1$ for all $n = 1, 2, \ldots$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that $\limsup_{n\to\infty} \|x_n\| \le r$, $\limsup_{n\to\infty} \|y_n\| \le r$ and $\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r$ hold for some $r \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

1991 *Mathematics Subject Classification.* Primary: 47H09, 49M05.

Key words and phrases. Asymptotically nonexpansive mapping, common fixed point, weak and strong convergence, iteration scheme.

We recall that a Banach space E is said to satisfy Opial's condition [5] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$. Moreover, we also know that a mapping $T : C \to E$ is called demiclosed with respect to $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E, x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

Now we state another lemma due to J. Górnicki [3] which we shall use in our weak convergence theorem.

Lemma 2. *Let E be a uniformly convex Banach space satisfying Opial's condition and let C be a nonempty, closed and convex subset of E. Let T be asymptotically nonexpansive mapping of C into itself. Then $I - T$ is demiclosed with respcet to zero.*

We shall now prove the following lemma which plays a vital role in our later work. This lemma generalizes the corresponding lemma of [7] where it was proved for one mapping case. However, we not only prove it for two mappings case but also the calculations are made much simpler.

Lemma 3. *Let E be a normed space and let C be a nonempty bounded, closed and convex subset of E. Let, for $k > 0$, S and T be uniformly k-Lipschitzian mappings of C into itself. Define a sequence $\{x_n\}$ as in* (1.1). *If*

$$\lim_{n\to\infty} \|x_n - S^n x_n\| = 0 = \lim_{n\to\infty} \|x_n - T^n x_n\|,$$

then

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0 = \lim_{n\to\infty} \|x_n - Tx_n\|.$$

Proof. Set

$$c_n = \|x_n - S^n x_n\|$$

and

$$d_n = \|x_n - T^n x_n\|$$

for all $n = 1, 2, \ldots$. Also put, for simplicity, $y_n = (1 - b_n)x_n + b_n T^n x_n$, $n = 1, 2, \ldots$ so that (1.1) becomes

$$x_{n+1} = (1 - a_n)x_n + a_n S^n y_n$$

and

$$\begin{aligned}
\|x_n - x_{n+1}\| &= a_n \|x_n - S^n y_n\| \\
&\leq \|x_n - S^n y_n\| \\
&\leq \|x_n - S^n x_n\| + \|S^n x_n - S^n y_n\| \\
&\leq c_n + k\|x_n - y_n\| \\
&= c_n + kb_n \|x_n - T^n x_n\| \\
&\leq c_n + kd_n
\end{aligned}$$

That is,

$$\|x_n - x_{n+1}\| \leq c_n + kd_n. \tag{2.1}$$

Using (2.1), we get

$$\begin{aligned}\|x_{n+1} - Sx_{n+1}\| &\le \|x_{n+1} - S^{n+1}x_{n+1}\| + \|S^{n+1}x_{n+1} - Sx_{n+1}\| \\ &\le c_{n+1} + k\|x_{n+1} - S^n x_{n+1}\| \\ &\le c_{n+1} + k(\|x_n - x_{n+1}\| + \|x_n - S^n x_n\| \\ &\quad + \|S^n x_n - S^n x_{n+1}\|) \\ &\le c_{n+1} + k[(k+1)\|x_n - x_{n+1}\| + c_n] \\ &= c_{n+1} + kc_n + (k^2+k)\|x_n - x_{n+1}\| \\ &\le c_{n+1} + kc_n + (k^2+k)(c_n + kd_n) \\ &= c_{n+1} + (k^2+2k)c_n + (k^3+k^2)d_n\end{aligned}$$

which gives

$$\limsup_{n\to\infty} \|x_{n+1} - Sx_{n+1}\| \le 0$$

because $\lim_{n\to\infty} c_n = 0 = \lim_{n\to\infty} d_n$. Hence

$$\text{(A)} \qquad \lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

Similarly,

$$\begin{aligned}\|x_{n+1} - Tx_{n+1}\| &\le \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - Tx_{n+1}\| \\ &\le d_{n+1} + kd_n + (k^2+k)\|x_n - x_{n+1}\| \\ &\le d_{n+1} + kd_n + (k^2+k)(c_n + kd_n) \\ &= d_{n+1} + (k^3+k^2+k)d_n + (k^2+k)c_n\end{aligned}$$

which implies

$$\limsup_{n\to\infty} \|x_{n+1} - Tx_{n+1}\| \le 0.$$

Consequently,

$$\text{(B)} \qquad \lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

By (A) and (B), we get the desired result. □

3. Weak and Strong Convergence Theorems

We first prove the following lemma which, in fact, forms a major part of the proofs of both weak and strong convergence theorems.

Lemma 4. *Let E be a uniformly convex Banach space and let C be its bounded, closed and convex subset. Let S and T from C into itself be two mappings satisfying*

$$\|S^n x - S^n y\| \le k_n \|x - y\|$$

and

$$\|T^n x - T^n y\| \le k_n \|x - y\|,$$

for all $n = 1, 2, \ldots$, where $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Define a sequence $\{x_n\}$ in C as:

$$x_1 \in C, \;\; x_{n+1} = (1-a_n)x_n + a_n S^n[(1-b_n)x_n + b_n T^n x_n],$$

for all $n = 1, 2, \ldots$, where $\{a_n\}$ and $\{b_n\}$ are sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. If $F(S) \cap F(T) \ne \phi$ then

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0 = \lim_{n\to\infty} \|Tx_n - x_n\|.$$

Proof. Let $p \in F(S) \cap F(T)$ and put $y_n = (1-b_n)x_n + b_n T^n x_n$ for the sake of simplicity. A straightforward calculation gives

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-a_n)(x_n - p) + a_n(S^n y_n - p)\| \\ &\leq [(1-a_n) + a_n k_n(1-b_n) + a_n k_n^2 b_n]\|x_n - p\|.\end{aligned}$$

Setting $V_n = (1-a_n) + a_n k_n(1-b_n) + a_n k_n^2 b_n$, we can write $\|x_{n+1} - p\| \leq V_n \|x_n - p\|$ for all $n = 1, 2, \ldots$. By mathematical induction, $\|x_{n+m} - p\| \leq \left(\prod_{i=n}^{n+m-1} V_i\right) \|x_n - p\|$ for all $m, n = 1, 2, \ldots$. Also noting $\sum_{n=1}^{\infty}(V_n - 1) < \infty$, we obtain $\lim_{n\to\infty} \prod_{i=n}^{\infty} V_i = 1$ and hence $\lim_{n\to\infty} \|x_n - p\|$ exists. Let $\lim_{n\to\infty} \|x_n - p\| = c$ where $c \geq 0$ is a real number. If $c = 0$, the result is obvious. So we assume $c > 0$. Now

$$\|T^n x_n - p\| \leq k_n \|x_n - p\|$$

for all $n = 1, 2, \ldots$, so

$$\limsup_{n\to\infty} \|T^n x_n - p\| \leq c.$$

Also

$$\begin{aligned}\|y_n - p\| &= \|(1-b_n)(x_n - p) + b_n(T^n x_n - p)\| \\ &\leq (1-b_n)\|x_n - p\| + k_n b_n \|x_n - p\| \\ &= \|x_n - p\| + (k_n - 1)b_n \|x_n - p\| \\ &\leq \|x_n - p\| + (1-\delta)(k_n - 1)\|x_n - p\|\end{aligned}$$

gives

$$\limsup_{n\to\infty} \|y_n - p\| \leq c. \tag{3.1}$$

Next,

$$\|S^n y_n - p\| \leq k_n \|y_n - p\|$$

gives by virtue of (3.1) and $k_n \to 1$ as $n \to \infty$ that

$$\limsup_{n\to\infty} \|S^n y_n - p\| \leq c.$$

Moreover, $c = \lim_{n\to\infty} \|x_{n+1} - p\|$ means that

$$\lim_{n\to\infty} \|(1-a_n)(x_n - p) + a_n(S^n y_n - p)\| = c.$$

Applying Lemma 1,

$$\lim_{n\to\infty} \|S^n y_n - x_n\| = 0. \tag{3.2}$$

Now

$$\begin{aligned}\|x_n - p\| &\leq \|x_n - S^n y_n\| + \|S^n y_n - p\| \\ &\leq \|x_n - S^n y_n\| + k_n \|y_n - p\|\end{aligned}$$

yields that

$$c \leq \liminf_{n\to\infty} \|y_n - p\|. \tag{3.3}$$

By (3.1) and (3.3), we obtain

$$\lim_{n\to\infty} \|y_n - p\| = c. \tag{3.4}$$

That is

$$\lim_{n\to\infty} \|(1-b_n)(x_n - p) + b_n(T^n x_n - p)\| = c.$$

Again by Lemma 1, we get

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{3.5}$$

Then

$$\begin{aligned}\|S^n x_n - x_n\| &\le \|S^n x_n - S^n y_n\| + \|S^n y_n - x_n\| \\ &\le k_n\|x_n - y_n\| + \|S^n y_n - x_n\| \\ &= k_n b_n\|T^n x_n - x_n\| + \|S^n y_n - x_n\| \\ &\le k_n(1-\delta)\|T^n x_n - x_n\| + \|S^n y_n - x_n\|\end{aligned}$$

implies together with (3.5) and (3.2) that

$$\lim_{n\to\infty} \|S^n x_n - x_n\| = 0 = \lim_{n\to\infty} \|T^n x_n - x_n\|. \tag{3.6}$$

Lemma 3 now reveals that

$$\lim_{n\to\infty} \|S x_n - x_n\| = 0 = \lim_{n\to\infty} \|T x_n - x_n\| \tag{3.7}$$

which is the desired result. □

Theorem 1. *Let E be a uniformly convex Banach space satisfying Opial's condition and let C, S, T and $\{x_n\}$ be as taken in Lemma 4. If $F(S) \cap F(T) \neq \phi$ then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Proof. Let p be a common fixed point of S and T. Then $\lim_{n\to\infty} \|x_n - p\|$ exists as proved in Lemma 4. We prove that $\{x_n\}$ has a unique weak subsequential limit in $F(S) \cap F(T)$. For, let u and v be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemma 4, $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$ and $I - S$ is demiclosed with respect to zero by Lemma 2, therefore we obtain $Su = u$. Similarly, $Tu = u$. Again in the same fashion, we can prove that $v \in F(S) \cap F(T)$. Next, we prove the uniqueness. To this end, if u and v are distinct then by Opial's condition,

$$\begin{aligned}\lim_{n\to\infty} \|x_n - u\| &= \lim_{n_i\to\infty} \|x_{n_i} - u\| \\ &< \lim_{n_i\to\infty} \|x_{n_i} - v\| \\ &= \lim_{n\to\infty} \|x_n - v\| \\ &= \lim_{n_j\to\infty} \|x_{n_j} - v\| \\ &< \lim_{n_j\to\infty} \|x_{n_j} - u\| \\ &= \lim_{n\to\infty} \|x_n - u\|.\end{aligned}$$

This is again a contradiction whereby completing the proof. □

Remark 1. Above theorem contains Theorem 2.1 of J. Schu [6] as a special case when $T = I$, the identity mapping.

Finally, we approximate common fixed points by the following strong convergence theorem.

Theorem 2. *Let E be a uniformly convex Banach space and let C be its compact convex subset and S, T and $\{x_n\}$ as in Lemma 4. If $F(S) \cap F(T) \neq \phi$ then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Proof. By Lemma 4, $\lim_{n\to\infty} \|Sx_n - x_n\| = 0 = \lim_{n\to\infty} \|Tx_n - x_n\|$. Since C is compact so there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to q$ (say) in C. Continuity of S and T gives $Sx_{n_i} \to Sq$ and $Tx_{n_i} \to Tq$ as $n_i \to \infty$. Then by (3.7),

$$\|Sq - q\| = 0 = \|Tq - q\|.$$

This yields $q \in F(S) \cap F(T)$ so that $\{x_{n_i}\}$ converges strongly to q in $F(S) \cap F(T)$. But again by Lemma 4, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(S) \cap F(T)$ therefore $\{x_n\}$ must itself converge to $q \in F(S) \cap F(T)$. This completes the proof. □

References

[1] G. Das and J. P. Debata, *Fixed points of quasi-nonexpansive mappings*, Indian J. Pure Appl. Math., **17** (1986), 1263–1269.

[2] K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **35(1)** (1972), 171–174.

[3] J. Górnicki, *Weak convergence theorems for asymptotically nonexpansive mappings in uniformly convex Banach spaces*, Comment. Math. Univ. Carolin. **30** (1989), 249–252.

[4] S. H. Khan and W. Takahashi, *Iterative approximation of fixed points of asymptotically nonexpansive mappings with compact domains*, to appear in "PanAmerican Mathematical Journal".

[5] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc., **73** (1967), 591–597.

[6] J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc. **43** (1991), 153–159.

[7] J. Schu, *Iterative construction of fixed points of asymptotically nonexpansive mappings*, J. Math. Anal. Appl. **158** (1991), 407–413.

[8] W. Takahashi and T. Tamura, *Convergence theorems for a pair of nonexpansive mappings*, J. Convex Analysis, **5(1)** (1995), 45–58.

Department of Mathematical and Computing Sciences, Tokyo Institute of Technology,
O-okayama, Meguro-ku, Tokyo 152–8552, Japan.
email(Safeer Hussain Khan): r970930@is.titech.ac.jp
email(Wataru Takahashi): wataru@is.titech.ac.jp