A RESULT CONCERNING DERIVATIONS IN NONCOMMUTATIVE BANACH ALGEBRAS

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Abstract. In this paper we show that under certain conditions the range of derivations of Banach algebras is contained in the radical. Our results extend those obtained in [14] and [2]. For example, let A be a Banach algebra over the complex field. Suppose there exists a derivation $D : A \to A$, such that $\alpha D^3 + D^2$ is a derivation for some $\alpha \in \mathbb{C}$. Then D maps into its radical.

1. Introduction

Let A be a complex Banach algebra. For elements a, b in A, we shall denote by $[a, b]$ the commutator $ab - ba$. A *derivation* on an algebra A is a linear mapping $D : A \to A$ that satisfies $D(ab) = aD(b) + D(a)b$ for all $a, b \in A$. A mapping f on a ring R is said to be *centralizing* on R if $[f(x), x] \in Z(R)$ holds for all $x \in R$, where $Z(R)$ is the center of the ring R. Throughout this paper, the Jacobson radical (prime radical) of A will be denoted by $\mathrm{rad}(A)$ $(\mathrm{prad}(A))$. Recall that $\mathrm{rad}(A)(\mathrm{prad}(A))$ is the intersection of all primitive ideals(prime ideals) of A and that $\mathrm{prad}(A) \subseteq \mathrm{rad}(A)$.

In 1955, I. M. Singer and J. Wermer [12] proved that every continuous derivation on a commutative Banach algebra maps the algebra into its Jacobson radical. They conjectured that the assumption of continuity can be removed. After more than thirty years, M. P. Thomas[13] confirmed this conjecture. A noncommutative version of the Singer–Wermer theorem was proved by B. Yood[15] who showed that a continuous derivation D on a Banach algebra A satisfying $[D(a), b] \in \mathrm{rad}(A)$ for all $a, b \in A$ has the range in $\mathrm{rad}(A)$. Somewhat later, M. Brešar and J. Vukman [1] generalized Yood's result by showing that the same conclusion holds under the weaker assumption $[D(a), a] \in \mathrm{rad}(A)$ for all $a \in A$. A similar result, treating continuous centralizing derivations on Banach algebras, was obtained later by M. Mathieu and G. J. Murphy [8]. Recently M. Brešar [3] generalized the theorems mentioned by showing that a continuous derivation D must map into $\mathrm{rad}(A)$ provided that $[D(a), a] \in Q(A)$ for all $a \in A$, where $Q(A)$ is the set of quasinilpotent elements of A. In view of Thomas' theorem it seems reasonable to conjecture that in all these results the continuity assumption is superfluous. M. Mathieu and V. Runde [9] showed that this is indeed so in a result on centralizing derivations, that is, they showed that the result in [8] holds true without assuming the continuity. The author [5] showed that the result of [1] holds true without assuming the continuity if one replaces the assumption $[D(a), a] \in \mathrm{rad}(A)$ by a somewhat stronger assumption $[D(a), a] \in \mathrm{prad}(A)$. The same result was also obtained by Mathieu[6]. We remark that the question whether all (possibly discontinuous) derivations on noncommutative Banach algebras satisfying $[D(a), a] \in \mathrm{rad}(A)$ for all $a \in A$

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have their range in $\operatorname{rad}(A)$ is equivalent to the so-called noncommutative Singer–Wermer conjecture, that is, to the question whether every primitive ideal is invariant under any (possibly discontinuous) derivation [7]. The aim of this paper is to prove similar results to those obtained in [14] and [2], however, without assuming the continuity of derivations.

Recall the concept of separating space of a linear operator. Let X, Y be Banach spaces, and $T : X \to Y$ be a linear mapping. Then

$$S(T) = \{y \in Y | \text{there is a sequence } x_n \text{ in } X \text{ with } x_n \to 0 \text{ and } Tx_n \to y\}$$

is said to be the *separating space* of T; it is a closed linear subspace of Y. By the closed graph theorem, $S(T)$ is zero if and only if T is continuous. A closed 2-sided ideal J of A is a *separating ideal* if, for each sequence a_n in A, there exists N such that $\overline{(Ja_n \cdots a_1)} = \overline{(Ja_N \cdots a_1)}$ for all $n \geqslant N$. By the Stability lemma[11], it is easy to see that the separating space of every derivation is a separating ideal.

The following results will be used in the proof of the theorems.

Lemma 1 [9]. *Let D be a derivation on a ring R. Then D fixes each minimal prime ideal P of R such that R/P is torsion-free.*

Lemma 2 [4]. *Let P be a minimal prime ideal of A such that a separating ideal $J \nsubseteq P$. Then P is closed.*

We now state three theorems which present the motivation for the present article.

Theorem A [14, theorem 1]. *Let A be a Banach algebra over the complex field $\mathbb{C}$. Suppose $D : A \to A$ is a continuous derivation such that $\alpha D^3 + D^2$ is a derivation for some $\alpha \in \mathbb{C}$. In this case D maps A into its radical.*

Theorem B [2, theorem 1]. *Let D and G be continuous derivations of a Banach algebra A such that $[D^2(x) + G(x), x] \in \operatorname{rad}(A)$ for all x in A. Then both D and G map A into $\operatorname{rad}(A)$.*

Theorem C [2, theorem 3]. *Let D be a continuous derivation on a Banach algebra A. If $[D(x), x]^2 \in \operatorname{rad}(A)$ for all x in A, then D maps A into $\operatorname{rad}(A)$.*

Our goal is to remove the continuity assumption in all these theorems, however, in Theorems B and C we shall have to assume the stronger assumption that $[D^2(x)+G(x), x]$ and $[D(x), x]^2$ always lies in $\operatorname{prad}(A)$ instead of in $\operatorname{rad}(A)$.

2. Results

The proofs of our results rest heavily on the proofs of the original theorems and we will repeatedly use the same method as in the proof of Theorem 1.

Theorem 1. *Let A be a Banach algebra over the complex field $\mathbb{C}$. Suppose $D : A \to A$ is a derivation such that $\alpha D^3 + D^2$ is a derivation for some $\alpha \in \mathbb{C}$. In this case D maps A into its radical.*

Proof. Let Q be a primitive ideal of A. Using Zorn's lemma, we can choose a minimal prime ideal $P \subseteq Q$ of A. Since D leaves all such minimal prime ideals invariant by Lemma 1, D

induces a derivation D_P on the prime algebra A/P, defined by $D_P(x+P) = D(x)+P$, for all $x \in A$.

First we consider the case that P is closed, so that A/P is a prime Banach algebra. In the case A/P is commutative, and so Thomas' theorem [13] implies that $D_P(A/P) \subseteq \operatorname{rad}(A/P) \subseteq Q/P$, so $D(A) \subseteq Q$.

We consider the case A/P is noncommutative. The assumption of the theorem implies that $\alpha {D_P}^3 + {D_P}^2$ is a derivation. Let us assume that $\alpha = 0$. In the case we have ${D_P}^2$ is a derivation, and since A/P is prime, it follows from Theorem 1 of [10] that $D_P = 0$. In the case $\alpha \neq 0$, all the assumptions of Theorem 2 of [14] are fulfilled(with $D_1 = D_P$, and $D_2 = D_P/\alpha$). Thus we have $D_P = 0$ or $D_P/\alpha = 0$. In any case $D_P = 0$.

Secondly we consider the case that P is not closed, so then the separating space $S(D) \subseteq P$ by Lemma 2. Let $\pi : A \to A/\overline{P}$ denote the canonical epimorphism. We have, by [11, Chapter 1],

$$S(\pi \circ D) = \overline{\pi(S(D))} = 0$$

whence $\pi \circ D$ is continuous. As a result, $(\pi \circ D)\overline{P} = 0$, that is, $D(\overline{P}) \subseteq \overline{P}$. Hence we induce a derivation $D_{\overline{P}}$ on $A/\overline{P}$ as above. Then $D_{\overline{P}}$ is continuous by Lemma 1.4 [11]. Using again the assumption of the theorem, we see that $D_{\overline{P}} : A/\overline{P} \to A/\overline{P}$ is a continuous derivation such that $\alpha {D_{\overline{P}}}^3 + {D_{\overline{P}}}^2$ is a derivation for some $\alpha \in C$. Hence Theorem A implies that $D_{\overline{P}}(A/\overline{P}) \subseteq \operatorname{rad}(A/\overline{P})$, so $D(A) \subseteq Q$.

We proved that in any case $D(A) \subseteq Q$ for every primitive ideal Q, that is, $D(A) \subseteq \operatorname{rad}(A)$.

Corollary 1 [14, Theorem 3]**.** *Let A be a semisimple Banach algebra over the complex field C. Suppose there exists a derivation $D : A \to A$, such that $\alpha D^3 + D^2$ is a derivation for some $\alpha \in C$. In this case $D = 0$.*

Theorem 2. *Let D and G be derivations of a Banach algebra A such that $[D^2(x) + G(x), x] \in \operatorname{prad}(A)$ for all x in A. Then both D and G map A into $\operatorname{rad}(A)$.*

Proof. For each minimal prime ideal $P \subseteq Q$, Q a primitive ideal, we argue as in the proof of Theorem 1 for each of D and G to see that $[{D_P}^2(x) + G_P(x), x] = 0$ for all x in A/P.

First consider the case that P is closed. Then A/P is a prime Banach algebra. If A/P is commutative, then $D(A) \subseteq Q$ as before. Similarly we get $G(A) \subseteq Q$. If A/P is noncommutative, then we see that there is no loss of generality in assuming that A is prime, and that $[D^2(x) + G(x), x] = 0$.

We must show that $D = 0$ and $G = 0$. The proof of Theorem B remains valid under the hypothesis that the algebra is prime, without the use of continuity of D and G. It follows that $D = G = 0$.

Secondly, consider the case that P is not closed. Proceeding as in Theorem 1, each of the induced derivations $D_{\overline{P}}$ and $G_{\overline{P}}$ is continuous on the Banach algebra $A/\overline{P}$.

From the hypothesis of the theorem, we obtain that $[{D_{\overline{P}}}^2(x) + G_{\overline{P}}(x), x]$ is contained $\operatorname{prad}(A/\overline{P})$. Hence all the conditions in the proof of the original theorem hold, so it follows that $D_{\overline{P}}(A/\overline{P}) \subseteq \operatorname{rad}(A/\overline{P})$ and $G_{\overline{P}}(A/\overline{P}) \subseteq \operatorname{rad}(A/\overline{P})$, and so $D(A) \subseteq Q$ and $G(A) \subseteq Q$. In any case it follows that $D(A) \subseteq Q$ for every primitive ideal Q. The proof of the theorem is complete.

Theorem 3. *Let D be a derivation on a Banach algebra A. If $[D(x), x]^2$ $\in \operatorname{prad}(A)$ for all x in A, then D maps A into $\operatorname{rad}(A)$.*

Proof. For each minimal prime ideal $P \subseteq Q$, Q a primitive ideal, the same argument as before leads to $[D_P(x), x]^2 = 0$ for all x in A/P.

First we consider the case that P is closed. If A/P is commutative, then this case is the same as in Theorem 1.

In the case that A/P is noncommutative, we see that there is no loss of generality in assuming that A is prime, noncommutative, and $[D(x), x]^2 = 0$ for all x in A. Now, in this situation one can show that $D = 0$ in the same manner as in the proof Theorem C without using the continuity of D. Thus, $D(A) \subseteq P \subseteq Q$.

Secondly, we consider the case that P is not closed. An argument analogous to that used in Theorem 1 shows that the new derivation $D_{\overline{P}}$ induced from D is continuous with $[D_{\overline{P}}(x), x]^2 \in \operatorname{rad}(A/\overline{P})$ for all x in $A/\overline{P}$. Hence the hypothesis of Theorem C holds, and so it follows that $D_{\overline{P}}(A/\overline{P}) \subseteq \operatorname{rad}(A/\overline{P})$, and hence $D(A) \subseteq Q$. Thus, in any case we have that $D(A) \subseteq Q$ where for Q is an arbitrary primitive ideal. The proof of the theorem is complete.

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