INTEGRALS, SUMS OF RECIPROCAL POWERS AND MITTAG-LEFFLER EXPANSIONS

J.M. AMIGÓ

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Abstract. The zeta series with odd exponents and the alternating series of the positive odd integers to an even power can be expressed as infinite integrals involving derivatives of some hyperbolic functions. These integral formulas can be derived in a straightforward way from the Mittag-Leffler series of the corresponding hyperbolic function.

1. Introduction

Let $\zeta(n)$ denote, as usual, the zeta series of exponent $n = 2, 3, ...$,

$$\zeta(n) = \sum_{\nu=1}^{\infty} \frac{1}{\nu^n}$$

Other related series of reciprocal powers are [1, Ch. 23]

$$\eta(n) = \sum_{\nu=1}^{\infty} \frac{(-)^{\nu+1}}{\nu^n} = (1 - 2^{1-n})\zeta(n) \tag{1}$$

$$\lambda(n) = \sum_{\nu=0}^{\infty} \frac{1}{(2\nu+1)^n} = (1 - 2^{-n})\zeta(n) \tag{2}$$

for $n = 2, 3, ...$ ($\eta(1) = \ln 2$) and

$$L(n) = \sum_{\nu=0}^{\infty} \frac{(-)^{\nu}}{(2\nu+1)^n} \quad (n = 1, 2, ...) \tag{3}$$

Remember that

$$\zeta(2n) = (-)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n} = \frac{(2\pi)^{2n}}{2(2n)!} |B_{2n}| \qquad (n = 1, 2, ...) \tag{4}$$

where $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30, ...$ are the Bernoulli numbers, and

$$L(2n+1) = (-)^n \frac{(\pi/2)^{2n+1}}{2(2n)!} E_{2n} = \frac{(\pi/2)^{2n+1}}{2(2n)!} |E_{2n}| \qquad (n = 0, 1, ...) \tag{5}$$

where $E_0 = 1$, $E_4 = 5, ...$ (all integers) are the Euler numbers.

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No similar numerical formulas for $\zeta(2n+1)$ and $L(2n)$ are known. Instead, some integral expressions can be derived in a variety of ways, usually from properties of the transcendental functions $\zeta(z)$ and $\Gamma(z)$, $z \in \mathbb{C}$, or (in a more involved way) via Jacobian elliptic functions. But there are also more elementary techniques based, for example, on the Mittag-Leffler expansions of some hyperbolic functions [5, Ch. 7]. Indeed, it can be shown that

1. from
$$\operatorname{sech} x = \pi \sum_{k=0}^{\infty} (-)^k \frac{2k+1}{\left(\frac{(2k+1)\pi}{2}\right)^2 + x^2}$$
it follows
$$L(2n) = (-)^n \frac{\pi^{2n-1}}{2^{2n}(2n-1)!} \int_0^{\infty} \frac{dx}{x} \frac{d^{2n-1} \operatorname{sech} x}{dx^{2n-1}} \tag{6}$$

2. from
$$\tanh x = 2x \sum_{k=0}^{\infty} \frac{1}{\left(\frac{(2k+1)\pi}{2}\right)^2 + x^2}$$
it follows
$$\zeta(2n+1) = (-)^n \frac{\pi^{2n}}{(2^{2n+1}-1)(2n)!} \int_0^{\infty} \frac{dx}{x} \frac{d^{2n} \tanh x}{dx^{2n}} \tag{7}$$

3. from
$$x \operatorname{csch} x = 1 - 2x^2 \sum_{k=1}^{\infty} \frac{(-)^{k+1}}{k^2\pi^2 + x^2}$$
it follows
$$\zeta(2n+1) = (-)^{n+1} \frac{(2\pi)^{2n}}{(2^{2n}-1)(2n+1)!} \int_0^{\infty} \frac{dx}{x} \frac{d^{2n+1}(x \operatorname{csch} x)}{dx^{2n+1}} \tag{8}$$

4. from
$$x \coth x = 1 + 2x^2 \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2 + x^2}$$
it follows
$$\zeta(2n+1) = (-)^n \frac{\pi^{2n}}{(2n+1)!} \int_0^{\infty} \frac{dx}{x} \frac{d^{2n+1}(x \coth x)}{dx^{2n+1}} \tag{9}$$

This paper can be considered a continuation of [3], which contains a proof of (8) based on the corresponding Mittag-Leffler expansion. The proofs of (7) and (9) are formally analogous. In this paper we extend this technique to the L-series by showing (6). Other interesting integral expressions for $L(2n)$ will be also derived (v.g. Eq. (19)).

2. First step: $L(2)$

From the Mittag-Leffler expansion of $\operatorname{sech} x$ [5, Ch. 7]

$$\operatorname{sech} x = \pi \sum_{k=0}^{\infty} (-)^k \frac{2k+1}{\left(\frac{(2k+1)\pi}{2}\right)^2 + x^2} \qquad (x \in \mathbb{R})$$

and the decomposition in simple fractions ($x \neq 0$)

$$(10)\qquad \frac{(-)^k \pi(2k+1)}{x^2\left[\left(\frac{(2k+1)\pi}{2}\right)^2+x^2\right]} = (-)^k \frac{4}{\pi}\left(\frac{1}{(2k+1)x^2} - \frac{1}{(2k+1)\left[\left(\frac{(2k+1)\pi}{2}\right)^2+x^2\right]}\right)$$

it follows

$$(11)\qquad \begin{aligned} \frac{\operatorname{sech} x}{x^2} &= \frac{4}{\pi}\sum_{k=0}^{\infty}\left(\frac{(-)^k}{(2k+1)x^2} - \frac{(-)^k}{(2k+1)\left[\left(\frac{(2k+1)\pi}{2}\right)^2+x^2\right]}\right) \\ &= \frac{4}{\pi x^2}\sum_{k=0}^{\infty}\frac{(-)^k}{(2k+1)} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-)^k}{(2k+1)\left[\left(\frac{(2k+1)\pi}{2}\right)^2+x^2\right]} \end{aligned}$$

Now,

$$\sum_{k=0}^{\infty}\frac{(-)^k}{(2k+1)} = L(1) = \frac{(\pi/2)\,|E_0|}{2} = \frac{\pi}{4}$$

so that

$$(12)\qquad \frac{\operatorname{sech} x}{x^2} = \frac{1}{x^2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-)^k}{(2k+1)\left[\left(\frac{(2k+1)\pi}{2}\right)^2+x^2\right]} \qquad (x \neq 0)$$

and

$$(13)\qquad \frac{1-\operatorname{sech} x}{x^2} = \frac{1}{x^2} - \frac{\operatorname{sech} x}{x^2} = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-)^k}{(2k+1)\left[\left(\frac{(2k+1)\pi}{2}\right)^2+x^2\right]} \qquad (x \neq 0)$$

The rhs of (13) extends by continuity the function on the lhs to all $x \in \mathbb{R}$.

On the other hand,

$$\begin{aligned} &\sum_{k=1}^{\infty}\int_0^{\infty} dx \left|\frac{(-)^k}{(2k+1)\left[\left(\frac{(2k+1)\pi}{2}\right)^2+x^2\right]}\right| = \sum_{k=1}^{\infty}\frac{1}{2k+1}\int_0^{\infty}\frac{dx}{\left(\frac{(2k+1)\pi}{2}\right)^2+x^2} \\ = \;& \sum_{k=1}^{\infty}\frac{1}{2k+1}\left[\frac{2}{(2k+1)\pi}\arctan\frac{2x}{(2k+1)\pi}\right]_0^{\infty} = \sum_{k=1}^{\infty}\frac{1}{(2k+1)^2} = \lambda(2) = \frac{\pi^2}{8} \end{aligned}$$

for all $x \in \mathbb{R}$, so that (13) can be integrated termwise [2, Th. 10.26] to obtain

$$\begin{aligned}
\frac{\pi}{4}\int_0^\infty \frac{dx}{x^2}(1-\operatorname{sech} x) &= \sum_{k=0}^\infty \frac{(-)^k}{(2k+1)}\int_0^\infty \frac{dx}{\left(\frac{(2k+1)\pi}{2}\right)^2 + x^2} \\
&= \sum_{k=0}^\infty \frac{(-)^k}{(2k+1)}\left[\frac{2}{(2k+1)\pi}\arctan\frac{2x}{(2k+1)\pi}\right]_0^\infty \\
&= \sum_{k=0}^\infty \frac{(-)^k}{(2k+1)^2} \\
&= L(2)
\end{aligned}$$

The integral on the lhs can be eventually simplified by integration by parts with the following result:

Corollary 2.1. *We have*

$$L(2) = \frac{\pi}{4}\int_0^\infty \frac{dx}{x}\operatorname{sech} x \tanh x = -\frac{\pi}{4}\int_0^\infty \frac{dx}{x}\frac{\operatorname{sech} x}{dx}$$

3. Generalization: $L(2n)$

In order to generalize the previous result, we need the following lemma.

Lemma 3.1. *For $n = 1, 2, \ldots$ and $x \neq 0$*

$$\begin{aligned}
\frac{\operatorname{sech} x}{x^{2n}} &= \frac{1}{x^{2n}} + \frac{E_2}{2!}\frac{1}{x^{2n-2}} + \ldots + \frac{E_{2n-2}}{(2n-2)!}\frac{1}{x^2} + \\
&\quad +(-)^n \frac{2^{2n}}{\pi^{2n-1}}\sum_{k=0}^\infty \frac{(-)^k}{(2k+1)^{2n-1}\left[\left(\frac{(2k+1)\pi}{2}\right)^2 + x^2\right]} \\
&= \sum_{k=0}^{n-1}\frac{E_{2k}}{(2k)!}\frac{1}{x^{2n-2k}} + (-)^n\frac{2^{2n}}{\pi^{2n-1}}\sum_{k=0}^\infty \frac{(-)^k}{(2k+1)^{2n-1}\left[\left(\frac{(2k+1)\pi}{2}\right)^2 + x^2\right]}
\end{aligned} \tag{14}$$

Proof. The proof is by induction. For $n = 1$, we get Eq.(12). Suppose now the formula is true for $n \in \mathbb{N}$ and let us prove it for $n+1$.

$$\begin{aligned}
\frac{\operatorname{sech} x}{x^{2n+2}} &= \frac{1}{x^2}\frac{\operatorname{sech} x}{x^{2n}} = \sum_{k=1}^{n-1}\frac{E_{2k}}{(2k)!}\frac{1}{x^{2n-2k+2}} \\
&\quad +(-)^n\frac{2^{2n}}{\pi^{2n-1}}\sum_{k=0}^\infty \frac{(-)^k}{(2k+1)^{2n-1}x^2\left[\left(\frac{(2k+1)\pi}{2}\right)^2 + x^2\right]}
\end{aligned} \tag{15}$$

To decompose in simple fractions the general term of the last series, resort to (10),

$$\frac{1}{x^2\left[\left(\frac{(2k+1)\pi}{2}\right)^2 + x^2\right]} = \frac{4}{\pi^2}\left(\frac{1}{(2k+1)^2x^2} - \frac{1}{(2k+1)^2\left[\left(\frac{(2k+1)\pi}{2}\right)^2 + x^2\right]}\right)$$

hence

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(-)^k}{(2k+1)^{2n-1} x^2 \left[\left(\frac{(2k+1)\pi}{2}\right)^2 + x^2\right]} \\
= \quad & \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{(-)^k}{(2k+1)^{2n+1} x^2} - \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{(-)^k}{(2k+1)^{2n+1} \left[\left(\frac{(2k+1)\pi}{2}\right)^2 + x^2\right]}
\end{aligned}$$

Substitution in Eq. (15) leads to

$$\begin{aligned}
\frac{\operatorname{sech} x}{x^{2n+2}} \quad = \quad & \sum_{k=0}^{n-1} \frac{E_{2k}}{(2k)!} \frac{1}{x^{2n-2k+2}} + (-)^n \frac{2^{2n}}{\pi^{2n-1}} \frac{4}{\pi^2} \frac{L(2n+1)}{x^2} \\
& + (-)^{n+1} \frac{2^{2n}}{\pi^{2n-1}} \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{(-)^k}{(2k+1)^{2n+1} \left[\left(\frac{(2k+1)\pi}{2}\right)^2 + x^2\right]}
\end{aligned}$$

Finally, using (5),

$$\begin{aligned}
&\frac{\operatorname{sech} x}{x^{2n+2}} \\
= \quad & \sum_{k=0}^{n-1} \frac{E_{2k}}{(2k)!} \frac{1}{x^{2n-2k+2}} + \frac{E_{2n}}{(2n)!} + (-)^{n+1} \frac{2^{2n+2}}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{(-)^k}{(2k+1)^{2n+1} \left[\left(\frac{(2k+1)\pi}{2}\right)^2 + x^2\right]} \\
= \quad & \sum_{k=0}^{n} \frac{E_{2k}}{(2k)!} \frac{1}{x^{2(n+1)-2k}} + (-)^{n+1} \frac{2^{2(n+1)}}{\pi^{2(n+1)-1}} \sum_{k=0}^{\infty} \frac{(-)^k}{(2k+1)^{2n+1} \left[\left(\frac{(2k+1)\pi}{2}\right)^2 + x^2\right]}
\end{aligned}$$

□

Eq. (14) defines by continuity the function

$$\frac{\operatorname{sech} x}{x^{2n}} - \sum_{k=0}^{n-1} \frac{E_{2k}}{(2k)!} \frac{1}{x^{2n-2k}} \tag{16}$$

at $x = 0$.

Corollary 3.2. *For* $n = 1, 2, \ldots$

$$L(2n) = (-)^n \frac{\pi^{2n-1}}{2^{2n}} \int_0^{\infty} \frac{dx}{x^{2n}} \left(\operatorname{sech} x - \sum_{k=0}^{n-1} \frac{E_{2k}}{(2k)!} x^{2k} \right) \tag{17}$$

Proof. Integration of (14) from 0 to ∞ yields

$$\begin{aligned}
&\int_0^\infty dx \left(\sum_{k=0}^{n-1} \frac{E_{2k}}{(2k)!} \frac{1}{x^{2n-2k}} - \frac{\operatorname{sech} x}{x^{2n}} \right) \\
= \quad & (-)^{n+1} \frac{2^{2n}}{\pi^{2n-1}} \sum_{k=0}^{\infty} \frac{(-)^k}{(2k+1)^{2n-1}} \int_0^\infty \frac{dx}{\left(\frac{(2k+1)\pi}{2}\right)^2 + x^2} \\
= \quad & (-)^{n+1} \frac{2^{2n}}{\pi^{2n-1}} \sum_{k=0}^{\infty} \frac{(-)^k}{(2k+1)^{2n-1}} \left[\frac{2}{(2k+1)\pi} \arctan \frac{2x}{(2k+1)\pi} \right]_0^\infty \\
= \quad & (-)^{n+1} \frac{2^{2n}}{\pi^{2n-1}} L(2n)
\end{aligned}$$

□

Another expression can be derived from Corollary 3.2 by subdividing $[0,\infty) = [0,\xi) \cup [\xi,\infty)$ with $0 < \xi < \pi/2$ and using the Taylor series [1, 4.5.66]

$$\operatorname{sech} x = \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)!} x^{2k} \quad (|x| < \pi/2) \tag{18}$$

as follows:

$$\begin{aligned}
&\int_0^\infty \frac{dx}{x^{2n}} \left(\operatorname{sech} x - \sum_{k=0}^{n-1} \frac{E_{2k}}{(2k)!} x^{2k} \right) \\
= \quad & \int_0^\xi \sum_{k=n}^{\infty} \frac{E_{2k}}{(2k)!} x^{2k-2n} dx + \int_\xi^\infty dx \left(\frac{\operatorname{sech} x}{x^{2n}} - \sum_{k=0}^{n-1} \frac{E_{2k}}{(2k)!} x^{2k-2n} \right) \\
= \quad & \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k-2n+1)(2k)!} \xi^{2k-2n+1} + \int_\xi^\infty dx \frac{\operatorname{sech} x}{x^{2n}}
\end{aligned}$$

Hence, for $0 < \xi < \pi/2$,

$$L(2n) = (-)^n \frac{\pi^{2n-1}}{2^{2n}} \left(\sum_{k=0}^{\infty} \frac{E_{2k}}{(2k-2n+1)(2k)!} \xi^{2k-2n+1} + \int_\xi^\infty dx \frac{\operatorname{sech} x}{x^{2n}} \right)$$

In particular, choosing $\xi = 1$,

$$L(2n) = (-)^n \frac{\pi^{2n-1}}{2^{2n}} \left(\sum_{k=0}^{\infty} \frac{E_{2k}}{(2k-2n+1)(2k)!} + \int_1^\infty dx \frac{\operatorname{sech} x}{x^{2n}} \right) \tag{19}$$

which can be considered a generalization of (5) for even exponents.

Finally, let us integrate by parts the rhs of (17).

Corollary 3.3. *For* $n = 1, 2, \ldots$

$$L(2n) = (-)^n \frac{\pi^{2n-1}}{2^{2n}(2n-1)!} \int_0^\infty \frac{dx}{x} (\operatorname{sech} x)^{(2n-1)} \tag{20}$$

Proof. A first integration by parts of (17) gives

$$
\begin{aligned}
&(-)^n \frac{2^{2n}}{\pi^{2n-1}} L(2n) \\
= \;& \int_0^\infty dx \left(\frac{-1}{(2n-1)x^{2n-1}} \right)^{(1)} \left(\operatorname{sech} x - \sum_{k=0}^{n-1} \frac{E_{2k}}{(2k)!} x^{2k} \right) \\
= \;& \frac{-1}{(2n-1)} \left[\frac{1}{x^{2n-1}} \left(\operatorname{sech} x - \sum_{k=0}^{n-1} \frac{E_{2k}}{(2k)!} x^{2k} \right) \right]_0^\infty \\
& + \frac{1}{(2n-1)} \int_0^\infty \frac{dx}{x^{2n-1}} \left((\operatorname{sech} x)^{(1)} - \sum_{k=1}^{n-1} \frac{E_{2k}}{(2k-1)!} x^{2k-1} \right)
\end{aligned}
$$

As before, the integrated term vanishes at infinity. On the other hand, the Taylor expansion of $\operatorname{sech} x$ for $|x| < \pi/2$ (Eq. (18)) shows that this term behaves near the origin as

$$
\frac{-1}{(2n-1)} \left(\frac{E_{2n}}{(2n)!} x + \frac{E_{2n+2}}{(2n+2)!} x^3 + ... \right) = O(x)
$$

so that it also vanishes at the lower limit $x = 0$.

Integrate a second time by parts,

$$
\begin{aligned}
&(-)^n \frac{2^{2n}}{\pi^{2n-1}} L(2n) \\
= \;& \frac{1}{(2n-1)} \int_0^\infty dx \left(\frac{-1}{(2n-2)x^{2n-2}} \right)^{(1)} \left((\operatorname{sech} x)^{(1)} - \sum_{k=1}^{n-1} \frac{E_{2k}}{(2k-1)!} x^{2k-1} \right) \\
= \;& \frac{-1}{(2n-1)(2n-2)} \left[\frac{1}{x^{2n-2}} \left((\operatorname{sech} x)^{(1)} - \sum_{k=1}^{n-1} \frac{E_{2k}}{(2k-1)!} x^{2k-1} \right) \right]_0^\infty \\
& + \frac{1}{(2n-1)(2n-2)} \int_0^\infty \frac{dx}{x^{2n-2}} \left((\operatorname{sech} x)^{(2)} - \sum_{k=1}^{n-1} \frac{E_{2k}}{(2k-2)!} x^{2k-2} \right)
\end{aligned}
$$

Since $y = \operatorname{sech} x$ has the flat asymptote $y = 0$ at infinity, $\lim_{x\to\infty} (\operatorname{sech} x)^{(\nu)} = 0$ for all $\nu = 1, 2, ...$, so that the integrated term vanishes at infinity. As for the lower limit, if $|x| < \pi/2$

$$
\begin{aligned}
& \frac{1}{x^{2n-2}} \left((\operatorname{sech} x)^{(1)} - \sum_{k=1}^{n-1} \frac{E_{2k}}{(2k-1)!} x^{2k-1} \right) \\
= \;& \frac{1}{x^{2n-2}} \left(\sum_{k=1}^{\infty} \frac{E_{2k}}{(2k-1)!} x^{2k-1} - \sum_{k=1}^{n-1} \frac{E_{2k}}{(2k-1)!} x^{2k-1} \right) \\
= \;& \frac{1}{x^{2n-2}} \left(\sum_{k=n}^{\infty} \frac{E_{2k}}{(2k-1)!} x^{2k-1} \right) = O(x)
\end{aligned}
$$

and, consequently, it also vanishes.

Following in this way, after $2n-2$ integrations by parts one arrives at the final expression

$$
\begin{aligned}
& (-)^n \frac{2^{2n}}{\pi^{2n-1}} L(2n) \\
= \quad & \frac{1}{(2n-1)!} \int_0^\infty \frac{dx}{x^2} \left((\operatorname{sech} x)^{(2n-2)} - \sum_{k=n-1}^{n-1} \frac{E_{2k}}{(2k-2n+2)!} x^{2k-2n+2} \right) \\
= \quad & \frac{1}{(2n-1)!} \int_0^\infty dx \left(\frac{-1}{x} \right)^{(1)} \left((\operatorname{sech} x)^{(2n-2)} - E_{2n-2} \right) \\
= \quad & \frac{-1}{(2n-1)!} \left[\frac{1}{x} \left((\operatorname{sech} x)^{(2n-2)} - E_{2n-2} \right) \right]_0^\infty \\
& + \frac{1}{(2n-1)!} \int_0^\infty \frac{dx}{x} \left((\operatorname{sech} x)^{(2n-1)} \right)
\end{aligned}
$$

The claim follows since

$$
(\operatorname{sech} x)^{(2n-2)} - E_{2n-2} \approx \frac{E_{2n}}{2!} x^2
$$

near the origin. □

Observe that by L'Hôpital,

$$
\lim_{x \to 0} \frac{(\operatorname{sech} x)^{(2n-1)}}{x} = (\operatorname{sech} x)^{(2n)} \Big|_{x=0} = E_{2n}
$$

i.e. the integrand in (20) has no singularity at the origin.

References

[1] M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions*, Dover.

[2] T. Apostol, *Mathematical Analysis* (2nd edition), Addison-Wesley.

[3] J.M. Amigó, Some integral expressions for Riemann's zeta series with odd exponents, *Scientiae Mathematicae* 3 (2000), 167-172.

[4] K. Knopp, *Infinite Sequences and Series*, Dover.

[5] M. R. Spiegel, *Complex Variables*, McGraw-Hill.

Operations Research Center. Miguel Hernández University. 03202 Elche (Alicante). Spain

E-mail address: jm.amigo@umh.es