TOPOLOGICAL STRUCTURE FOR THE SPACE CONSISTING OF IMAGES

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Abstract. The theory of mathematical morphology was developed for binary images and extensively uses set-theoretic operations such as union, intersection, and set complement, and many algorithms for pattern analysis critically depend on an accurate geometrical and topological image description. In the present paper, we shall eliminate some theoretical shortcoming in [3], and give a perfect topological structure for the space consisting of images.

1 Introduction The theory of mathematical morphology was developed for binary images and extensively uses set-theoretic operations such as union, intersection, and set complement, because an image under consideration is always considered as a set in mathematical morphology. Since many algorithms for pattern analysis, which process noisy data, critically depend on an accurate geometrical and topological image description. We have to provide a precise mathematical description of an image X under consideration, and have to provide its crucial geometrical and topological structure.

In the present paper, we shall eliminate some theoretical shortcoming in [3], and give a perfect topological structure for the space consisting of images.

Let R^n be the n-dimensional Euclidean space, and for $x, y \in R^n$, $\rho(x,y)$ is the distance between the points x and y. If ϵ is a positive real number and $x \in R^n$, then we denote $U(x,\epsilon) = \{y \in R^n : \rho(x,y) < \epsilon\}$ which is called to be open ball centred at x of radius ϵ. If $x = O$ is the origin of R^n, we denote $U(x,\epsilon)$ by ϵD.

In general, $\overline{A}$ means the closure of A in X, where A is a subset of a space X. We denote the set of natural numbers by letter Ω. If a sequence $\{x_i : i \in \Omega\}$ converges to x^* in a space X, then we write it by $x_i \xrightarrow{X} x^*$. When (X,ρ) is a metric space, and a sequence $\{x_i : i \in \Omega\}$ converges to x^* in X, then $x_i \xrightarrow{X} x^*$ will be denoted for simplicity by the symbol $x_i \to x^*$ or $\rho(x_i, x^*) \to 0$.

If y is a point in R^n and A, B are subsets of R^n, we let $A[y]$ be its translation by the point y, i.e., $A[y] = \{a + y : a \in A\}$, and $\check{A}$ be the symmetric set of A with respect to the origin, i.e., $\check{A} = \{-a : a \in A\}$. $A \oplus B = \{a + b : a \in A, b \in B\}$ is called the dilation of set A by set B, and $A \ominus \check{B} = \cap_{b \in \check{B}} A[b]$ is the erosion of set A by set B. It is clear that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$, and $U(A,\epsilon) = A \oplus \epsilon\mathrm{D}$, and $A \ominus \check{B} = \{x : B[x] \subset A\}$.

For A and $B \subset R^n$, let $\rho(A,B) = inf\{\epsilon : B \subset U(A,\epsilon)\}$. $H(A,B) = max\{\rho(A,B), \rho(B,A)\}$.

Let $Y \subset R^n$ be a 'very big' bounded closed set which contains the origin as its interior point. We shall construct three families as follows.

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If we let $\mathcal{F} = \{F \subset R^n : F \neq \emptyset, F \text{ is closed}\}$, $\mathcal{K} = \{K \subset R^n : K \neq \emptyset, K \text{ is compact}\}$ and $\mathcal{P} = \{K \subset Y : K \neq \emptyset, K \text{ is compact}\}$, then it is clear that $\mathcal{K} \subset \mathcal{F}$ and $\mathcal{P} = \{K \cap Y : K \in \mathcal{K}$ and $K \cap Y \neq \emptyset\}$.

Definition 1.1 ([3]). *Let $G_1, G_2, \cdots, G_m$ be finite non-empty open sets, and $K_1, K_2, \cdots, K_p$ finite compact sets of R^n (K_j can be empty). We set*

$N(\{G_i\}, \{K_j\}) = \{F \in \mathcal{F} : F \cap G_i \neq \emptyset$ *for each* $i = 1, 2, \cdots, m; F \cap K_j = \emptyset$ *for each* $j = 1, 2, \cdots, p\}$

$\mathcal{B}(\mathcal{F}) = \{N(\{G_i\}, \{K_j\}) : \{G_i\}$ is a finite family of non-empty open sets of R^n, and $\{K_j\}$ is a finite family of compact sets of $R^n\}$, *and* $\mathcal{T}(\mathcal{F}) = \{\cup \mathcal{B}^* : \mathcal{B}^* \subset \mathcal{B}(\mathcal{F})\}$.

It can be easily proved that $(\mathcal{F}, \mathcal{T}(\mathcal{F}))$ is a topological space. Similarly we know that $(\mathcal{K}, \mathcal{T}(\mathcal{K}))$, $(\mathcal{P}, \mathcal{T}(\mathcal{P}))$ are topological spaces. $\mathcal{T}(\mathcal{F})$ is called HM-topology.

2. Some Lemmas

In this section, we shall give some lemmas which are useful for the main theorems.

Lemma 2.1. *Topological spaces $(\mathcal{F}, \mathcal{T}(\mathcal{F}))$ ($(\mathcal{K}, \mathcal{T}(\mathcal{K}))$ and $(\mathcal{P}, \mathcal{T}(\mathcal{P}))$) are Hausdorff spaces with countable bases.*

Proof. We shall prove for only the case of $(\mathcal{F}, \mathcal{T}(\mathcal{F}))$. Ler $r_1, r_2, \cdots, r_m, \cdots$ be the rational points of R^n, and $U_{m,l} = \{x \in R^n : \rho(r_m, x) < \frac{1}{l}\}$ and $\mathcal{V} = \{U_{m,l} : m, l \in \Omega\} = \{V_1, V_2, \cdots, i \in \Omega\}$. Then for any open set $U \subset R^n$ and $x \in U$, there is some V in $\mathcal{V}$ with $x \in V \subset U$.

We let $F \in \mathcal{F}$ and $F \in N(\{G_i\}, \{K_j\})$, where $i = 1, \cdots, m$, and $j = 1, \cdots, p$. Since every K_j is compact, we can have $V_1^*, \cdots, V_l^* \in \mathcal{V}$, such that $\cup_{j=1}^p K_j \subset \cup_{k=1}^l V_k^*$ and $F \cap (\cup_{k=1}^l \overline{V}_k^*) = \emptyset$. On the other hand, we can pick $V_i^1 \in \mathcal{V}$ with $F \cap V_i^1 \neq \emptyset$ and $V_i^1 \subset G_i$ for each $i = 1, \cdots, m$. Then $F \in N(\{V_i^1\}, \{\overline{V}_k^*\}) \subset N(\{G_i\}, \{K_j\})$ where $i = 1, \cdots, m$, $k = 1, \cdots, l$, and $j = 1, \cdots, p$. Since $\mathcal{V}$ is countable it is seen that $\mathcal{F}$ has a countable base.

Secondly we shall prove that $\mathcal{F}$ is a Hausdorff space. For $F, G \in \mathcal{F}$ with $F \neq G$, we suppose that $F - G \neq \emptyset$. Let $x \in F - G$ and $y \in G$. Then there is an open set $V \in \mathcal{V}$ with $x \in V$ and $\overline{V} \cap G = \emptyset$, and there is a compact set $K \subset R^n$ with $K \cap F = \emptyset$. We get $F \in N(V, K)$. Similarly we have an open set U with $y \in U$ and $U \cap \overline{V} = \emptyset$ and then $G \in N(U, \overline{V})$. It is clear that $N(V, K) \cap N(U, \overline{V}) = \emptyset$. Hence $\mathcal{F}$ is a Hausdorff space.

From Lemma 2.1 we know that the convergency can be characterized by sequence in above spaces.

Lemma 2.2 ([3]). *Let $\{F_i : i \in \Omega\}$ be a sequence in $\mathcal{F}$. Then $F_i \xrightarrow{\mathcal{F}} F$ if and only if the following two conditions are satisfied:*

1). *If G is an open set in R^n and $G \cap F \neq \emptyset$, then G intersects eventually $\{F_i : i \in \Omega\}$ (that is, there is an $N \in \Omega$, such that $G \cap F_i \neq \emptyset$ for each $i > N$).*

2). *If K is a compact set in R^n with $K \cap F = \emptyset$, then K does not cofinally intersect $\{F_i : i \in \Omega\}$ (that is, there is an $N \in \Omega$, such that $K \cap F_i = \emptyset$ for each $i > N$).*

Lemma 2.3. *If $F_i \xrightarrow{\mathcal{F}} F$ in $(\mathcal{F}, \mathcal{T}(\mathcal{F}))$, $y_i \in F_i (i \in \Omega)$ and $y_i \to y$ in R^n, then $y \in F$.*

Proof. Suppose that $y \notin F$, then there is some $\delta > 0$, such that $\overline{U(y, \delta)} \cap F = \emptyset$. Since $F_i \xrightarrow{\mathcal{F}} F$, $\overline{U(y, \delta)}$ can not eventually intersect $\{F_i : i \in \Omega\}$ by Lemma 2.2. This contradicts to that $y_i \in F_j (i \in \Omega)$.

Lemma 2.4. *If $F_i \xrightarrow{\mathcal{F}} F$ in $(\mathcal{F}, \mathcal{T}(\mathcal{F}))$ and $y \in F$, then we can pick $y_i \in F_i (i \in \Omega)$ with $y_i \to y$.*

Proof. Let $\{U_i(y) : i \in \Omega\}$ be a decreasing neighborhood base of y in Y. By Lemma 2.2, for $U_1(y)$ there is an $i_1 \in \Omega$, such that $U_1(y) \cap F_i \neq \emptyset$ for any $i > i_1$. Similarly for $U_2(y)$, there is an $i_2 > i_1$, such that $U_2(y) \cap F_i \neq \emptyset$ for any $i > i_2$. Inductively we can get a sequence $\{y_i : i \in \Omega\}$ with $y_i \in F_i$ with $y_i \to y$.

It is clear that the arguments of Lemmas 2.2, 2.3 and 2.4 are also true for spaces $(\mathcal{K}, \mathcal{T}(\mathcal{K}))$ and $(\mathcal{P}, \mathcal{T}(\mathcal{P}))$.

Lemma 2.5 ([6]). *$H(A, B)$ is a metric function on the space $\mathcal{P}$.*

Proof. Since $A, B \in \mathcal{P}$ are compact in R^n, we have that $H(A, B) = 0$ if and only if $A = B$, and $H(A, B) \geq 0$. It is clear that $H(A, B) = H(B, A)$.

If $B \subset A \oplus \epsilon_1 D$ and $C \subset B \oplus \epsilon_2 D$, then we have that $C \subset A \oplus \epsilon_1 D \oplus \epsilon_2 D$, that is $\rho(A, C) \leq \rho(A, B) + \rho(B, C)$. Similarly we have that $\rho(C, A) \leq \rho(C, B) + \rho(B, A)$. Hence $H(A, B)$ is a metric.

Lemma 2.6. *Let $\{F_i : i \in \Omega\}$ be a sequence in $\mathcal{P}$ and $F \in \mathcal{P}$. If $H(F_i, F) \to 0$, then $F_i \xrightarrow{\mathcal{P}} F$.*

Proof. Since $H(F_i, F) \to 0$, we have that $\delta_i = \rho(F_i, F) \to 0$. For any $x \in F$, there is an $a_i \in F_i$ with $\rho(a_i, x) < \delta_i + \frac{1}{i}$, and hence $a_i \to x$. This means that if G is an open set in R^n with $G \cap F \neq \emptyset$, then for some i_0, $G \cap F_i \neq \emptyset$ for each $i > i_0 \in \Omega$.

Suppose K is compact in R^n with $K \cap F = \emptyset$, then $\delta(F, K) > 0$, where $\delta(F, K) = inf\{\rho(x, y) : x \in F \text{ and } y \in K\}$. Furthermore $\eta_i = \rho(F, F_i) \to 0$, because $H(F_i, F) \to 0$. Hence there is an $i_0 \in \Omega$, such that $2\eta_i < \delta(F, K)$ for $i > i_0$, and then $F_i \subset F \oplus \frac{1}{2}\delta(F, K)D$. That is $F_i \cap K = \emptyset$ for each $i > i_0$. Therefore $F_i \xrightarrow{\mathcal{P}} F$ by Lemma 2.2.

Lemma 2.7. *Let $\{F_i : i \in \Omega\}$ be a sequence in $\mathcal{P}$ and $F \in \mathcal{P}$. If $F_i \xrightarrow{\mathcal{P}} F$, then $H(F_i, F) \to 0$.*

Proof. Let $x \in F$ and U be open in Y with $x \in U$, then U intersects eventually with $\{F_i : i \in \Omega\}$ by Lemma 2.2, and hence we can get some $a_i \in F_i$ such that $\rho(a_i, x) \to 0$ and $\delta(F_i, x) \leq \rho(a_i, x)$.

If we suppose that $\rho(F_i, F) \to 0$ is not true, then there are some $\epsilon > 0$ and some subsequence $\{F_{i_k} : k \in \Omega\} \subset \{F_i : i \in \Omega\}$ with $\rho(F_{i_k}, F) \geq \epsilon$ for each $k \in \Omega$. We have $x_k \in F$ with $\delta(F_{i_k}, x_k) \geq \epsilon$. Since F is compact, the sequence $\{x_k : k \in \Omega\}$ has a converging subsequence. Without loss of generality we can let that $x_k \to x \in F$. Since $\delta(F_{i_k}, x) \to 0$ and $\mid \delta(F_{i_k}, x_k) - \delta(F_{i_k}, x) \mid \leq \rho(x, x_k)$, we have that $\delta(F_{i_k}, x_k) \to 0$. This is a contradiction.

We prove furthermore that $\rho(F, F_i) \to 0$. Suppose not, that is, there are some $\epsilon > 0$ and some subsequence $\{F_{i_k} : k \in \Omega\} \subset \{F_i : i \in \Omega\}$ with $\rho(F, F_{i_k}) > \epsilon$, and then we can pick out some $a_k \in F_{i_k}$ for each $k \in \Omega$ such that $\delta(F, a_k) \geq \epsilon$. Since $\{a_k : k \in \Omega\} \subset Y$, $\{a_k : k \in \Omega\}$ has a subsequence converging to some $a^* \in Y$. From Lemma 2.3, $a^* \in F$. This contradicts to $\delta(F, a_k) \geq \epsilon$.

Therefore we have that $H(F_i, F) \to 0$.

3. Main results

Theorem 3.1. *The space $\mathcal{P}$ is metrizable and its metric is $H(A, B)$.*

Proof. The proof can be completed by means of Lemmas 2.1, 2.6 and 2.7.

Definition 3.2. *For a sequence $\{F_i : i \in \Omega\}$ of $\mathcal{F}$, let*

$\widetilde{F}= \{x$: *For each neighborhood U_x of x, there is a subsequence $\{F_{i_k} : k \in \Omega\}$ such that $U_x \cap F_{i_k} \neq \emptyset$.$\}$ and*

$\underline{F} = \{x$: *For each neighborhood U_x of x, U_x intersects eventually with $\{F_i : i \in \Omega\}\}$,*

where U_x is the neighborhood of x in R^n. We call $\widetilde{F}$ upper closed limit and $\underline{F}$ lower closed limit.

Theorem 3.3. *Let $\{F_i : i \in \Omega\}$ be a sequence in $\mathcal{F}$. Then $F_i \xrightarrow{\mathcal{F}} F$ if and only if $F = \widetilde{F} = \underline{F}$.*

Proof. Necessity. For $x \in \widetilde{F}$, let $\{U_i(x) : i \in \Omega\}$ be a decreasing neighborhood base of x in R^n.

For $U_1(x)$, we can get a subsequence $\{F_i^1 : i \in \Omega\} \subset \{F_i : i \in \Omega\}$ with $U_1(x) \cap F_i^1 \neq \emptyset$ for each $i \in \Omega$. Then we have a sequence $x_1^1, x_2^1, \cdots$, where $x_i^1 \in U_1(x) \cap F_i^1$.

For $U_2(x)$, we can get a subsequence $\{F_i^2 : i \in \Omega\} \subset \{F_i : i \in \Omega\}$ with $U_2(x) \cap F_i^2 \neq \emptyset$ for $i \in \Omega$. So we have a sequence $x_1^2, x_2^2, \cdots$, with $x_i^2 \in U_2(x) \cap F_i^2$.

By induction, we can get a sequence $\{x_i^i : i \in \Omega\}$ with $x_i^i \to x$. Since $F_i \xrightarrow{\mathcal{F}} F$, then $x \in F$ by Lemma 2.3. From Lemma 2.2 we have that if U_x is a neighborhood of x in R^n, then U_x intersects eventually with $\{F_i : i \in \Omega\}$ and hence $x \in \underline{F}$ by Definition 3.2. That is $\widetilde{F} \subset \underline{F}$. It is clear that $\underline{F} \subset \widetilde{F}$. Hence we have $\widetilde{F} = \underline{F}$.

Sufficiency. If $\widetilde{F} = \underline{F} = F$, we shall prove that $F_i \xrightarrow{\mathcal{F}} F$ by means of Lemma 2.2.

First we let U be open in R^n with $U \cap F \neq \emptyset$. Let x be any point in $U \cap F$, then $x \in F = \underline{F}$. By Definition 3.2, the open set U intersects eventually with $\{F_i : i \in \Omega\}$.

Secondly we let K be a compact set in R^n with $K \cap F = \emptyset$. Here we shall prove that K does not eventually intersect with $\{F_i : i \in \Omega\}$. Suppose not. So there were a subsequence $\{F_i^1 : i \in \Omega\} \subset \{F_i : i \in \Omega\}$ with $K \cap F_i^1 \neq \emptyset$ for each $i \in \Omega$. Then we have a sequence $\{x_i \in K \cap F_i^1 : i \in \Omega\}$ in the compact set K. Without loss of generality we can suppose $x_i \to x^* \in K$. For each neighborhood $U(x^*)$ of x^*, $U(x^*)$ intersects with infinitely many elements of $\{F_i : i \in \Omega\}$. Since $x^* \in \widetilde{F} = F$, we have $x^* \in K \cap F$. This is a contradiction.

Theorem 3.4. *The space $\mathcal{P}$ is compact.*

Proof. It is sufficient to prove that every sequence $\{F_i : i \in \Omega\}$ has a converging subsequence in $\mathcal{P}$, since $\mathcal{P}$ has a countable base.

Let $\{F_i : i \in \Omega\} \subset \mathcal{P}$, and $x_i \in F_i$ for each $i \in \Omega$. Since Y is closed and bounded, the sequence $\{x_i : i \in \Omega\}$ has an accumulation point x^*, and $x^* \in \widetilde{F}$. Let $\mathcal{V} = \{V_1, V_2, \cdots\}$ be a countable base for Y.

The first step. Let $\mathcal{V} = \mathcal{V}_0^1 \cup \mathcal{V}_0^2 \cup \mathcal{V}_0^3$. For $V \in \mathcal{V}_0^1$, V intersects with at most finitely many elements of $\{F_i : i \in \Omega\}$; For $V \in \mathcal{V}_0^2$, V intersects with infinitely many elements of $\{F_i : i \in \Omega\}$, but V does not intersect eventually with $\{F_i : i \in \Omega\}$; If V intersects eventually with $\{F_i : i \in \Omega\}$, then $V \in \mathcal{V}_0^3$. As mentioned above we know that $\mathcal{V}_0^2 \cup \mathcal{V}_0^3 \neq \emptyset$.

That $\mathcal{V}_0^2 = \emptyset$ implies that, if an open set U (in Y) intersects with infinitely many elements of $\{F_i : i \in \Omega\}$, then U intersects eventually with $\{F_i : i \in \Omega\}$. In this case we have $\widetilde{F} = \underline{F}$ and, since $\widetilde{F} \neq \emptyset$, we have $F_i \xrightarrow{\mathcal{P}} \widetilde{F}$ by Theorem 3.3.

If $\mathcal{V}_0^2 \neq \emptyset$, write $\mathcal{V}_0^2 = \{V_1^0, V_2^0, \cdots\} \subset \mathcal{V}$. For $V_1^0 \in \mathcal{V}_0^2$, we have a subsequence $\{F_i^1 : i \in \Omega\} \subset \{F_i : i \in \Omega\}$ with $V_1^0 \cap F_i^1 \neq \emptyset$ for each $i \in \Omega$. In this case, we let $\mathcal{V}_0^2 = \mathcal{V}_1^1 \cup \mathcal{V}_1^2 \cup \mathcal{V}_1^3$ satisfying, for $V \in \mathcal{V}_1^1$, V intersects with only finitely many elements of $\{F_i^1 : i \in \Omega\}$; If $V \in \mathcal{V}_1^2$, V intersects with infinitely many elements of $\{F_i^1 : i \in \Omega\}$, but V does not intersect eventually with $\{F_i^1 : i \in \Omega\}$; While V intersects eventually with $\{F_i^1 : i \in \Omega\}$, then $V \in \mathcal{V}_1^3$. It is clear that $V_1^0 \in \mathcal{V}_1^3$. If we let $\mathcal{V}^1 = \mathcal{V}_0^3 \cup \mathcal{V}_1^3$, then $\mathcal{V}^1 \neq \emptyset$.

The second step. Suppose that $\mathcal{V}_1^2 = \emptyset$. Then the open set U (in Y) which intersects with infinitely many elements of $\{F_i^1 : i \in \Omega\}$, intersects eventually with $\{F_i^1 : i \in \Omega\}$. Set

$\widetilde{F}\,(1) = \{x$: if U is a neighborhood of x in Y, then U intersects with infinitely many elements of $\{F_i^1 : i \in \Omega\}\}$, and

$\underline{F}(1) = \{x$: if U is a neighborhood of x, then U intersects eventually with $\{F_i^1 : i \in \Omega\}\}$.

Then we have $\widetilde{F}\,(1) = \underline{F}(1)$. Since $\mathcal{V}^1 \neq \emptyset$ we know that $\underline{F}(1) \neq \emptyset$, and $\{F_i^1 : i \in \Omega\}$ is convergent in $\mathcal{P}$ by Theorem 3.3.

If $\mathcal{V}_1^2 \neq \emptyset$, then $\mathcal{V}_1^2 = \{V_1^1, V_2^1, \cdots\} \subset \mathcal{V}_0^2$. For $V_1^1 \in \mathcal{V}_1^2$, we can get a subsequence $\{F_i^2 : i \in \Omega\} \subset \{F_i^1 : i \in \Omega\}$ with $V_1^1 \cap F_i^2 \neq \emptyset$ for each $i \in \Omega$. Let $\mathcal{V}_1^2 = \mathcal{V}_2^1 \cup \mathcal{V}_2^2 \cup \mathcal{V}_2^3$ such that, if $V \in \mathcal{V}_2^1$, then V intersects with at most finitely many elements of $\{F_i^2 : i \in \Omega\}$; for $V \in \mathcal{V}_2^2$, V intersects with infinitely many elements of $\{F_i^2 : i \in \Omega\}$, but V does not intersect eventually with $\{F_i^2 : i \in \Omega\}$, while V intersects eventually with $\{F_i^2 : i \in \Omega\}$, then $V \in \mathcal{V}_2^3$. Similarly we know that $V_1^1 \in \mathcal{V}_2^3$, and if let $\mathcal{V}^2 = \mathcal{V}^1 \cup \mathcal{V}_2^3$, then $\mathcal{V}^2 \neq \emptyset$.

The jth step. By induction, for each $j \in \Omega$ we define the following

1). $\{F_i^j : i \in \Omega\}$ is a subsequence of $\{F_i^{j-1} : i \in \Omega\}$,

2). $\widetilde{F}\,(j) = \{x$: if U is a neighborhood of x in Y, then U intersects with infinitely many elements of $\{F_i^j : i \in \Omega\}\}$,

3). $\underline{F}(j) = \{x$: if U is a neighborhood of x, then U intersects eventually with $\{F_i^j : i \in \Omega\}\}$,

4). $\mathcal{V}_{j-1}^2 = \mathcal{V}_j^1 \cup \mathcal{V}_j^2 \cup \mathcal{V}_j^3$ and $\mathcal{V}^j = \mathcal{V}^{j-1} \cup \mathcal{V}_j^3$.

If for some $j \in \Omega$ with $\mathcal{V}_j^2 = \emptyset$, then any open set U in Y which intersects with infinitely many elements of $\{F_i^j : i \in \Omega\}$, intersects eventually with $\{F_i^j : i \in \Omega\}$. Therefore $\widetilde{F}\,(j) = \underline{F}(j)$ and $F_i^j \xrightarrow{\mathcal{P}} \widetilde{F}\,(j)$, that is, $\{F_i : i \in \Omega\}$ has a convergent subsequence.

On the other hand if $\mathcal{V}_j^2 \neq \emptyset$ for each $j \in \Omega$, then we can pick out a sequence $\{F_i^i : i \in \Omega\}$ from the following sequences

$$\begin{array}{c} F_1^1, F_2^1, \cdots, F_n^1, \cdots \\ F_1^2, F_2^2, \cdots, F_n^2, \cdots \\ \cdots \\ F_1^j, F_2^j, \cdots, F_n^j, \cdots \\ \cdots \end{array}$$

For the sequence $\{F_i^i : i \in \Omega\}$, we know easily that if an open set U in Y intersects with infinitely many elements of $\{F_i^i : i \in \Omega\}$, then U intersects eventually with $\{F_i^i : i \in \Omega\}$. Hence $\{F_i^i : i \in \Omega\}$ is convergent in $\mathcal{P}$. That is, $\mathcal{P}$ is a compact space.

4. Two counter-examples and the conclusions

Example 4.1. The space $\mathcal{K}\,(\mathcal{F})$ has not necessarily the metric $H(A, B)$.

Let $R^n = R$ and $F = \{0\}$, $F_i = \{0, i\}$ for each $i \in \Omega$, then $F_i \in \mathcal{K}$, and $F_i \xrightarrow{\mathcal{K}} F$. But $H(F_i, F) \to +\infty$.

Example 4.2. The space $\mathcal{K}$ is not necessarily compact.

Let $R^n = R$ and $F_i = \{i\}$, then $F_i \in \mathcal{K}$. It is clear that $\{F_i : i \in \Omega\}$ is not convergent.

Open set and closed set are only mathematical concepts, are not physical concepts, hence we can consider an image as a closed set. In actual pattern recognition, the treated pattern is always limited in a large scope, and a pattern can be considered as a compact set in mathematics. In above we proved that the topological space $\mathcal{P}$ is a compact metric space with a countable base, and $H(A, B)$ is a metric on $\mathcal{P}$. The metric $H(A, B)$ coincides with our physical intuition. In [4] we proved the dilation operation is a continuous map and the erosion operation is a upper semicontinuous map on the space $\mathcal{P}$. From the erosion and dilation we can get all other operations of mathematical morphology. Hence we deem that the space $\mathcal{P}$ is a good mathematical space for pattern recognition.

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