ON FUZZY BCC-IDEALS

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Abstract. The aim of this paper is to give some characterizations of fuzzy BCC-ideals. Also we solve the problem of classifying fuzzy BCC-ideals by their family of level BCC-ideals in BCC-algebras.

The concept of fuzzy sets was introduced by Zadeh. O.G.Xi [6] applied the concept of fuzzy sets to BCK-algebras. A fuzzy ideal of a BCK-algebra was extensively investigated by Y.B.Jun, J.Meng et al. ([1],[2],[5]). In this paper we develop some results in [1], [2] to BCC-algebras.

By a BCC-algebra we mean an algebra $(X;^*,0)$ of type $(2,0)$ satisfying the axioms:

(1) $((x^*y)^*(z^*y))^*(x^*z) = 0$,

(2) $x^*x = 0$,

(3) $0^*x = 0$,

(4) $x^*0 = x$,

(5) $x^*y = y^*x = 0$ implies $x = y$,

for all $x, y, z \in X$. We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x^*y = 0$.

The avobe definition is a dual form of the ordinary definition (See [3]).

In any BCC-algebra X, the following hold:

(6) $x^*y \leq x$,

(7) $x \leq y$ implies $x^*z \leq y^*z$ and $z^*y \leq z^*x$.

Any BCK-algebra is a BCC-algebra, but not conversely. A BCC-algebra is a BCK-algebra iff it satisfies $(x^*y)^*z = (x^*z)^*y$ or $(x^*(x^*y))^*y = 0$.

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A non-empty subset A of a BCC-algebra X is called a BCC-ideal (See [7] and [8]) iff (i) $0\in A$ and (ii) $y,(x^*y)^*z \in A$ imply $x^*z \in A$. A non-empty subset S of a BCC-algebra X is called a subalgebra of X if, for any $x,y \in S$, we have $x^*y \in S$.

Definition 1 ([9]). *Let X be a BCC-algebra, a function $\mu : X \to [0,1]$ is called a fuzzy subalgebra of X, if, for any $x,y \in X$, we have:*

$$\mu(x^*y) \geq min(\mu(x),\mu(y)).$$

Definition 2 ([9]) *Let μ be a fuzzy set in a set X. For $t \in [0,1]$, the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called a level subset of μ.*

Theorem 3 ([9]). *Let X be a BCC-algebra and let μ be an arbitrary fuzzy subalgebra of X. Then $\mu(0) \geq \mu(X)$ for any $x \in X$.*

Theorem 4 ([9]).*Let X be a BCC-algebra. Then a fuzzy set μ in X is a fuzzy subalgebra of X if and only if, for every $t \in [0,1]$, μ_t is a subalgebra of X when $\mu_t \neq \emptyset$.*

Definition 5 ([9]). *Let X be a BCC-algebra. A fuzzy set μ $(\mu : X \to [0,1])$ in X is said to be a fuzzy BCC-ideal of X if it satisfies*

(i) *$\mu(0) \geq \mu(x)$ for any $x \in X$,*

(ii) *$\mu(x^*z) \geq min\{\mu((x^*y)^*z),\mu(y)\}$ for any $x,y,z \in X$.*

Theorem 6 ([9]). *A fuzzy set μ in a BCC-algebra X is a fuzzy BCC-ideal of X if and only if, for each $t \in [0,1]$, $\mu_t = \{x \in X : \mu(x) \geq t\}$ is a BCC-ideal of X, when $\mu_t \neq \emptyset$.*

Theorem 7 *For any fuzzy BCC-ideal μ of BCC-algebra X, if $x \leq y$ then $\mu(x) \geq (y)$.*

Proof. Let μ be a fuzzy BCC-ideal of BCC-algebra X. If $x \leq y$, then $x^*y = 0$. It follows that

$$\begin{aligned} \mu(x) = \mu(x^*0) &\geq min(\mu((x^*y)^*0),\mu(y)) \qquad (by\ \text{(ii)}) \\ &= min(\mu(x^*y),\mu(y)) \\ &= min(\mu(0),\mu(y)) \\ &= \mu(y) \end{aligned}$$

This completes the proof.

Theorem 8 *Any fuzzy BCC-ideal μ of BCC-algebra X must be a fuzzy-subalgebra of X.*

Proof. Since $x^*y \leq x$ (by (6)), it follows from Theorem 7 that

$$\mu(x) \leq \mu(x^*y)$$

so by (ii)

$$\begin{aligned} \mu(x^*y) \geq \mu(x) = \mu(x^*0) &\geq min(\mu((x^*y)^*0), \mu(y)) \\ &= min(\mu(x^*y), \mu(y)) \\ &\geq min(\mu(x), \mu(y)) \end{aligned}$$

This shows that μ is a fuzzy subalgebra of X, proving the theorem.

Theorem 9 *A fuzzy subalgebra of BCC-algebra X is a fuzzy BCC-ideal of X if and only if, for all $x, y, z, s \in X$, the inequality $(x^*y)^*z \leq s$ implies that $\mu(x^*z) \geq min\{\mu(y), \mu(s)\}$.*

Proof. ($\Leftarrow$) Suppose that μ is a fuzzy subalgebra of BCC-algebra X and satisfying that $(x^*y)^* \leq zs$ implies $\mu(x^*z) \geq min\{\mu(y), \mu(s)\}$. Since

$$(x^*y)^*z \leq (x^*y)^*z \qquad (by\ (2))$$

it follows that $\mu(x^*z) \geq min\{\mu(y), \mu((x^*y)^*z)\}$. Hence μ is a fuzzy BCC-ideal of X.
($\Rightarrow$) Suppose that μ is a fuzzy BCC-ideal of X and $(x^*y)^*z \leq s$, it follows from Theorem 7 that $\mu(s) \leq \mu((x^*y)^*z)$, so by (ii)

$$\mu(x^*z) \geq min\{\mu(y), \mu((x^*y)^*z)\} \geq min\{\mu(y), \mu(s)\}.$$

The ploof is complete.

Definition 10 *Let X be a BCC-algebra and let μ be a fuzzy BCC-ideal of X. The BCC-ideals μ_t, $t \in [0, 1]$, are called level BCC-ideals of μ.*

Theorem 11 *Any BCC-ideal of a BCC-algebra X can be realized as a level BCC-ideal of some fuzzy BCC-ideal of X.*

Proof. Let A be a BCC-ideal of a BCC-algebra X and let μ be a fuzzy sets in X defined by

$$\mu(x) = \begin{cases} t, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

where t is a fixed number in $(0,1)$. Note that $0 \in A$, so that $\mu(0) = t \geq \mu(x)$ for all $x \in X$. Let $x, y, z \in X$. If $y \notin A$, then $\mu(y) = 0$ and so

$$\mu(x^*z) \geq 0 = min\{\mu((x^*y)^*z), \mu(y)\}.$$

Assume that $y \in A$:

If $x^*z \in A$, then $(x^*y)^*z$ may or may not belong to A. In any cases,

$$\mu(x^*z) = t \geq min\{\mu((x^*y)*z), \mu(y)\}.$$

If $x^*z \notin A$, then $(x^*y)^*z \notin A$ because A is a BCC-ideal. Hence

$$\mu(x^*z) = 0 = min\{\mu((x^*y)^*z), \mu(y)\}.$$

This shows that μ is a fuzzy BCC-ideal of X. For the fuzzy BCC-ideal μ, obviously $\mu_t = A$.

Note that if X is a finite BCC-algebra, then the number of ideals of X is finite whereas the number of level BCC-ideals of a fuzzy BCC-ideal μ appears to be infinite. But, since every level BCC-ideal is indeed a BCC-ideal of X, not all these level BCC-idals are distinct. The next theorem characterizes this aspect.

Theorem 12 *Let μ be a fuzzy BCC-ideal of a BCC-algebra X. Two level BCC-ideals μ_{t_1}, μ_{t_2}(with $t_1 < t_2$) of μ are equal if and only if there is no $x \in X$ such that $t_1 1 < \mu(x) < t_2$.*

Proof. Assume that $\mu_{t_1} = \mu_{t_2}$ for $t_1 < t_2$ and that there exists $x \in X$ such that $t_1 < \mu(x) < t_2$. Then μ_{t_2} is a proper subset of μ_{t_1}, with contradicts the hypothesis. Conversely suppose that there is no $x \in X$ such that $t_1 < \mu(x) < t_2$. Since $t_1 < t_2$, we have $\mu_{t_2} \subseteq \mu_{t_1}$. Let $x \in \mu_{t_1}$, then $\mu(x) \geq t_1$, and hence $\mu(x) \geq t_2$, because $\mu(x)$ dose not lie between t_1 and t_2. Hence $x \in \mu t_2$, which implies that $\mu_{t_1} \subseteq \mu_{t_2}$. This completes the proof.

Theorem 13 *Let μ and ν be two fuzzy BCC-ideals of a finite BCC-algebra X such that the families of level BCC-idals of μ and ν are identical. Then $\mu = \nu$ if and only if $Im(\mu) = Im(\nu)$, where $Im(\mu)$ denotes the image set of μ.*

Proof. the proof is similar to that of $[1; Theorem\ 2.11]$.

Let Γ denote the class of fuzzy BCC-ideals of a finite BCC-algebra X. Define a relation $\sim$ on Γ as follows: for any $\mu, \nu \in \Gamma$, $\mu \sim \nu$ if and only if μ and ν have the identical family of level BCC-ideals.

Theorem 14 *The relation $\sim$ is an equivalence relation.*

Proof. The proof is similar to that of [1; *Lemma* 2.12].

Theorem 15 *Let Γ be the collection of all fuzzy BCC-ideals of a finite BCC-algebra X and let Π be the collection of all level BCC-ideals of member of Γ. Then there is a $1-1$ cprrespondence between the BCC-ideals of X and the equivalence classes of level BCC-ideals under a suitable equivalence relation on Π.*

Proof. The proof is similar to that of [1; *Theorem* 2.14].

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