NOTE ON A LEMMA OF KOMORI

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Abstract. A lemma stated without proof by Y. Komori in a paper on a class of algebras related to BCK-algebras (to show that this class is not a variety) is here given a proof and applied to a broader class of algebras, which we call Komori algebras. The idea Komori had in mind for a proof can be gathered from a paper by M. Nagayama in which the lemma is proved for the class of BCK-algebras and used to show that this class is not a variety. The Komori algebras of the present note are defined by abstracting away everything not essential to this proof. This permits a formulation of the Lemma in the following general form: a non-trivial quasivariety of Komori algebras is a variety only if it satisfies some non-degenerate alien identity. Here an alien identity is one in which the rightmost variables of the two terms involves are distinct, and such an identity is non-degenerate if neither of these terms is equal to 1 (a constant in Komori algebras) over the quasivariety concerned.

1 Introduction Our purpose here is to give a clear proof of a generalization of Lemma 2 of [Ko], where the result is stated without proof, by abstracting from the proof of a similar result as it appears in [Na]. The latter paper deals with BCK-algebras and the former with (what Komori calls) BCC-algebras. (See the definitions below. The latter terminology is ill chosen, making no sense from the combinatory logical origins of labels like "BCK", as explained, e.g., in [Bu]. In [AR] the BCC-algebras are called left residuation algebras and the associated logic, BK-logic.) A form of Komori's Lemma is given in [Id], where it is attributed to Komori on the basis of its having been explained by H. Ono on a visit to Cracow in June 1981. Our version of the lemma appears as Proposition 3 below. The author is grateful to Matthew Spinks for much bibliographical and other information on BCK-algebras, and to Professor Komori for suggesting, in response to a request for a proof of [Ko]'s Lemma 2, that the relevant argument could be extracted from [Na].

The abstraction we make from the class of BCK-algebras is to define what we shall call Komori algebras, since it is these that permit Nagayama's proof of a Komori-type Lemma without any redundant hypotheses. A *Komori algebra* is an algebra $< A, \to, 1 >$ of type $< 2, 0 >$ in which equations $(K1)$-$(K3)$ and quasi-equation $(K4)$ are satisfied:

$(K1)$ $1 \to x \approx x$

$(K2)$ $x \to x \approx 1$

$(K3)$ $x \to 1 \approx 1$

$(K4)$ If $x \to y$ and $y \to x \approx 1,$ then $x \approx y$

The combined effect of (K1)-(K3) is to make the subalgebra generated by any element $a \neq 1$ of a Komori algebra isomorphic to the "implicational reduct" of the two-element boolean algebra (identifying a with 0) of Figure 1, which itself also satisfies (K4):

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$\to$	1	a
1	1	a
a	1	1

Figure 1

A well known class of Komori algebras consists of the BCK-algebras, satisfying in addition

$(K5)\quad (x \to y) \to ((y \to z) \to (x \to z)) \approx 1$

and Komori himself ([Ko]) considered what he called BCC-algebras, satisfying instead the weaker

$(K6)\quad (y \to z) \to ((x \to y) \to (x \to z)) \approx 1$

Komori proved that neither of these quasivarieties of Komori algebras is a variety (a result already known in the former case from [Wr]) by means of the Lemma with which we are concerned. In the latter case, this is given as Lemma 2 of [Ko], which introduces a partition-which we describe presently-of the class of terms constructed out of the variables x, y (and the constant 1), whose blocks are called $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{1}$, reading as follows: if any class K of BCC-algebras is a variety, then there exist $s \in \mathbf{X}$, $t \in \mathbf{Y}$ such that $s \approx t \in Id(K)$. Here $Id(K)$ is the class of identities holding in all algebras in K. Below, we shall write "$K \vDash s \approx t$" for "$s \approx t \in Id(K)$". The partition in question is defined by induction on the construction of these x, y-terms, as we shall call them, thus:

(1) $x \in \mathbf{X}$, $y \in \mathbf{Y}$, $1 \in \mathbf{1}$.

(2) If $s \in \mathbf{X}$ or $s \in \mathbf{Y}$, then $s \to t \in \mathbf{1}$.

(3) If $s \in \mathbf{1}$ then $s \to t$ belongs to whichever block of the partition $\{\mathbf{X}, \mathbf{Y}, \mathbf{1}\}$ t belongs to.

The effect of conditions (2) and (3) can be depicted in the following table:

$\to$	$\mathbf{1}$	$\mathbf{X}$	$\mathbf{Y}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{X}$	$\mathbf{Y}$
$\mathbf{X}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{Y}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

Figure 2

This tabular representation makes particularly evident two important facts about Komori's partition, collected here as Proposition 0:

PROPOSITION 0. *(i). If x,y-terms s_1 and s_2 lie in the same block of the partition $\{\mathbf{X}, \mathbf{Y}, \mathbf{1}\}$ and x,y- terms t_1 and t_2 also lie in the same block (as each other), then $s_1 \to t_1$ and $s_2 \to t_2$ also lie in the same block.*

(ii) If Figure 2 is viewed as the multiplication table for a binary operation $\to$ on the three-element set $\{\mathbf{X}, \mathbf{Y}, \mathbf{1}\}$, with $\mathbf{1}$ as the 1-element, then the algebra depicted is not a Komori algebra since although $(K1)$-$(K3)$ are satisfied, $(K4)$ is not, as $\mathbf{X} \to \mathbf{Y} = \mathbf{Y} \to \mathbf{X} = \mathbf{1}$ although $\mathbf{X} \neq \mathbf{Y}$.

The following further facts about this partition given with the classes of BCK- and BCC-algebras in mind [Na] and [Ko] do not require this specific focus. Proposition 1 here is Lemma 3 of [Ko] and Lemma 2.6 in [Na]; Proposition 2 is the transposition of Lemma 2.5 from [Na] to the more general setting of Komori algebras; the proof for Prop. 2(i) is by a simultaneous induction on the complexity of terms for both claims, with 2(ii) guaranteed by the symmetry to 2(i).This inductive argument requires (K1)-(K3) of the definition of Komori algebras. Ko denotes the class of all Komori algebras:

PROPOSITION 1. *The rightmost variable of any term in* $\mathbf{X}$ *(resp.* $\mathbf{Y}$*) is* x *(resp.* y*).*

PROPOSITION 2. *If* t^x *(resp.* t^y*) is the result of replacing every occurrence of* y *(of* x*) in the x,y-term* t *by* x *(resp., by* y*) then:*

(i) *for* $t \in \mathbf{1}$ *we have* $Ko \vDash t^x \approx 1$ *and for* $t \in \mathbf{X} \cup \mathbf{Y}$, *we have* $Ko \vDash t^x \approx x$.

(ii) *for* $t \in \mathbf{1}$ *we have* $Ko \vDash t^y \approx 1$ *and for* $t \in \mathbf{X} \cup \mathbf{Y}$, *we have* $Ko \vDash t^y \approx y$.

Terms s and t are *alien* if they have different rightmost variables. Equations between alien terms s, t - alien identities, as we shall call them - figure prominently in the BCK-algebraic literature as 'varietizing identities', i.e., identities $s \approx t$ such that any quasivariety of BCK-algebras satisfying them is a variety of BCK-algebras. Examples of such varietizing - or as we should more explicitly say, "BCK-varietizing" - identities are:

(1) $(x \to y) \to y \approx (y \to x) \to x$

(2) $(x \to y) \to ((y \to x) \to x) \approx (x \to y) \to ((y \to x) \to y)$

(3) $(((x \to y) \to y) \to x) \to x \approx (((y \to x) \to x) \to y) \to y$

which have received attention in, respectively, [RT], [Di], and [Co]. ((1) was first considered in the context of BCK-algebras by S. Tanaka in the 1970s, but the reference given provides an extended discussion. And as for (2), we should note that it is in fact a minor variation - in the context of BCK-algebras - on the identity which appears in [Di], namely

$$(x \to y) \to ((y \to x) \to x) \approx (y \to x) \to ((x \to y) \to y)$$

but we did not want all of our examples to possess the additional distracting feature of having the form $s \approx t$ with the term t resulting from interchanging x and y in s.) Such alien identities absorb the effect of the quasi-identity (K4), because - to illustrate with the first of those listed - given from the assumption that $a \to b = b \to a = 1$, in some BCK-algebra satisfying (1), we have

$(a \to b) \to b = (b \to a) \to a$, and so, reducing both sides using the assumption given:
$1 \to b = 1 \to a$, reducing again to:
$a = b$

thus by-passing any need for an explicit appeal to (K4). (As far as the author is aware, this reasoning first appears in [Ly], in the proof of Theorem 2, where (1) is considered along with (K1) and (K2) from our definition of Komori algebras.)

A similar point holds for the other examples, only with more 'reduction steps' involved in the verification. Notice that in each case the alien identity is not only BCK-varietizing, but Ko-varietizing (i.e. the class of all Komori algebras satisfying the identity is variety). Now, not just any old alien identity can be expected to have this effect, because we must immediately set aside the degenerate case of such identities as

$$x \to (y \to y) \approx x \to x$$

which are satisfied by all BCK-algebras, or indeed (as here) by all Komori algebras. Komori's lemma, as we shall formulate it, says that some non-degenerate alien identity must always be satisfied in any quasivariety of Komori algebras if that quasivariety is to be a variety (setting aside the case in which because the quasivariety contains only the one-element algebra, all identities are degenerate). It does not say that for every quasivariety K of Komori algebras, every K-varietizing identity is a non-degenerate alien identity, and this is indeed not the case. For example, taking K as the quasivariety of all BCK-algebras, the non-alien identity (4)

$$x \to (x \to y) \approx x \to y \tag{4}$$

is a K-varietizing identity, defining the variety of "positive implicative" BCK-algebras. The lemma promises only that amongst the consequences of such an identity (taken together with the quasi-equational theory of BCK-algebras) there is to be found at least one non-degenerate alien identity. In the present case, (2) and (3) from our list are such consequences. (Another example of the same phenomenon: the non-alien BCK-varietizing identity

$$(x \to y) \to x \approx x \tag{5}$$

giving the variety of "implicative" BCK-algebras - a proper subvariety of positive implicative BCK-algebras rather than the other way round, as the terminology might suggest - which has all of (1)-(3) as consequences. Note that in the sentential logic tradition (4) and (5) are versions of the Contraction Law and Peirce's Law, respectively, and that positive implicative and implicative BCK-algebras represent algebraizations of the implicational fragments of positive - or equivalently intuitionistic - logic and classical logic respectively. The identities satisfied by all implicative BCK-algebras are exactly those satisfied by the algebra depicted in Figure 1 above. Warning: much of the literature on BCK-algebras uses a dual notation, writing '0' for our '1' and writing '$y - x$', '$y * x$', or just 'yx', for our '$x \to y$'. This is because the authors are thinking of BCK-algebras as abstracting from the properties of arithmetical subtraction and set-theoretic difference rather than of an implicational connective.)

Call a quasivariety of algebras *non-trivial* if it contains at least one non-trivial algebra. Our version of 'Komori's Lemma' runs as follows:

PROPOSITION 3. *If K is a non-trivial quasivariety of Komori algebras, then K is a variety only if there exist alien x,y-terms s and t with $K \nvDash s \approx 1$ and $K \vDash s \approx t$.*

Proof. Suppose K is a non-trivial quasivariety of Komori algebras. We establish the result contrapositively. Suppose $(*)$ that for all alien x, y-terms s and t, if $K \vDash s \approx t$ then $K \vDash s \approx 1$. We will show that in that case K is not a variety. The supposition suffices to establish the following Claim: for every pair of x, y-terms t_1, t_2, if $K \vDash t_1 \approx t_2$, then t_1 and t_2 lie in the same block of the partition $\{\mathbf{X}, \mathbf{Y}, \mathbf{1}\}$. We content ourselves with ruling out the representative potential counterexamples (a) that although $K \vDash t_1 \approx t_2$, we have $t_1 \in 1$ but $t_2 \in \mathbf{X} \cup \mathbf{Y}$, and (b), that although $K \vDash t_1 \approx t_2$, we have $t_1 \in \mathbf{X}$, but $t_2 \in \mathbf{Y}$. To rule out (a), suppose for definiteness that $t_2 \in \mathbf{X}$ and note that since $K \subseteq Ko$, Prop.2(i) gives $K \vDash t_1^x \approx 1$ and $K \vDash t_2^x \approx x$, which, since according to (a) we have $K \vDash t_1 \approx t_2$, would imply $K \vDash 1 \approx x$, contradicting the assumption that K is a non-trivial quasivariety. (If, on the other hand, $t_2 \in \mathbf{Y}$, then we appeal to Prop.2(ii) instead.) Having ruled out (a), we turn to the task of ruling out (b) as a possibility. In this case, we have $t_1 \in \mathbf{X}$ and $t_2 \in \mathbf{Y}$, so by Prop.1 the identity $t_1 \approx t_2$ holding over K is an identity between alien x, y-terms and so our supposition $(*)$ gives the conclusion that $K \vDash t_1 \approx 1$, which would be an instance of (a), already ruled out as a possibility. We have thus established the Claim above - that for every pair of x, y-terms t_1, t_2, if $K \vDash t_1 \approx t_2$, then t_1 and t_2 lie in the same block of the partition $\{\mathbf{X}, \mathbf{Y}, \mathbf{1}\}$. This means that the latter partition can be regarded as a partition not just of the set of x, y-terms but of the set of "K-equivalence-classes" (congruence classes) of these terms, such a class $[t]$ being $\{s | K \vDash s \approx t\}$. (We should really write "$[t]^K$" but will take the dependence on K as read here.) These $[t]$ make up the universe of the free algebra $\mathscr{F}_K(2)$ on two generators (x and y) in the quasivariety K on which the binary operation $\rightarrow$, which we shall now write as $\rightarrow^*$ to distinguish it from the operation in the algebra (not even a Komori algebra) of terms, is defined by: $[s] \rightarrow^* [t] = [s \rightarrow t]$. Define the mapping φ from $\mathscr{F}_K(2)$ to the three-element algebra depicted in Figure 1 by setting $\varphi([t])$ =the block of the partition $\{\mathbf{X}, \mathbf{Y}, \mathbf{1}\}$ to which the term t belongs. We have already seen - in the Claim above - that this block does not depend on the selection of t as an element of $[t]$, so φ is well-defined. Next we define a binary $\rightarrow$ operation, which to avoid confusion we will write as $\rightarrow^{③}$ on this three-element algebra, by stipulating that $\varphi([s]) \rightarrow^{③} \varphi([t]) = \varphi([s] \rightarrow^* [t])$, recalling that the latter is in turn $= \varphi([s \rightarrow t])$. We must check the consistency of this stipulation. We need to know that it never happens that although $\varphi([s_1]) = \varphi([s_2])$, and $\varphi([t_1]) = \varphi([t_2])$, we have $\varphi([s_1] \rightarrow^* [t_1]) \neq \varphi([s_2] \rightarrow^* [t_2])$, or the stipulation would be inconsistent. But we recall again that $\varphi([s_1] \rightarrow^* [t_1]) = \varphi([s_1 \rightarrow t_1])$ and $\varphi([s_2] \rightarrow^* [t_2]) = \varphi([s_2 \rightarrow t_2])$. Now by the Claim above, for any term $u, \varphi([u])$ is just whichever of $\mathbf{X}, \mathbf{Y}, \mathbf{1}$ the term u belongs to in the Komori partition of terms, so the possibility to be ruled out is that s_1 and s_2 are in the same block, and so are t_1 and t_2, while $s_1 \rightarrow t_1$ and $s_2 \rightarrow t_2$ lie in different blocks, and as noted in Prop. 0(i), this is indeed impossible. The way φ has been defined makes the three-element algebra of Figure 2 a homomorphic image of the free 2-generated algebra $\mathscr{F}_K(2)$ in the quasivariety K, but since the three-element algebra is not even a Komori algebra (Prop. 0(ii)), it is certainly not in $K \subseteq Ko$, concluding the proof that K is not a variety. □

As we have said, Proposition 3 is a generalized form of Komori's lemma: a lemma because it is used to show that a particular quasi-variety of Komori algebras is not a variety, by showing that the necessary condition it supplies for variety status - the non-existence of non-degenerate alien identities - is not satisfied for the quasi-variety in question. [Ko] and [Na] use a technique of Komori's to demonstrate the non-existence of such identities, which consists of the provision of a Gentzen system for which a cut elimination theorem holds, from which the result follows. It would be interesting to know if such a Gentzen system could be provided for the larger quasi-variety of all Komori algebras. Be that as it may, the fact that the latter is not a variety follows in any case directly from [Wr], since there Wroński, to show that the BCK-algebras do not form a variety, exhibits a BCK-algebra - and thus a Komori algebra - with a homomorphic image not satisfying the quasi-identity (K4). A second question is raised by the discussion of varietizing identities in the preamble to Proposition 3. It was pointed out that although Komori's lemma does not say every varietizing identity is a non-degenerate alien identity, it does say that every varietizing identity implies such an identity. This leaves open the possibility, however, that every varietizing identity is equivalent to such an identity. More precisely, let us say that for a quasivariety X of Komori algebras, identities $s_1 \approx t_1$ and $s_2 \approx t_2$ are strongly X-equivalent if every algebra in X satisfies the quasi-identities "if $s_1 \approx t_1$ then $s_2 \approx t_2$" and its converse, and weakly X-equivalent if any algebra in X satisfying $s_1 \approx t_1$ satisfies $s_2 \approx t_2$, and conversely. Then we can ask if there are interesting quasivarieties X (for instance, $X = Ko$, or X = the class of all BCK-algebras) for which it holds that every X-varietizing identity is weakly - or perhaps even strongly - X-equivalent to an alien identity. (Alien identities need not be constructed from two variables here, but from any number, as long as the rightmost variables in the two terms are distinct.) The author has no information on this question, however.

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