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A GENERALIZED CAUCHY-KOVALEVSKAJA-NAGUMO THEOREM WITH SHRINKINGS

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Abstract. The unique existence of the solution of the Cauchy problem for PDE of the form

$$\partial_1 u(t,x) = f(t,x,u(t,x),\partial_2 u(t,x),\partial_2^p u(\alpha(t)t,x),\partial_2^q u(t,\beta(t,x)x))$$

is proved. t is in $\mathbb{R}$, x is in $\mathbb{C}$ and $u(t,x)$ is in $\mathbb{C}$. p and q are positive integers. ∂_1 and ∂_2 denote differentiations with respect to the 1^{st} and 2^{nd} variables, respectively. $f(t,x,u_1,\cdots,u_4)$ is assumed to be holomorphic in $(x,u_1,\cdots,u_4)$. α and β are called ***shrinking functions***. It is assumed that $\sup|\alpha(t)|<1$ and $\sup|\beta(t,x)|<1$.

1. Introduction

In the preceding note [4] the author studied the following two types of Cauchy problems:

$$\partial_1 u(t,x) = f(t,x,u(t,x),\partial_2^q u(t,\beta(t,x)x)),\quad u(0,x)=0, \tag{1.1}$$

$$\partial_1 u(t,x) = f(t,x,u(t,x),\partial_2^p u(\alpha(t)t,x)),\quad u(0,x)=0. \tag{1.2}$$

In (1.1) and (1.2) $u(t,x)$ denotes a complex valued unknown function of the variable $(t,x)\in\mathbb{R}\times\mathbb{C}$. ∂_i denotes partial differentiation with respect to the ith variable. p and q are positive integers. f, α and β are given continuous functions. It is assumed that $f(t,x,u_1,u_2)$ is holomorphic in (x,u_1,u_2). The functions α,β are called *shrinkings*. The reason for the use of this term is that they satisfy the conditions $\sup|\alpha(t)|<1$ and $\sup|\beta(t,x)|<1$, respectively. It is assumed that $\beta(t,x)$ is holomorphic in x. Under these conditions the Cauchy problems (1.1) and (1.2) were solved in [4]. The results in [4] are regarded as generalizations of those by Augustynowicz *et al.*[2], [3] for linear PDEs.

In the present note the author intends to unify and generalize the theories in [4] for two PDEs (1.1) and (1.2). The differential equation we consider here is of the form

$$\partial_1 u(t,x) = f(t,x,u(t,x),\partial_2 u(t,x),\partial_2^p u(\alpha(t)t,x),\partial_2^q u(t,\beta(t,x)x)). \tag{1.3}$$

It is obvious that the differential equations in (1.1) and (1.2) are special cases of the equation (1.3). Note, however, that on the right-hand side of the equation (1.3) there appears the first order partial derivative $\partial_2 u(t,x)$ of $u(t,x)$ with respect to the 'space variable' x, while it doesn't in (1.1) nor in (1.2). For this reason the result given in the present note is regarded as a generalization of the theorem of Cauchy-Kovalevskaja-Nagumo [1] (in the case where $\dim(x)=\dim(u(t,x))=1$). Since $\partial_2 u(t,x)$ does not appear in (1.1) nor in (1.2), the results in [4] cannot be regarded as generalizations of the C-K-N theorem, although they are not included in the C-K-N theorem. It is not difficult to generalize the result of this note to the multi-dimensional case. We omit it here, however, for the sake of simplicity of notation.

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Now we want to solve the differential equation (1.3) under the initial condition

$$u(0,x)=0. \tag{1.4}$$

In order to solve the Cauchy problem (1.3)-(1.4), we first transform this problem into some kind of integral equation, that is in the same manner as in [4]. But we do not use the principle of the contraction mapping in order to solve the integral equation as in [4]. We shall use in this note Tychonoff's fixed point theorem. This follows Nagumo[1]'s method who used Schauder's fixed point theorem. It may be possible to use the principle of the contraction mapping for the present case, too. In that way, however, much more complex calculation is expected to be needed. In order to use Schauder's theorem, Nagumo introduced in [1] a famous trick called *Nagumo's lemma.* We use a similar trick, too.

In order to state the result of this note we need to make some notational preparation. If T, R and S are positive constants, we write

$$\begin{aligned} A(T,R) &= \{(t,x)\in\mathbb{R}\times\mathbb{C}\ ;\ |t|<T, |x|<R\},\\ B(T,R,S) &= \{(t,x,u_1,\dots,u_4)\in\mathbb{R}\times\mathbb{C}^5\ ;\\ &\qquad (t,x)\in A(T,R),\quad |u_i|<S\ (i=1,\dots,4)\}. \end{aligned}$$

Put

$$r=p+q.$$

We fix the value of r at this value throughout this note. By means of this value of r we define the domain $\Omega(T,R)$ by

$$\Omega(T,R)=\{(t,x)\in\mathbb{R}\times\mathbb{C}\ ;|t|<T,\ R-|x|-|t|^{1/(r+1)}>0\}.$$

The main purpose of this note is to prove the following theorem.

Theorem 1.1. *Let T, R, S, m and n be positive constants. Assume that m and n are less than 1. Let p and q be positive integers. In the partial differential equation* (1.3) *assume that*

(i) *$f(t,x,u_1,\dots,u_4)$ is a complex valued bounded continuous function of $(t,x,u_1,\dots,u_4)\in B(T,R,S)$,*
(ii) *$f(t,x,u_1,\dots,u_4)$ is holomorphic in $(x,u_1,\dots,u_4)$,*
(iii) *$\partial_i f(t,x,u_1,\dots,u_4)$ $(i=2,\dots,5)$ is bounded in $B(T,R,S)$,*
(iv) *$\alpha(t)$ is a real valued continuous function of $t\in[-T,T]$ satisfying the inequality*

$$|\alpha(t)|\le m,$$

(v) *$\beta(t,x)$ is a complex valued continuous function of $(t,x)\in A(T,R)$ that is holomorphic in x and satisfies the inequality*

$$|\beta(t,x)|\le n.$$

Then there is a positive constant a such that the Cauchy problem (1.3)-(1.4) *has a unique C^1 solution $u\colon \Omega(a,R)\to\mathbb{C}$.*

The 'existence part' of the theorem will be proved in §2. The 'uniqueness part' of the theorem will be proved in §3.

2. Existence of a solution

In this section we prove the 'existence part' of the assertion of Theorem 1.1. Solving the Cauchy problem (1.3)-(1.4) is equivalent to solving the integral equation

$$w(t,x)=f\Big(t,x,\int_0^t w(\tau,x)d\tau,\int_0^t \partial_2 w(\tau,x)d\tau,\int_0^{\alpha(t)t}\partial_2^p w(\tau,x)d\tau,\int_0^t \partial_2^q w(\tau,\beta(t,x)x)d\tau\Big). \tag{2.1}$$

We want to solve the integral equation (2.1) by Tychonoff's fixed point theorem. For this purpose we first define the following function spaces. Write

$$\mathcal{C}(a)=\{w\colon \Omega(a,R)\to\mathbb{C}\ ;\ w \text{ is bounded and continuous}\},$$
$$\mathcal{D}(a)=\{w\in\mathcal{C}(a)\ ;\ w(t,x) \text{ is holomorphic in } x\}.$$

We define the positive constant K by

$$K=\sup\{|f(t,x,u_1,\dots,u_4)|,|\partial_i f(t,x,u_1,\dots,u_4)|\ (i=2,\dots,6)\ ;\ (t,x,u_1,\dots,u_4)\in B(T,R,S)\}, \tag{2.2}$$

and put

$$M=K(1+4\sqrt{R}). \tag{2.3}$$

Next we define the set of functions $\mathcal{E}(a)$ by

$$\mathcal{E}(a)=\{w\in\mathcal{D}(a)\ ;\ |w(t,x)|\le M,\ |\partial_2 w(t,x)|\le M/\sqrt{R-|x|-|t|^{1/(r+1)}} \text{ for all } (t,x)\in\Omega(a,R)\}.$$

We define the topology in $\mathcal{D}(a)$ by the uniform convergence on each compact sets of the domain $\Omega(a,R)$. Then $\mathcal{D}(a)$ becomes a complete locally convex linear topological space and $\mathcal{E}(a)$ is a closed convex set of $\mathcal{D}(a)$.

The following lemma given in Nagumo[1] is very important for the present note.

Lemma 2.1 (Nagumo's lemma)**.** *Let $\mathcal{D}$ be a bounded open domain in $\mathbb{C}$. For each element x of $\mathcal{D}$, we denote by $\rho(x)$ the distance from x to the boundary of $\mathcal{D}$. If f is a holomorphic function in $\mathcal{D}$ such that*

$$|f(x)|\le\frac{C}{\rho(x)^\alpha}$$

where C and α are given positive constants. Then the inequality

$$\left|\frac{df}{dx}(x)\right|\le\frac{(\alpha+1)^{\alpha+1}}{\alpha^\alpha}\frac{C}{\rho(x)^{\alpha+1}}$$

holds.

From this lemma follows the following Corollary.

Corollary 2.1.1. *Let w be an element of $\mathcal{E}(a)$ and l an integer ≥ 1. Then $\partial_2^l w(t,x)$ satisfies the inequality*

$$|\partial_2^l w(t,x)|\le\sqrt{2}(l-1/2)^{l-1/2}\frac{M}{(R-|x|-|t|^{1/(r+1)})^{l-1/2}}. \tag{2.4}$$

Further the inequality

$$|\partial_2^l w(mt,x)|\le\frac{\sqrt{2}(l-1/2)^{l-1/2}}{\tilde{m}^{l-1/2}}\frac{M}{|t|^{(l-1/2)/(r+1)}} \tag{2.5}$$

holds, where

$$\tilde{m} = 1 - m^{1/(r+1)}. \tag{2.6}$$

Next write

$$\tilde{n} = R(1-n)/2 \tag{2.7}$$

and let a be a constant satisfying

$$0 < a \leq \tilde{n}^{r+1}, \tag{2.8}$$

then, if $(\tau, x) \in \Omega(a, R)$ and $|t| < T$, the inequality

$$|\partial_2^l w(\tau, \beta(t,x)x)| \leq \frac{\sqrt{2}(l-1/2)^{l-1/2}}{\tilde{n}^{l-1/2}} M \tag{2.9}$$

holds.

Proof. If $w \in \mathcal{E}(a)$, then the inequality

$$|\partial_2 w(t,x)| \leq \frac{M}{\sqrt{R - |x| - |t|^{1/(r+1)}}}$$

holds. So we have (2.4) by the Lemma 2.1. If $(t,x) \in \Omega(a,R)$, then from the inequality

$$\begin{aligned} R - |x| - |mt|^{1/(r+1)} &= (R - |x| - |t|^{1/(r+1)}) + (1 - m^{1/(r+1)})|t|^{1/(r+1)} \\ &\geq \tilde{m}|t|^{1/(r+1)}, \end{aligned}$$

we see that

$$\begin{aligned} |\partial_2^l w(mt, x)| &\leq \sqrt{2}(l-1/2)^{l-1/2} \frac{M}{(R - |x| - |mt|^{1/(r+1)})^{l-1/2}} \\ &\leq \frac{\sqrt{2}(l-1/2)^{l-1/2}}{\tilde{m}^{l-1/2}} \frac{M}{|t|^{(l-1/2)/(r+1)}} \end{aligned}$$

holds, which shows (2.5). Next suppose $0 < a \leq \tilde{n}^{r+1}$. Then we have

$$R - |\beta(t,x)x| - |\tau|^{1/(r+1)} \geq R - nR - \tilde{n} = \tilde{n}$$

for each $(\tau, x) \in \Omega(a,R)$ and each $t \in [-T, T]$. Hence we obtain the inequality

$$\begin{aligned} |\partial_2^l w(\tau, \beta(t,x)x)| &\leq \sqrt{2}(l-1/2)^{l-1/2} \frac{M}{(R - |\beta(t,x)x| - |\tau|^{1/(r+1)})^{l-1/2}} \\ &\leq \frac{\sqrt{2}(l-1/2)^{l-1/2}}{\tilde{n}^{l-1/2}} M. \end{aligned}$$

This shows that (2.9) holds. □

Here we prepare a list of the estimates of the integrals appearing in the equation (2.1).

Lemma 2.2. *Let $\tilde{m}, \tilde{n}$ be the positive constants defined by (2.6) and (2.7), respectively. For each $w \in \mathcal{E}(a)$ and $(t,x) \in \Omega(a,R)$ the following three inequalities hold:*

$$\left| \int_0^t w(\tau, x) d\tau \right| \leq M|t|, \tag{2.10}$$

$$\left| \int_0^t \partial_2 w(\tau, x) d\tau \right| \leq 2(r+1)M\sqrt{R}|t|^{r/(r+1)}, \tag{2.11}$$

$$\left| \int_0^t \partial_2^2 w(\tau, x) d\tau \right| \leq 3\sqrt{3}(r+1)M \frac{|t|^{r/(r+1)}}{\sqrt{R - |x| - |t|^{1/(r+1)}}}. \tag{2.12}$$

If $l = p$ or $l = p+1$, then

$$\left|\int_0^{\alpha(t)t} \partial_2^l w(\tau,x)d\tau\right| \le \frac{m\sqrt{2}(l-1/2)^{l-1/2}(r+1)}{\tilde{m}^{l-1/2}(r-l+3/2)} M|t|^{(r-l+3/2)/(r+1)}. \tag{2.13}$$

Further, if $0 < a \le \tilde{n}^{r+1}$, then the inequality

$$\left|\int_0^{t} \partial_2^l w(\tau,\beta(t,x)x)d\tau\right| \le \frac{\sqrt{2}(l-1/2)^{l-1/2}}{\tilde{n}^{l-1/2}} M|t| \tag{2.14}$$

holds for each $(t,x) \in \Omega(a,R)$ and any nonnegative integer l.

Proof. (2.10) is clear. (2.11) also holds, since

$$\begin{aligned}
\left|\int_0^t \partial_2 w(\tau,x)d\tau\right| &\le \int_0^{|t|} \frac{M}{\sqrt{R-|x|-\tau^{1/(r+1)}}}d\tau \\
&= \int_0^{|t|^{1/(r+1)}} \frac{(r+1)s^r M}{\sqrt{R-|x|-s}}ds \\
&\le (r+1)M|t|^{r/(r+1)} \int_0^{|t|^{1/(r+1)}} \frac{1}{\sqrt{R-|x|-s}}ds \\
&\le 2(r+1)M\sqrt{R}|t|^{r/(r+1)}.
\end{aligned}$$

Next, by (2.4) of the Corollary 2.1.1, the inequality

$$|\partial_2^2 w(t,x)| \le \frac{3\sqrt{3}}{2}\frac{M}{(R-|x|-|t|^{1/(r+1)})^{3/2}}$$

holds. Therefore, we have

$$\begin{aligned}
\left|\int_0^t \partial_2^2 w(\tau,x)d\tau\right| &\le \frac{3\sqrt{3}}{2}M\int_0^{|t|} \frac{1}{(R-|x|-\tau^{1/(r+1)})^{3/2}}d\tau \\
&\le \frac{3\sqrt{3}}{2}(r+1)M|t|^{r/(r+1)} \int_0^{|t|^{1/(r+1)}} \frac{1}{(R-|x|-s)^{3/2}}ds \\
&\le 3\sqrt{3}(r+1)M|t|^{r/(r+1)}\frac{1}{\sqrt{R-|x|-|t|^{1/(r+1)}}},
\end{aligned}$$

which shows (2.12). If $l = p$ or $l = p+1$, then $l - 1/2 \le p + 1/2 < r+1$. Hence, by (2.5) of the Corollary 2.1.1, we have

$$\begin{aligned}
\left|\int_0^{\alpha(t)t} \partial_2^l w(\tau,x)d\tau\right| &\le \int_0^{m|t|} |\partial_2^l w(\tau,x)|d\tau \\
&= m\int_0^{|t|} |\partial_2^l w(ms,x)|ds \\
&\le \frac{m\sqrt{2}(l-1/2)^{l-1/2}}{\tilde{m}^{l-1/2}}M\int_0^{|t|} \frac{1}{s^{(l-1/2)/(r+1)}}ds \\
&= \frac{m\sqrt{2}(l-1/2)^{l-1/2}}{\tilde{m}^{l-1/2}}M\frac{r+1}{r-l+3/2}|t|^{(r-l+3/2)/(r+1)},
\end{aligned}$$

which shows (2.13). It is easy to see that (2.14) holds, in virtue of (2.9) of the Corollary 2.1.1. □

Now we suppose that a is a number that satisfies (2.8) and the following inequality:

$$\max\left\{Ma, 2(r+1)M\sqrt{R}a^{r/(r+1)}, \frac{m\sqrt{2}(p-1/2)^{p-1/2}}{\tilde{m}^{p-1/2}}\frac{r+1}{r-p+3/2}Ma^{(r-p+3/2)/(r+1)}, \frac{\sqrt{2}(q-1/2)^{q-1/2}}{\tilde{n}^{q-1/2}}Ma\right\} \leq \frac{S}{2}. \tag{2.15}$$

Then, by the list of the estimates of integrals in Lemma 2.2, for each element w in $\mathcal{E}(a)$, an element $\tilde{w}$ in $\mathcal{D}(a)$ is defined by

$$\tilde{w}(t,x) = f(t,x,\int_0^t w(\tau,x)d\tau, \int_0^t \partial_2 w(\tau,x)d\tau, \int_0^{\alpha(t)t} \partial_2^p w(\tau,x)d\tau, \int_0^t \partial_2^q w(\tau,\beta(t,x)x)d\tau).$$

We denote the map $w \mapsto \tilde{w}$ by Φ. We shall show that there is a positive number a such that

$$\Phi(\mathcal{E}(a)) \subset \mathcal{E}(a).$$

This is true, if the inequality

$$|\partial_2 \Phi(w)(t,x)| \leq \frac{M}{\sqrt{R-|x|-|t|^{1/(r+1)}}} \tag{2.16}$$

holds for all $(t,x) \in \Omega(a,R)$ whenever w is in $\mathcal{E}(a)$. In order to check if the inequality (2.16) holds write $\tilde{w}(t,x) = \Phi(w)(t,x)$. Then we have

$$\begin{aligned}\partial_2\tilde{w}(t,x) = \partial_2 f(t,x,\int_0^t w(\tau,x)d\tau,\cdots) &\\ + \partial_3 f(\cdots)\int_0^t \partial_2 w(\tau,x)d\tau &\\ + \partial_4 f(\cdots)\int_0^t \partial_2^2 w(\tau,x)d\tau &\\ + \partial_5 f(\cdots)\int_0^{\alpha(t)t} \partial_2^{p+1} w(\tau,x)d\tau &\\ + \partial_6 f(\cdots)\int_0^t \partial_x\{\partial_2^q w(\tau,\beta(t,x)x)\}d\tau, &\end{aligned}$$

where ∂_x denotes the partial differentiation with respect to x. It follows from the above expression of $\partial_2\tilde{w}(t,x)$ that

$$|\partial_2\tilde{w}(t,x)| \leq K\left\{1 + \left|\int_0^t \partial_2 w(\tau,x)d\tau\right| + \left|\int_0^t \partial_2^2 w(\tau,x)d\tau\right| + \left|\int_0^{\alpha(t)t} \partial_2^{p+1} w(\tau,x)d\tau\right| + \left|\int_0^t \partial_x\{\partial_2^q w(\tau,\beta(t,x)x)\}d\tau\right|\right\}, \tag{2.17}$$

where K is the positive constant defined by (2.2). By the inequalities from (2.11) to (2.13) in Lemma 2.2 we see that the inequalities

$$\left|\int_0^t \partial_2 w(\tau,x)d\tau\right| \le 2(r+1)M\sqrt{R}a^{r/(r+1)},$$
$$\left|\int_0^t \partial_2^2 w(\tau,x)d\tau\right| \le 3\sqrt{3}(r+1)M\frac{a^{r/(r+1)}}{\sqrt{R-|x|-|t|^{1/(r+1)}}},$$
$$\left|\int_0^{\alpha(t)t} \partial_2^{p+1} w(\tau,x)d\tau\right| \le \frac{m\sqrt{2}(p+1/2)^{p+1/2}}{\tilde{m}^{p+1/2}}\frac{r+1}{r-p+1/2}Ma^{(r-p+1/2)/(r+1)}$$

hold for $(t,x)\in\Omega(a,R)$. As for the fourth integral in (2.17) note that the inequality

$$\begin{aligned}|\partial_2^q w(\tau,\beta(t,x)x)| &\le \frac{\sqrt{2}(q-1/2)^{q-1/2}}{\tilde{n}^{q-1/2}}M\\ &\le \frac{\sqrt{2}(q-1/2)^{q-1/2}}{\tilde{n}^{q-1/2}}\frac{M\sqrt{R}}{\sqrt{R-|x|-|\tau|^{1/(r+1)}}}\end{aligned}$$

holds in virtue of (2.9). It follows, by Lemma 2.1, that

$$|\partial_x\{\partial_2^q w(\tau,\beta(t,x)x)\}| \le \frac{3\sqrt{3}}{\sqrt{2}}\frac{(q-1/2)^{q-1/2}}{\tilde{n}^{q-1/2}}\frac{M\sqrt{R}}{(R-|x|-|\tau|^{1/(r+1)})^{3/2}}.$$

Therefore, we see that the inequality

$$\begin{aligned}&\left|\int_0^t \partial_x\{\partial_2^q w(\tau,\beta(t,x)x)\}d\tau\right|\\ &\quad\le \frac{3\sqrt{3}}{\sqrt{2}}\frac{(q-1/2)^{q-1/2}M\sqrt{R}}{\tilde{n}^{q-1/2}}\int_0^{|t|}\frac{1}{(R-|x|-\tau^{1/(r+1)})^{3/2}}d\tau\\ &\quad\le 3\sqrt{6}\frac{(q-1/2)^{q-1/2}M\sqrt{R}(r+1)}{\tilde{n}^{q-1/2}}\frac{a^{r/(r+1)}}{\sqrt{R-|x|-|t|^{1/(r+1)}}}\end{aligned}$$

holds. Now take a number $a>0$ such that

$$\begin{aligned}&2(r+1)M\sqrt{R}a^{r/(r+1)}\\ &\qquad+\frac{m\sqrt{2}(p+1/2)^{p+1/2}}{\tilde{m}^{p+1/2}}\frac{r+1}{r-p+1/2}Ma^{(r-p+1/2)/(r+1)}\le 1,\end{aligned}\tag{2.18}$$

and

$$\left\{3\sqrt{3}(r+1)M+3\sqrt{6}\frac{(q-1/2)^{q-1/2}M\sqrt{R}(r+1)}{\tilde{n}^{q-1/2}}\right\}a^{r/(r+1)}\le\frac{M}{2K}.\tag{2.19}$$

Then, by (2.17) and (2.3), we have

$$\begin{aligned}|\partial_2\tilde{w}(t,x)| &\le 2K+\frac{M}{2}\frac{1}{\sqrt{R-|x|-|t|^{1/(r+1)}}}\\ &\le\left(2K\sqrt{R}+\frac{M}{2}\right)\frac{1}{\sqrt{R-|x|-|t|^{1/(r+1)}}}\\ &\le\frac{M}{\sqrt{R-|x|-|t|^{1/(r+1)}}},\end{aligned}$$

which shows that (2.16) holds and that $\Phi(\mathcal{E}(a))$ is included by $\mathcal{E}(a)$.

By the above discussion we see also that the map $\Phi\colon \mathcal{E}(a) \to \mathcal{E}(a)$ is continuous with respect to the relative topology induced by the locally convex topology of $\mathcal{D}(a)$. The following lemma summarizes the results obtained so far.

Lemma 2.3. *If $a > 0$ satisfies* (2.8), (2.15), (2.18) *and* (2.19), *then the relation*

$$\Phi(\mathcal{E}(a)) \subset \mathcal{E}(a)$$

holds and the map

$$\Phi\colon \mathcal{E}(a) \to \mathcal{E}(a)$$

is continuous.

We want now to solve the integral equation (2.1) by applying Tychonoff's fixed point theorem to the map $\Phi\colon \mathcal{E}(a) \to \mathcal{E}(a)$. Tychonoff's fixed point theorem says that, if G is a compact convex set of a locally convex topological linear space E, then the given continuous map $\phi\colon G \to G$ has a fixed point. But it is difficult to apply this theorem directly to the map $\Phi : \mathcal{E}(a) \to \mathcal{E}(a)$ itself, since $\mathcal{E}(a)$ is not compact in $\mathcal{D}(a)$, in general. In order to use Tychonoff's theorem, we need some preparations. We shall prove that the closed convex hull of the set $\Phi(\mathcal{E}(a))$ is compact in $\mathcal{D}(a)$. For this purpose we prepare first a lemma.

Lemma 2.4. *Let l be a non-negative integer. Then the set of functions*

$$\left\{\Omega(a,R) \ni (t,x) \mapsto \int_0^t \partial_2^l w(\tau,x)d\tau \in \mathbb{C} \ ; \ w \in \mathcal{E}(a)\right\}$$

defined on $\Omega(a,R)$ is equicontinuous at each point of $\Omega(a,R)$.

Proof. Take an arbitrary element (t_0, x_0) of $\Omega(a,R)$ and fix it. Let δ_0 be a positive constant such that, if $|t-t_0| \le \delta_0, |x-x_0| \le \delta_0$, then (t,x) belongs to $\Omega(a,R)$. We assume that the variables t,τ,x appearing in this proof always satisfy the inequalities $|t-t_0| \le \delta_0,\ |\tau - t_0| \le \delta_0$ and $|x-x_0| \le \delta_0$, respectively. If $w \in \mathcal{E}(a)$, then by the Cauchy's integral formula we have

$$\begin{aligned}|\partial_2^l w(\tau,x) - \partial_2^l w(\tau,x_0)| &\le \frac{l!M}{2\pi}\int_{|\xi|=R-|\tau|^{1/(r+1)}}\left|\frac{1}{(\xi-x)^{l+1}} - \frac{1}{(\xi-x_0)^{l+1}}\right||d\xi| \\ &\le \frac{l!M}{2\pi}\frac{1}{(R-|x_0|-\delta_0-|\tau|^{1/(r+1)})^{2(l+1)}} \\ &\qquad \times \int_{|\xi|=R-|\tau|^{1/(r+1)}} |(\xi-x)^{l+1} - (\xi-x_0)^{l+1}|\,|d\xi|\,.\end{aligned}$$

On the other hand the inequality

$$|(\xi-x)^{l+1} - (\xi-x_0)^{l+1}| \le (l+1)(2R)^l|x-x_0|$$

holds. Hence we have

$$|\partial_2^l w(\tau,x) - \partial_2^l w(\tau,x_0)| \le \frac{(l+1)!(2R)^l MR}{(R-|x_0|-\delta_0-|\tau|^{1/(r+1)})^{2(l+1)}}|x-x_0|$$

and

$$\begin{aligned}(2.20)\qquad &\left|\int_0^t \partial_2^l(w(\tau,x) - w(\tau,x_0))d\tau\right| \\ &\qquad \le \frac{(l+1)!(2R)^l MR(|t_0|+\delta_0)}{(R-|x_0|-\delta_0-(|t_0|+\delta_0)^{1/(r+1)})^{2(l+1)}}|x-x_0|.\end{aligned}$$

Next, using the Cauchy's integral formula again, we obtain

$$|\partial_2^l w(\tau,x_0)| \le \frac{l!MR}{(R-|x_0|-|\tau|^{1/(r+1)})^{l+1}}.$$

Therefore the inequality

$$\left|\int_{t_0}^{t} \partial_2^l w(\tau, x_0)d\tau\right| \leq \frac{l!MR}{(R-|x_0|-(|t_0|+\delta_0)^{1/(r+1)})^{l+1}}|t-t_0| \tag{2.21}$$

holds. It follows from (2.20) and (2.21) that we have

$$\begin{aligned}
&\left|\int_0^t \partial_2^l w(\tau,x)d\tau - \int_0^{t_0} \partial_2^l w(\tau,x_0)d\tau\right| \\
&\leq \left|\int_0^t \partial_2^l w(\tau,x)d\tau - \int_0^{t} \partial_2^l w(\tau,x_0)d\tau\right| + \left|\int_0^t \partial_2^l w(\tau,x_0)d\tau - \int_0^{t_0} \partial_2^l w(\tau,x_0)d\tau\right| \\
&\leq \frac{(l+1)!(2R)^l MR(|t_0|+\delta_0)}{(R-|x_0|-\delta_0-(|t_0|+\delta_0)^{1/(r+1)})^{2(l+1)}}|x-x_0| \\
&\qquad\qquad + \frac{l!MR}{(R-|x_0|-(|t_0|+\delta_0)^{1/(r+1)})^{l+1}}|t-t_0|,
\end{aligned}$$

which shows that the set of functions of the lemma is equicontinuous at the point (t_0, x_0). □

Corollary 2.4.1. *Let l be a non-negative integer. Then the set of functions*

$$\left\{\Omega(a,R) \ni (t,x) \mapsto \int_0^{\alpha(t)t} \partial_2^l w(\tau,x)d\tau \in \mathbb{C}\ ;\ w \in \mathcal{E}(a)\right\},$$

and

$$\left\{\Omega(a,R) \ni (t,x) \mapsto \int_0^{t} \partial_2^l w(\tau,\beta(t,x)x)d\tau \in \mathbb{C}\ ;\ w \in \mathcal{E}(a)\right\}$$

are equicontinuous at each point of $\Omega(a,R)$.

From these results follows the next lemma.

Lemma 2.5. *If a is a positive constant such that* (2.8), (2.15), (2.18) *and* (2.19) *hold, then the closed convex hull of $\Phi(\mathcal{E}(a))$ is compact in $\mathcal{D}(a)$.*

Proof. Since f is bounded, the set of functions $\Phi(\mathcal{E}(a))$ is uniformly bounded. Further, since f is continuous, $\Phi(\mathcal{E}(a))$ is equicontinuous at each point of $\Omega(a,R)$ by Lemma 2.4 and Corollary 2.4.1. Let $\langle\Phi(\mathcal{E}(a))\rangle$ be the convex hull of $\Phi(\mathcal{E}(a))$. Then any element w of $\langle\Phi(\mathcal{E}(a))\rangle$ is written as

$$w = \beta_1 w_1 + \cdots + \beta_k w_k,$$

where $\beta_1, \ldots, \beta_k$ are non negative numbers such that $\sum \beta_i \leq 1$ and $w_1, \ldots, w_k$ are elements of $\Phi(\mathcal{E}(a))$. Therefore we see that $\langle\Phi(\mathcal{E}(a))\rangle$ is uniformly bounded as a set of functions and equicontinuous at each point of $\Omega(a,R)$. It follows by the theorem of Ascoli-Arzelà that the closure $\overline{\langle\Phi(\mathcal{E}(a))\rangle}$ of the set $\langle\Phi(\mathcal{E}(a))\rangle$ in $\mathcal{D}(a)$ is compact. □

Since

$$\Phi(\overline{\langle\Phi(\mathcal{E}(a))\rangle}) \subset \Phi(\mathcal{E}(a)) \subset \overline{\langle\Phi(\mathcal{E}(a))\rangle},$$

we obtain the continuous map

$$\Phi\colon \overline{\langle\Phi(\mathcal{E}(a))\rangle} \to \overline{\langle\Phi(\mathcal{E}(a))\rangle}.$$

Therefore we can apply Tychonoff's fixed point theorem to the above map and we conclude that the integral equation (2.1) has a solution. Thus we have proved the following proposition.

Proposition 2.1. *If the number a satisfies* (2.8), (2.15), (2.18) *and* (2.19), *then the Cauchy problem* (1.3)-(1.4) *has a C^1 solution in $\Omega(a,R)$.*

3. Uniqueness of the solution

In this section we prove the 'uniqueness part' of Theorem 1.1. For this purpose we first prepare a simple lemma.

Lemma 3.1. *If* $(t,x,u_1,\dots,u_4),(t,x,v_1,\dots,v_4)\in B(T,R,S)$ *, then we have*

$$|f(t,x,u_1,\dots,u_4)-f(t,x,v_1,\dots,v_4)|\le K\{|u_1-v_1|+\cdots+|u_4-v_4|\},$$

where K *is the positive constant defined by* (2.2).

Proof. Omitted. □

Let a be a positive number and $u_1,u_2\colon\Omega(a,R)\to\mathbb{C}$ be C^1solutions of the Cauchy problem (1.3)-(1.4). We want to show that, if a is small enough, then $u_1(t,x)=u_2(t,x)$ holds for each $(t,x)\in\Omega(a,R)$. To do so it is enough to show that

$$w(t,x):=u_1(t,x)-u_2(t,x)$$

is equal to 0 for $(t,x)\in\Omega(a,R)$.

For the moment take a positive number a arbitrarily and define the constant M_0 by

$$M_0=\sup\{(R-|x|-|t|^{1/(r+1)})|w(t,x)|\ ;\ (t,x)\in\Omega(a,R)\}. \tag{3.1}$$

Then, for each $(t,x)\in\Omega(a,R)$, the inequality

$$|w(t,x)|\le\frac{M_0}{R-|x|-|t|^{1/(r+1)}}$$

holds. By Lemma 2.1, the above inequality leads to the following inequality

$$|\partial_2^l w(t,x)|\le(l+1)^{l+1}\frac{M_0}{(R-|x|-|t|^{1/(r+1)})^{l+1}}, \tag{3.2}$$

where l is an arbitrary non-negative integer and $(t,x)\in\Omega(a,R)$.

Since $w=u_1-u_2$, we have, by Lemma 3.1 and (3.2),

$$\begin{aligned}|\partial_1 w(t,x)|&=|\partial_1u_1(t,x)-\partial_1u_2(t,x)|\\&\le K\{|w(t,x)|+|\partial_2w(t,x)|+|\partial_2^pw(\alpha(t)t,x)|+|\partial_2^qw(t,\beta(t,x)x)|\}\\&\le KM_0\left\{\frac{1}{R-|x|-|t|^{1/(r+1)}}+\frac{4}{(R-|x|-|t|^{1/(r+1)})^2}\right.\\&\qquad\left.+\frac{(p+1)^{p+1}}{(R-|x|-|\alpha(t)t|^{1/(r+1)})^{p+1}}+\frac{(q+1)^{q+1}}{(R-|\beta(t,x)x|-|t|^{1/(r+1)})^{q+1}}\right\}\end{aligned} \tag{3.3}$$

for $(t,x)\in\Omega(a,R)$.

In the right-hand side of the above inequality note that

$$R-|x|-|\alpha(t)t|^{1/(r+1)}\ge R-|x|-m^{1/(r+1)}|t|^{1/(r+1)}$$

for $(t,x)\in\Omega(a,R)$. Note further that, if a satisfies (2.8), then

$$R-|\beta(t,x)x|-|t|^{1/(r+1)}\ge R-nR-|t|^{1/(r+1)}\ge\tilde n.$$

Hence, by (3.3), we have

$$\begin{aligned}|\partial_1 w(t,x)|\le KM_0&\left\{\frac{1}{R-|x|-|t|^{1/(r+1)}}+\frac{4}{(R-|x|-|t|^{1/(r+1)})^2}\right.\\&\left.+\frac{(p+1)^{p+1}}{(R-|x|-m^{1/(r+1)}|t|^{1/(r+1)})^{p+1}}+\frac{(q+1)^{q+1}}{\tilde n^{q+1}}\right\}\end{aligned}$$

for $(t,x)\in\Omega(a,R)$.

If we integrate both sides of the above inequality with respect to t, we obtain

$$
\begin{aligned}
(3.4)\qquad |w(t,x)| &\le \left| \int_0^t |\partial_1 w(\tau,x)| d\tau \right| \\
&\le KM_0 \left\{ \int_0^{|t|} \frac{1}{R-|x|-\tau^{1/(r+1)}} d\tau + \int_0^{|t|} \frac{4}{(R-|x|-\tau^{1/(r+1)})^2} d\tau \right. \\
&\quad \left. + \int_0^{|t|} \frac{(p+1)^{p+1}}{(R-|x|-m^{1/(r+1)}\tau^{1/(r+1)})^{p+1}} d\tau + \frac{(q+1)^{q+1}}{\tilde{n}^{q+1}} \int_0^{|t|} d\tau \right\}.
\end{aligned}
$$

As for the first three integrals among the last four ones appearing in (3.4) we have the following estimation:

$$
\begin{aligned}
\int_0^{|t|} \frac{1}{R-|x|-\tau^{1/(r+1)}} d\tau &\le \frac{|t|}{R-|x|-|t|^{1/(r+1)}}, \\
\int_0^{|t|} \frac{1}{(R-|x|-\tau^{1/(r+1)})^2} d\tau &= \int_0^{|t|^{1/(r+1)}} \frac{(r+1)s^r}{(R-|x|-s)^2} ds \\
&\le \frac{(r+1)|t|^{r/(r+1)}}{R-|x|-|t|^{1/(r+1)}}, \\
\int_0^{|t|} \frac{1}{(R-|x|-m^{1/(r+1)}\tau^{1/(r+1)})^{p+1}} d\tau &\le \frac{|t|}{(R-|x|-m^{1/(r+1)}|t|^{1/(r+1)})^{p+1}} \\
&\le \frac{1}{(1-m^{1/(r+1)})^{p+1}} |t|^{(r-p)/(r+1)} \\
&\le \frac{R}{\tilde{m}^{p+1}} \frac{|t|^{(r-p)/(r+1)}}{R-|x|-|t|^{1/(r+1)}}.
\end{aligned}
$$

As for the last integral on the right-hand side of (3.4) note that the inequality

$$
\int_0^{|t|} d\tau = |t| \le R \frac{|t|}{R-|x|-|t|^{1/(r+1)}}
$$

holds. These estimations lead, by (3.4), to the inequality

$$
\begin{aligned}
(3.5)\quad |w(t,x)| \le KM_0 \left\{ |t| + 4(r+1)|t|^{r/(r+1)} + \frac{(p+1)^{p+1}R}{\tilde{m}^{p+1}} |t|^{(r-p)/(r+1)} \right. \\
\left. + \frac{(q+1)^{q+1}R}{\tilde{n}^{q+1}} |t| \right\} \frac{1}{R-|x|-|t|^{1/(r+1)}}.
\end{aligned}
$$

We see therefore that, if the constant a satisfies (2.8) and

$$
(3.6)\qquad \left\{ a + 4(r+1)a^{r/(r+1)} + \frac{(p+1)^{p+1}R}{\tilde{m}^{p+1}} a^{(r-p)/(r+1)} + \frac{(q+1)^{q+1}R}{\tilde{n}^{q+1}} a \right\} \le \frac{1}{2K},
$$

then, by (3.5), the inequality

$$
(3.7)\qquad \sup\{(R-|x|-|t|^{1/(r+1)})|w(t,x)|\ ;\ (t,x)\in\Omega(a,R)\} \le \frac{M_0}{2}
$$

holds.

In order that (3.1) and (3.7) hold simultaneously the constant M_0 must be equal to 0. Thus we have proved the following proposition.

Proposition 3.1. *If the number a satisfies* (2.8) *and* (3.6), *then there is at most one C^1 solution in $\Omega(a,R)$ of the Cauchy problem* (1.3)-(1.4).

Combining the results of Propositions 2.1 and 3.1 we see that, if the number a satisfies (2.8), (2.15), (2.18), (2.19) and (3.6), then the Cauchy problem (1.3)-(1.4) has a unique C^1solution in $\Omega(a, R)$. This completes the proof of Theorem 1.1.

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