CONTINUOUS SELECTIONS ON ALMOST COMPACT SPACES

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ABSTRACT. We prove that (1) for a Tychonoff space X with pseudocompactness of X^2, βX is orderable if and only if X has a weak selection. (2) Let X be an almost compact space. Then the following are equivalent: (i)X is orderable, (ii) X has a continuous selection, (iii) X has a weak selection. (3) For a connected Tychonoff space X with a continuous selection, X^2 is pseudocompact then X is either compact or almost compact.

1. INTRODUCTION

In this paper all topological spaces are assumed to be Tychonoff. In what follows βX denotes the Čech-Stone compactification of a space X. Let X be a topological space, and let $\mathcal{F}(X)$ be the set of all non-empty closed subsets of X. Let us recall the definition of the *Vietoris topology* τ_V on $\mathcal{F}(X)$ [7]. The base for τ_V is defined by the collection of sets

$$\langle \mathcal{V} \rangle = \{F \in \mathcal{F}(X) : F \subset \cup\mathcal{V} \text{ and } F \cap V \neq \emptyset \text{ for } V \in \mathcal{V}\},$$

where $\mathcal{V}$ runs over all finite families of non-empty open subsets of X. Let $\mathcal{E} \subset \mathcal{F}(X)$. A map $\sigma : \mathcal{E} \to X$ is a *selection* for $\mathcal{E}$ if $\sigma(F) \in F$ for every $F \in \mathcal{E}$. A *continuous selection on* X is a selection for $\mathcal{F}(X)$ that is continuous with respect to the Vietoris topology τ_V on $\mathcal{F}(X)$. Let $\mathcal{F}_2(X)$ be the set of all non-empty closed subsets of X containing at most two points. It is clear that $\mathcal{F}_2(X)$ is a closed subset of $\mathcal{F}(X)$, the latter set being equipped with the Vietoris topology. A map $\sigma : \mathcal{F}_2(X) \to X$ will be called a *weak selection on* X if it is a selection for $\mathcal{F}_2(X)$ that is continuous with respect to the subspace topology on $\mathcal{F}_2(X)$ induced by τ_V. The following result is folklore:

Theorem 1.1. *A space X has a weak selection if and only if there exists a continuous map $\sigma : X^2 \to X$ such that, for all $x, y \in X$, one has*
(i) $\sigma(x, y) = \sigma(y, x)$ *and*
(ii) $\sigma(x, y) \in \{x, y\}$.

It is obvious that a space having a continuous selection also has a weak selection. A space X is *weakly orderable* if there exists a linear order $<$ on X such that all sets of the form $(-\infty, p) = \{x \in X : x < p\}$ and $(p, +\infty) = \{x \in X : x > p\}$ are open in X. It is well-known (and easy to see) that a weakly orderable space has a weak selection, and moreover, if a space X is weakly orderable via $<$ in such a way that every non-empty closed subset of X has a $<$-minimal element, then X has a continuous selection. Furthermore, the converse is true for connected spaces: if X is a connected space which has a continuous selection, then X must be weakly orderable via some order $<$ so that every non-empty closed subset of X has a $<$-minimal element [7]. It is interesting to look for conditions that are equivalent to the existence of a continuous selection on a given space. The following well known result gives such an equivalent condition in the class of compact spaces [8]:

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Theorem 1.2. *Let X be a compact Hausdorff space. Then the following conditions are equivalent.*

(i) *X is orderable.*

(ii) *X has a continuous selection.*

(iii)*X has a weak selection.*

This theorem is no longer valid for a wider class of Lindelöf spaces: for example, that real line $\mathbf{R}$ has no continuous selection [3]. There exists also a scattered linearly ordered space without a continuous selection [4]. The above theorem cannot be easily generalized to locally compact spaces. In fact, no equivalent condition for the existence of a continuous selection is known even for a space obtained by removing a single point from a compact space. A non-compact Tychonoff space X is *almost compact* if $\beta X - X$ has exactly one point. Note that, given any two disjoint zero-sets of an almost compact space X, at least one of them must be compact, and if in addition one assumes that X is normal, then "zero-sets" can be weakened to "closed sets" [1]. In section 2 we generalize Theorem 1.2 by showing that a Tychonoff space X with $X \times X$ pseudocompact has a weak selection if and only if βX is orderable. This implies that a pseudocompact locally compact space with a weak selection must be is orderable. In section 3 we will prove that Theorem 1.2 remains valid if one replaces "compact" by "almost compact". In section 4 we use results of section 2 and [6] to show that a connected space with a continuous selection X^2 is pseudocompact if and only if it is either compact or almost compact.

2. SELECTIONS ON PSEUDCOMPACT SPACES

Theorem 2.1. *Let X be a space such that X^2 is pseudocompact. Then the following conditions are equivalent.*

(1) *X is weakly orderable.*

(2) *X has a weak selection.*

(3) *βX has a weak selection.*

(4) *βX has a continuous selection.*

(5) *βX is orderable.*

Proof. Implication (1) $\to$ (2) is trivial. Implications (3) $\to$ (4) and (4) $\to$ (5) follow from Theorem 1.2. Implication (5) $\to$ (1) holds because a subspace of an orderable space is weakly orderable. So it remains only to prove that (2) $\to$ (3). Assume that X has a weak selection. Let $\sigma : X^2 \to X$ be a continuous map satisfying conditions (i) and (ii) of Theorem 1.1. Without loss of generality σ can be assumed to be a map into βX, so there exists a continuous extension $\varphi : \beta X^2 \to \beta X$ of σ. Since X^2 is pseudocompact, $\beta X^2 = \beta X \times \beta X$ [5], i.e. φ is a continuous map from $\beta X \times \beta X$ to βX. We are going to show that φ is a weak selection on βX by checking that φ satisfies conditions (i) and (ii) of Theorem 1.1 (with σ replaced by φ). Let $x, y \in \beta X$. Due to symmetry of x and y, it suffices to consider two cases.

Case 1: $x, y \in X$. In this case (i) and (ii) hold for φ because they hold for σ and φ is an extension of σ.

Case 2: $x \in X$ and $y \in \beta X - X$. There is a net $\{y_\alpha\}_{\alpha \in \Gamma} \subset X$ converging to y in βX. Then $\{(x, y_\alpha)\}_{\alpha \in \Gamma}$ is a net contained in X^2 converging to (x, y) in $(\beta X)^2$ and $\varphi(x, y_\alpha) \in \{x, y_\alpha\}$. Assume that $\varphi(x, y) = z$ for some $z \in \beta X$ such that $x \neq z \neq y$. Choose disjoint open sets $U_x,\ U_y,\ U_z$ containing $x,\ y,\ z$ respectively in βX. By $z = \lim \varphi(x, y_\alpha)$, there exists $\alpha \in \Gamma$ such that $\varphi(x, y_{\alpha'}) \in U_z$ for every $\alpha' \geq \alpha$. On the other hand by $(x, y) = \lim(x, y_\alpha)$ there exists $\beta \in \Gamma$ such that $(x, y_{\beta'}) \in U_x \times U_y$ for every $\beta' \geq \beta$ i.e. $\varphi(x, y_{\beta'}) \in U_x$ or U_y. Then for $\gamma \geq \alpha, \beta$ one has $\varphi(x, y_\gamma) \in (U_x \cap U_z) \cup (U_y \cap U_z)$, a contradiction with the choice of U_x, U_y, U_z. This contradicition implies that $\varphi(x,y) \in \{x,y\}$. Finally, $\varphi(x,y) = \lim\varphi(x,y_\alpha) = \lim\varphi(y_\alpha,x) = \varphi(y,x)$.

Case 3: $x,y \in \beta X - X$. It is possible to show as the similar way of Case 2. □

An orderable space is hereditarily collectionwise normal and hereditarily countably paracompact. Together with Theorem 2.1 this implies the following.

Corollary 2.2. *If X^2 is pseudocompact and X has a weak selection, then X is hereditarily collectionwise normal, hereditarily countably paracompact and countably compact.*

For an orderable space pseudocompactness, countably compactness and sequentially compactness are equivalent [6]. If X is a pseudocompact k-space then X^2 is pseudocompact [2]. Moreover each open subset of an orderable space is orderable, therefore;

Corollary 2.3. *Let X be a pseudocompact locally compact space with a weak selection. Then X is orderable and sequentially compact.*

Remark 2.4. If one glues together the first point 0 of two copies of the long line, then the resulting space L is a linearly ordered and L^2 is pseudocompact, so has a weak selection. However it does not have a continuous selection [9]. Note that $|\beta L - L| = 2$. We will discuss the case $|\beta X - X| = 1$ in the next section.

3. SELECTIONS ON ALMOST COMPACT SPACES

Let $<$ be a linear order on X. For every $p,q \in X$ define $(p,q)_< = \{r \in X : p < r < q\}, [\,p,q)_< = \{r \in X : p \le r < q\}, (p,q\,]_< = \{r \in X : p < r \le q\}, (-\infty,q)_< = \{r \in X : r < q\}, [\,p,\infty)_< = \{r \in X : p < r\}$.

Lemma 3.1. *Let X be an almost compact space with $\{x\} = \beta X - X$. Assume that βX is orderable. Then the topology of βX can be generated by a (possibly different) linear order $\ll$ such that x is a $\ll$-maximal element of βX.*

Proof. Let $<$ be a linear order on βX generating its topology. We will assume, without loss of generality, that x is not a maximal element of $(\beta X, <)$. If x is a minimal element of $(\beta X, <)$, then we can define $\ll$ to be the reverse order of $<$. So we may assume from now on that x is not a minimal element of $(\beta X, <)$. Therefore $X = (-\infty,x)_< \cup (x,\infty)_<$. Clearly $A = (-\infty,x)_<$ and $B = (x,\infty)_<$ are zero-sets in X with $X = A \oplus B$. Since X is almost compact, one of A or B, say A, must be compact. Obviously, x is a $<$-maximal element of $A \cup \{x\}$, and a $<$-minimal element of $\{x\} \cup B$. Since A is compact, $B \cup \{x\}$ is clopen in βX. Define new order $\ll$ on βX as follows. For every $p,q \in \beta X$

$$p \ll q \Leftrightarrow \begin{cases} p < q, & p,q \in A, \\ p > q, & p,q \in B \cup \{x\}, \\ p \in A, & q \in B \cup \{x\} \end{cases}$$

Since $B \cup \{x\}$ and A form a clopen cover of βX, the order topology of $\ll$ is equal to the order topology of $<$. □

Lemma 3.1 implies the following.

Lemma 3.2. *Let X be an almost compact space with $\{x\} = \beta X - X$. Assume that βX is orderable. Then X can be ordered via a linear order $<$ as follows.* (i) *Every closed subset of X has a $<$-minimal element.* (ii)$(-\infty, q\,]_<$ *is compact for every $q \in X$.*

Our next result demonstrates that Theorem 1.2 remains valid if one replaces "compact" by "almost compact".

Theorem 3.3. *For an almost compact space X the following conditions are equivalent.*

(1) *βX has a weak selection.*
(2) *βX has a continuous selection.*
(3) *βX is orderable.*
(4) *X is orderable.*
(5) *X is orderable via $<$ in such a way that every closed subset of X has a $<$-minimal element.*
(6) *X is weakly orderable via $<$ so that every closed subset of X has a $<$- minimal element.*
(7) *X has a continuous selection.*
(8) *X has a weak selection.*
(9) *X is weakly orderable.*

Proof. Almost compact spaces are pseudocompact and locally compact. Equivalence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (8) $\Leftrightarrow$ (9) follows from Theorem 2.1. Implication (3) $\Rightarrow$ (5) follows from Lemma 3.1. Finally, implications (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8), and (3) $\Rightarrow$ (4) $\Rightarrow$ (9) are trivial. $\square$

We now prove an anologue of Theorem 3.3 for the Fell topology on $\mathcal{F}(X)$. Recall that a base for the *Fell topology* τ_F on $\mathcal{F}(X)$ is the family of all sets

$$\langle \mathcal{V} \rangle = \{F \in \mathcal{F}(X) : F \subset \cup\mathcal{V} \text{ and } F \cap V \neq \emptyset \text{ for each } , V \in \mathcal{V}\}$$

where $\mathcal{V}$ runs over all finite families of non-empty open subsets of X such that $X - \cup\mathcal{V}$ is compact. The Fell topology is coarser than the Vietoris topology on $\mathcal{F}(X)$ and if X is compact then the Fell topology coincides with the Vietoris topology. A *Fell continuous selection* is a selection for $\mathcal{F}(X)$ that is continuous with respect to the Fell topology τ_F on $\mathcal{F}(X)$. A *weak Fell selection* is defined similarly to the weak selection.

Theorem 3.4. *For an almost compact space X the following conditions are equivalent.*
(1) *βX has a weak selection.*
(2) *βX has a continuous selection.*
(3) *βX has a Fell continuous selection.*
(4) *βX is orderable.*
(5) *X is orderable.*
(6) *X is orderable via $<$ in such a way that every closed subset of X has a minimal element.*
(7) *X has a Fell continuous selection.*
(8) *X has a weak Fell selection.*
(9) *X has a weak selection.*
(10) *X is weakly orderable.*

Proof. Theorem 3.3 implies (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (9) $\Leftrightarrow$ (10). (4) $\Rightarrow$ (7) holds by Lemma 3.2. (7) $\Rightarrow$ (8) $\Rightarrow$ (9) is trivial. $\square$

The following two examples demonstrate that our results are final to various degrees.

Example 3.5. (a) Both the Tychonoff plank T [2] and $\beta N - \{p\}$, where $p \in \beta N - N$, N is natural numbers have no weak (continuous) selection. These spaces are known to be almost compact and non-normal [2].

(b) $X = [0, \omega_1) \times [0, \omega_0]$ has no weak (continuous) selection (c.f. [4]). Since $[0, \omega_1) \times [0, \omega_0]$ is pseudocompact, $\beta X = [0, \omega_1] \times [0, \omega_0]$. Since βX contains non-normal subspace T, βX is not orderable.

4. SELECTIONS ON ORDERABLE CONNECTED SPACES

A locally compact pseudocompact space with a continuous selection need not be almost compact or compact; $[0,\omega_1) \oplus [0,\omega_1)$ is neither almost compact nor compact but is locally compact and pseudocompact with a continuous selection. We are going to prove in Corollary 4.5 that for connected spaces this becomes true.

For an orderable space X there exist a minimal orderable compactification X_0 and a maximal orderable compactification X_∞ of X. (Recall that an ordered space $(X', <')$ is an *orderable compactification* of $(X, <)$ provided that X is dense subset in X' and $<'$ coincides with $<$ on X.) The order of orderable compactifications of X coincides with one of compactifications of X, X_∞ and X_0 correspond to βX and the Alexandoroff compactification αX of X respectively. The following propositions are known [6].

Proposition 4.1. *Let X be an orderable space.*

1. *If X is connected, then $X_\infty = X_0$.*
2. *If X is connected and pseudocompact, then $\beta X = X_\infty = X_0$.*

Proposition 4.2. *Let X be a non-compact connected orderable space. Let $A = \{a, b\}$ be the set of a minimal and a maximal elements of X_0.*

1. *If $a \in X$, then $X_0 = X \cup \{b\}$.*
2. *$X \cap A = \emptyset$, then $X_0 = X \cup \{a, b\}$.*

These propositions and Corollary 2.3 imply the following.

Lemma 4.3. *Let X be orderable, connected and pseudocompact.*

1. *If X has either a minimal or a maximal element, then X is either compact or almost compact.*
2. *If X has neither minimal nor maximal element, then $\beta X = X \cup \{a, b\}$, where $\{a, b\}$ is the set of a minimal and a maximal elements of βX.*

An orderable connected space is locally compact. The next result follows:

Theorem 4.4. *For an orderable connected space X the following conditions are equivalent.*

(1) *X has a continuous selection.*
(2) *X has either a minimal or a maximal element.*

Proof. We only need to consider the case when that X is not compact. If X has either a minimal or a maximal element a, then $X_0 = X \cup \{b\}$, where $\{a, b\}$ is the set of a minimal and a maximal elements of X_0. Then X_0 is homeomorphic to αX, and αX has a continuous selection because it is orderable. Therefore X has a continuous selection. The converse follows from [9]. $\square$

A connected GO-space is orderable. Therefore;

Corollary 4.5. *For a connected space X which has a continuous selection the following conditions are equivalent.*

(1) *X^2 is pseudocompact.*
(2) *X is either compact or almost compact.*

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Added in proof : The equivalence (1) and (2) in Theorem 2.1 was independently obtained by G. Artico, U. Marconi, J. Pelant, L. Rotter and M. Tkachenko, and was announced in Topology Atlas.

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