REMARK ON CHAOTIC FURUTA INEQUALITY

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(*Dedicated to the memory of Professor Sz-Nagy Béla in deep sorrow.*)

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Abstract. Recently Uchiyama gave a nice comment on the implication of the Furuta inequality to the chaotic Furuta inequality, i.e., $\log A \ge \log B$ for positive invertible operators A and B if and only if $A^r \ge (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ for all $p,\ r \ge 0$. The purpose of this note is to show the converse implication. That is, the chaotic Furuta inequality and the Furuta inequality are equivalent.

1. Throughout this note, we use a capital letter as an operator on a Hilbert space H. An operator A is said to be positive (in symbol: $A \ge 0$) if $(Ax, x) \ge 0$ for all $x \in H$, and also an operator A is strictly positive (in symbol: $A > 0$) if A is positive and invertible.

One of recent developments in operator theory is the Furuta inequality [7]. Professor Sz-Nagy said that the Furuta inequality is a historical and beautiful extension of the Löwner-Heinz one. It also clarified the utility of operator means established by Kubo-Ando [14](cf.[3],[12]). We denote the α-power mean for $\alpha \in [0,1]$ by

$$A \sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}, \quad for\ A,\ B > 0.$$

The monotonicity of $\sharp_\alpha$ corresponds to the operator monotonicity of the function $f(t) = t^\alpha$. We use the notation $\natural_\alpha$ for $\alpha \notin [0,1]$ instead of $\sharp_\alpha$. In the light of the Kubo-Ando theory, the Furuta inequality is represented by the following way:

The Furuta inequality. *If $A \ge B > 0$, then*

$$A^{-r} \natural_{\frac{1+r}{p+r}} B^p \le A \tag{1}$$

for all $p \ge 1$ and $r \ge 0$.

For the sake of convenience, we cite (1) by the original form:

$$(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \le A^{\frac{p+r}{q}} \tag{2}$$

for $p,\ r \ge 0,\ q \ge 1$ with $(1+r)q \ge p+r$. It is easy to see that (1) exactly expresses the extremal case $(1+r)q = p+r$ in (2). We remark that the case of (1) is in fact the best possible by Tanahashi [15] and refer [8] for a one-page proof.

In the preceding note [13], one of the authors pointed out that (1) is partially satisfied under a weaker condition than the usual order $A \ge B > 0$. For this, the chaotic order was prepared: For positive invertible opretors $A,\ B > 0$, we denote by $A \gg B$ if $\log A \ge \log B$ (cf.[5]). Since $\log t$ is operator monotone, the chaotic order is really weaker than the usual one.

Another base is a satellite to the Furuta inequality [12]:

Theorem A. *If $A \ge B \ge 0$, then*

$$A^{-r} \natural_{\frac{1+r}{p+r}} B^p \le B \le A \le B^{-r} \natural_{\frac{1+r}{p+r}} A^p \tag{1'}$$

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for all $p \geq 1$ *and* $r \geq 0$.

We cite here our preceding result [13]:

Theorem B. *If* $A \gg B$ *for* $A,\ B > 0$, *then*

$$A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq B \tag{3}$$

for all $p \geq 1$ *and* $r \geq 0$.

2. At the begining of this section, we cite the following Furuta type characterization of the chaotic order [4]. We call it the chaotic Furuta inequality.

Theorem C. *For* $A,\ B > 0$, $A \gg B$ *if and only if*

$$A^{-r} \sharp_{\frac{r}{p+r}} B^p \leq I \tag{4}$$

or equivalently

$$B^p \sharp_{\frac{p}{p+r}} A^{-r} \leq I \tag{4'}$$

for all $p,\ r > 0$.

It is a 2-variable version of Ando's inequality [1].

The purpose of this short note is to propose the following equivalence between the Furuta inequality and the chaotic Furuta inequality.

Theorem 1. *The followings are equivalent.*

$$\text{(i)} \qquad If\ A \geq B > 0,\ then\ (1)\ holds\ for\ p \geq 1\ and\ r \geq 0.$$

$$\text{(ii)} \qquad If\ A \gg B\ for\ A,\ B > 0,\ then\ (3)\ holds\ for\ p \geq 1\ and\ r \geq 0.$$

There are some papers on the implication of (i) to (ii). Among others, Uchiyama's proof is fantastic [16]: Along with Furuta [11], the tool is mentioned:

$$\lim_{n \to \infty} (1 + \frac{\log X}{n})^n = X. \tag{5}$$

Assume that $A \gg B$. Then

$$A_n = 1 + \frac{\log A}{n} \geq B_n = 1 + \frac{\log B}{n}.$$

So one can apply the Furuta inequality for $A_n \geq B_n \geq 0$. That is,

$$(A_n^{\frac{nr}{2}} B_n^{np} A_n^{\frac{nr}{2}})^{\frac{\frac{1}{n}+r}{p+r}} \leq A_n^{1+nr}.$$

Taking $n \longrightarrow \infty$, we have (4).

To prove the converse (ii) $\Longrightarrow$ (i), we need the following lemma on $\sharp_\alpha$ which is also satisfied for $\natural_\alpha$:

Lemma 2. *For* $X,\ Y > 0$,

$$\text{(i)} \qquad X \sharp_\alpha Y = Y \sharp_{1-\alpha} X,$$

$$\text{(ii)} \qquad X \sharp_{\alpha\beta} Y = X \sharp_\alpha (X \sharp_\beta Y),$$

$$\text{(iii)} \qquad X \sharp_\alpha Y = X(X^{-1} \natural_{-\alpha} Y^{-1})X.$$

We remark that (i) is a rephrase of Furuta's lemma and (ii) is a kind of multiplicativity.

Suppose that the chaotic Furuta inequality (ii) and $A \geq B > 0$. Since $A \gg B$ is satisfied, we have (4') and so

$$\begin{aligned} A^{-r} \sharp_{\frac{1+r}{p+r}} B^p &= B^p \sharp_{1-\frac{1+r}{p+r}} A^{-r} = B^p \sharp_{\frac{p-1}{p+r}} A^{-r} \\ &= B^p \sharp_{\frac{p-1}{p}} (B^p \sharp_{\frac{p}{p+r}} A^{-r}) \leq B^p \sharp_{\frac{p-1}{p}} I = I \sharp_{\frac{1}{p}} B^p = B. \end{aligned}$$

Moreover, since $A \geq B$ is assumed, we have the Furuta inequality (1).

3. As a generalization of the Furuta inequality, Furuta [9](cf.[10]) had given an inequality which we called the grand Furuta inequality in [6]. It interpolates the Furuta inequality and the Ando-Hiai inequality [2] equivalent to the main result of log majorization. We here cite it in terms of operator mean:

Grand Furuta inequality: *If $A \geq B \geq 0$ and A is invertible, then for each $p \geq 1$ and $0 \leq t \leq 1$,*

$$\text{(6)} \qquad A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A$$

holds for $r \geq t$ and $s \geq 1$.

In [11], Furuta proposed the following question:

Let A and B be invertible positive operators. Then $A \gg B$ holds if and only if

$$\text{(Q)} \qquad A^{-r} \sharp_{\frac{r}{\beta+r}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \leq I$$

holds for any $\beta \geq p \geq 1$, $1 \geq t \geq 0$ and $r \geq 0$ **?**

Since (4) is a characterization of the chaotic order with the form corresponding to (1), it is natural to ask whether (Q) can be the corresponding form to (6). By taking $t = 0$ and $\beta = p$, it is obiously obtained from Theorem C that (Q) implies the chaotic order $A \gg B$. So the essential part of this question is the converse. Recently Furuta himself has given a counterexample which is very nice but needs very hard calculations. It is given by

$$\log A = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \quad \text{and} \quad \log B = \begin{pmatrix} 1 & 3 \\ 3 & -2 \end{pmatrix}.$$

Then $A \gg B$ and

$$A^{-r} \sharp_{\frac{r}{\beta+r}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) \not\leq I$$

for $r = 1$, $p = 2$, $\beta = 3$ and $t = 1$.

It seems that the condition $0 < t$ obstracts (Q) being the chaotic order. By shifting t to negative part, we can obtain the following:

Theorem 3. *Let A and B be positive invertible operators. Then the followings are equivalent.*

$$\text{(i)} \qquad A \gg B$$

$$\text{(ii)} \qquad A^{-r} \sharp_{\frac{r}{\beta+r}} (A^{-u} \natural_{\frac{\beta+u}{p+u}} B^p) \leq I \quad \textit{for} \quad 0 \leq u \leq r, \quad 0 \leq p \leq \beta \leq 2p.$$

To prove this theorem, we prepare the next lemma.

Lemma 4. *If $A \gg B$ and $0 \le p \le \beta \le 2p$, then for $0 \le u \le r$,*

$$A^{-u} \natural_{\frac{\beta+u}{p+u}} B^p \le A^{-r} \natural_{\frac{\beta+r}{p+r}} B^p.$$

Proof. Since $1 \le \frac{\beta+u}{p+u} \le 2$, by using Lemma 2 and Theorem C, we have

$$A^{-r} \sharp_{\frac{-u+r}{p+r}} B^p = A^{-r} \sharp_{\frac{-u+r}{r}} (A^{-r} \sharp_{\frac{r}{p+r}} B^p) \le A^{-r} \sharp_{\frac{-u+r}{r}} I = I \sharp_{\frac{u}{r}} A^{-r} = A^{-u}.$$

So we can show the result as follows:

$$\begin{aligned} A^{-u} \natural_{\frac{\beta+u}{p+u}} B^p &= B^p \natural_{\frac{p-\beta}{p+u}} A^{-u} = B^p(B^{-p} \sharp_{\frac{\beta-p}{p+u}} A^u)B^p \\ &\le B^p(B^{-p} \sharp_{\frac{\beta-p}{p+u}} (A^r \sharp_{\frac{-u+r}{p+r}} B^{-p}))B^p \\ &= B^p(B^{-p} \sharp_{\frac{\beta-p}{p+u}} (B^{-p} \sharp_{\frac{p+u}{p+r}} A^r))B^p \\ &= B^p(B^{-p} \sharp_{\frac{\beta-p}{p+r}} A^r)B^p = A^{-r} \natural_{\frac{\beta+r}{p+r}} B^p. \end{aligned}$$

Proof of Theorem 3. By Lemma 4 and Theorem C, we have

$$A^{-r} \sharp_{\frac{r}{\beta+r}} (A^{-u} \natural_{\frac{\beta+u}{p+u}} B^p) \le A^{-r} \sharp_{\frac{r}{\beta+r}} (A^{-r} \natural_{\frac{\beta+r}{p+r}} B^p) = A^{-r} \sharp_{\frac{r}{p+r}} B^p \le I.$$

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