ON A CONJECTURE BY ANDRZEJ WROŃSKI FOR BCK-ALGEBRAS AND SUBREDUCTS OF HOOPS

ISABEL M. A. FERREIRIM

Received February 7, 2000

ABSTRACT. We give a detailed proof of a conjecture by Andrzej Wroński on embedding BCK-algebras satisfying

$$(z \dot{-} x) \dot{-} (y \dot{-} x) \approx (z \dot{-} y) \dot{-} (x \dot{-} y)$$

into reducts of hoops (i.e., naturally ordered commutative integral monoids with residuation).

1. INTRODUCTION

BCK-algebras, introduced by Iséki [Ise66], are the equivalent algebraic semantics of the BCK logic of Meredith, in the sense of Blok and Pigozzi [BP89]. As is well known, every BCK-algebra may be viewed as a $\{\dot{-}, 0\}$-subreduct of a *pocrim* (partially ordered commutative residuated integral monoid, as coined by Blok and Raftery [BR95])—see Theorem 2.4 below.

Among pocrims, we distinguish those for which the order is *natural*, i.e., those for which $a \leq b$ if and only if $\exists c \;\; b = a + c$. They include for instance positive cones of Abelian lattice-ordered groups and are sometimes called naturally ordered pocrims. They have appeared in the literature also under the names *complementary semigroups* [Bos69], *hoops* [BO], *BCK-algebras with a supremum* [Cor82], *naturally ordered commutative integral monoids with residuation* [Wro85]. We shall use the name *hoops* because it is conveniently short.

It was first observed by Bosbach [Bos69], in a more general context, that the natural order could be expressed by the identity

(i) $x + (y \dot{-} x) \approx y + (x \dot{-} y)$.

Note that this identity expresses the fact that the binary term $x \vee y := y + (x \dot{-} y)$ is commutative, thus defining a join operation.

Identity (i) may be reformulated, using residuation only, by

(ii) $(z \dot{-} x) \dot{-} (y \dot{-} x) \approx (z \dot{-} y) \dot{-} (x \dot{-} y)$,

in the sense that a pocrim satisfies (i) if and only if it satisfies (ii). An immediate observation is that all residuation subreducts of hoops satisfy (ii).

Based on the argument above, Wroński formulated the conjecture that all BCK-algebras satisfying (ii) may be obtained as $\{\dot{-}, 0\}$-subreducts of hoops [Wro85].

The main aim of this paper is to provide a detailed proof of Wroński's conjecture. The main result (Theorem 5.1) with an outline of its proof, has appeared in [BF93].

1991 *Mathematics Subject Classification.* 06F05, 06F35, 03G25, 20M99.

Keywords. BCK-algebras, subreducts, hoops.

The author was partly supported by research project Praxis XXI "Álgebra e Matemáticas Discretas".

2. Preliminaries

Given a first order structure $\mathbf{A} = \langle A; L\rangle$ and $L_1 \subset L$ we say that $\mathbf{A}^{L_1} = \langle A; L_1\rangle$ is the L_1-*reduct of* $\mathbf{A}$. An L_1-structure $\mathbf{B} = \langle B; L_1\rangle$ is said to be an L_1-*subreduct* of $\mathbf{A}$ if $\mathbf{B}$ is a substructure of $\mathbf{A}^{L_1}$. Given a class $\mathcal{K}$ of L-structures, $\mathcal{K}^{L_1}$ and $\mathsf{S}\mathcal{K}^{\mathsf{L}_1}$ denote respectively the class of L_1-reducts and the class of L_1-subreducts of members of $\mathcal{K}$.

BCK-algebras. A BCK-algebra is an algebra $\mathbf{A} = \langle A; \dot{-}, 0\rangle$ of type $(2,0)$ which satisfies the following axioms:

(1) $((x \dot{-} y) \dot{-} (x \dot{-} z)) \dot{-} (z \dot{-} y) \approx 0$
(2) $x \dot{-} 0 \approx x$
(3) $0 \dot{-} x \approx 0$
(4) If $x \dot{-} y \approx 0$ and $y \dot{-} x \approx 0$ then $x \approx y$.

Note that the usual symbols for the binary operation on a BCK-algebra are either $*$ or $\cdot$; the latter is sometimes omitted or replaced by juxtaposition. Our choice of $\dot{-}$ (*monus*) is justified by the fact that in this paper we will be looking at certain BCK-algebras as $\{\dot{-}, 0\}$-subreducts of hoops.

The class of all BCK-algebras is a quasivariety. Wroński [Wro83] and, independently, Higgs [Hig84] have shown that this class is a proper quasivariety, i.e., it is not a variety.

The binary relation $\leq$ on A, defined by $a \leq b$ if and only if $a \dot{-} b = 0$, is a partial order with least element 0—see [IT78].

It is well known that in a BCK-algebra, the following always hold:

(5) $x \dot{-} x \approx 0$
(6) $x \dot{-} y \leq x$
(7) $x \dot{-} (x \dot{-} y) \leq y$
(8) $x \dot{-} (x \dot{-} (x \dot{-} y)) \approx x \dot{-} y$
(9) $(x \dot{-} y) \dot{-} z \approx (x \dot{-} z) \dot{-} y$.

The previous properties yield that $\dot{-}$ is isotone on the left and antitone on the right:

(10) If $x \leq y$ then $x \dot{-} z \leq y \dot{-} z$ and $z \dot{-} y \leq z \dot{-} x$.

The definition of BCK-algebras we adopted is slightly shorter than the one due to Iséki [IT78]. See [BR95] for a discussion on the equivalence between both.

For simplicity, we will abbreviate $(x \dot{-} y) \dot{-} z$ as $x \dot{-} y \dot{-} z$. For each natural number n, define $x \dot{-} ny$ inductively by $x \dot{-} 0y := x$; $x \dot{-} (n+1)y := x \dot{-} ny \dot{-} y$.

Observe that if $\mathbf{A}$ is a totally ordered BCK-algebra it follows from (2) and (3) that for all $a, b \in A$ $a \dot{-} b \dot{-} (b \dot{-} a) = a \dot{-} b$; a simple argument by induction on n shows that $\mathbf{A}$ also satisfies the identities:

(L_n) $x \dot{-} y \dot{-} n(y \dot{-} x) \approx x \dot{-} y$ for every natural number n.

A BCK-algebra, however, need not be totally ordered to satisfy (L_n). In fact it suffices to consider BCK-algebras in the relative subvariety defined by the identity

(11) $z \dot{-} x \dot{-} (y \dot{-} x) \approx z \dot{-} y \dot{-} (x \dot{-} y)$.

Lemma 2.1. *Let $\mathbf{A} = \langle A; \dot{-}, 0\rangle$ be a* BCK-*algebra satisfying* (11). *Then for every natural number n,* (L_n) *also holds in $\mathbf{A}$.*

Proof. We start by showing that (L) follows from (11). Our proof is based on an argument due to Bosbach [Bos69]. Let $a, b \in A$ and $c = a \dot{-} (b \dot{-} a)$. Then $c \leq a$, by (6), and $a \dot{-} c \leq b \dot{-} a \leq b \dot{-} c$, by (7) and (10). Thus, $a \dot{-} c \dot{-} (b \dot{-} c) = 0$ and, by (11), $a \dot{-} b \dot{-} (c \dot{-} b) = 0$, which implies that $a \dot{-} b \leq c \dot{-} b = a \dot{-} (b \dot{-} a) \dot{-} b$. Since, by (9), $a \dot{-} (b \dot{-} a) \dot{-} b = a \dot{-} b \dot{-} (b \dot{-} a)$, we conclude that $a \dot{-} b \leq a \dot{-} b \dot{-} (b \dot{-} a)$. On the other hand, $a \dot{-} b \dot{-} (b \dot{-} a) \leq a \dot{-} b$ by

(6) and therefore $a \dot{-} b = a \dot{-} b \dot{-} (b \dot{-} a)$ for all $a, b \in A$. So (L) holds, as claimed. An easy argument by induction on n now completes the proof of (L_n). □

In short, in any BCK-algebra, (11) implies (L). The converse need not hold, as the following example illustrates:

Example 2.2. Consider the 4-element set $A = \{0, a, b, c\}$, totally ordered by $0 < a < b < c$ and define a binary operation $\dot{-}$ on A by $x \dot{-} 0 = x$ for all $x \in A$, $x \dot{-} y = 0$ for all $x, y \in A$ such that $x \leq y$ and $b \dot{-} a = c \dot{-} a = c \dot{-} b = a$. It is easy to verify that $\mathbf{A} = \langle A; \dot{-}, 0\rangle$ is a BCK-algebra which satisfies (L). However, $c \dot{-} a \dot{-} (b \dot{-} a) = a \dot{-} a = 0 \neq a \dot{-} 0 = c \dot{-} b \dot{-} (a \dot{-} b)$ and therefore (11) fails in $\mathbf{A}$.

In [Cor80b], Cornish considered the variety of BCK-algebras defined by the identity

(J) $x \dot{-} (x \dot{-} (y \dot{-} (y \dot{-} x))) \approx y \dot{-} (y \dot{-} (x \dot{-} (x \dot{-} y)))$.

It is clear that the quasi-identity (4) follows from (J) and hence the class of BCK-algebras satisfying (J) is a variety. It is much harder to show that (J) follows from the quasi-equational basis for the class of all BCK-algebras satisfying (11), henceforth called *HBCK-algebras*.

Theorem 2.3. [Fer92, BF93, Kow94] *The class of all HBCK-algebras is a variety with equational basis* (1)–(3), (11) *and* (J).

An algebraic proof of this theorem was first obtained by the author in her Ph.D. Thesis [Fer92, BF93]. More recently, Kowalski [Kow94] obtained a syntactic derivation of (11) from the quasi-equational basis for the variety $\mathcal{HBCK}$ of all HBCK-algebras.

Pocrims and hoops. A structure $\mathbf{A} = \langle A; +, 0, \leq\rangle$ is a *partially ordered commutative monoid* if $\langle A; +, 0\rangle$ is a commutative monoid and $\leq$ is a partial order on A compatible with $+$, i.e., such that for all $x, y, z \in A$, if $x \leq y$, then $x + z \leq y + z$. $\mathbf{A}$ is *(dually) integral* if 0 is the least element of A. $\mathbf{A}$ is *(dually) residuated* if for all $x, y \in A$ there is a least element z such that $x \leq y + z$. Such z is called *the residual of* y *relative to* x and is denoted by $x \dot{-} y$. A partially ordered commutative residuated integral monoid $\langle A; +, 0, \leq\rangle$ can be treated as an algebra $\langle A; +, \dot{-}, 0\rangle$, since the partial order can be retrieved via $x \leq y$ iff $x \dot{-} y = 0$. Such algebras will be referred to by the acronym *pocrim*.

The $\{\dot{-}, 0\}$-subreducts of pocrims satisfy all of (1)–(4); hence they are BCK-algebras. The converse also holds, as was independently established by Pałasiński [Pal82], Ono and Komori [OK85] and Fleischer [Fle88].

Theorem 2.4. [Pal82, OK85, Fle88] *Let* $\mathbf{A} = \langle A; \dot{-}, 0\rangle$ *be an algebra of type* $(2, 0)$. $\mathbf{A}$ *is a BCK-algebra if and only if it is a* $\{\dot{-}, 0\}$*-subreduct of some pocrim.*

Pocrims can be traced back to research undertaken in the first half of this century on residuation in lattices of ideals of commutative rings with identity. In fact, if $\mathbf{R}$ is a commutative ring with identity 1 and $\mathrm{Id}(\mathbf{R})$ is the monoid of ideals of $\mathbf{R}$, with the usual ideal multiplication, ordered by *reversed* set inclusion, then for any two ideals I, J of $\mathbf{R}$ the residual of I relative to J exists and is given by $J \dot{-} I = \{x \in R : xI \subseteq J\}$. Hence, $\mathrm{Id}(\mathbf{R})$ is (the universe of) a pocrim.

The class $\mathcal{M}$ of all pocrims is a proper quasivariety [Hig84]. This class has been thoroughly investigated by Blok and Raftery [BR97].

Büchi and Owens [BO] introduced a special class of pocrims which they called hoops. A thorough study of hoops may be found in [BF00].

A partially ordered commutative monoid $\mathbf{A} = \langle A; +, 0, \leq\rangle$ is called *naturally ordered* if for all $x, y \in A$,

$$x \leq y \textit{ iff } (\exists z \in A)\ (y = x + z).$$

An algebra $\mathbf{A} = \langle A; +, \dot{-}, 0\rangle$ is called a *hoop* if it is a naturally ordered pocrim. We denote the class of hoops by $\mathcal{HO}$.

A word of caution is in order. 'Hoops' in this paper are the same as 'dual hoops' and are termwise equivalent to 'hoops' as considered in [BF00]. It should be noted that the only significant alteration is the reversal of the order relation considered. The author has adopted the new designation based on the traditional order relation for BCK-algebras and in accordance with the convention adopted by Blok and Raftery [BR97].

Given a hoop, it is easy to observe that $x \leq y$ if and only if $y = x + (y \dot{-} x)$. Moreover, $x + (y \dot{-} x) = y + (x \dot{-} y)$ holds for all x, y. As a consequence, every hoop is a join-semilattice with respect to its natural order, where the join operation is given by $x \vee y := x + (y \dot{-} x)$, see [BP94]. The underlying $\vee$-semilattice of any hoop is distributive, in the sense of Grätzer [Gra87], i.e., whenever $a \leq b \vee c$, there exist $b' \leq b$, $c' \leq c$ such that $a = b' \vee c'$—see [Bos69, BO].

Conversely any pocrim satisfying the equation $x + (y \dot{-} x) = y + (x \dot{-} y)$ is naturally ordered and hence a hoop. In fact, if $x \leq y$ then $y = y + 0 = y + (x \dot{-} y) = x + (y \dot{-} x)$; and if, for some z, $y = x + z$, then $x = x + 0 \leq x + z = y$. Hence we may say that the class of hoops consists of those pocrims satisfying

(H) $x + (y \dot{-} x) \approx y + (x \dot{-} y)$.

More precisely, it is known that an algebra $\mathbf{A} = \langle A; +, \dot{-}, 0\rangle$ is a hoop if and only if $\langle A; +, 0\rangle$ is a commutative monoid that satisfies the following identities:

(12) $(x \dot{-} y) \dot{-} z \approx x \dot{-} (y + z)$,
(5) $x \dot{-} x \approx 0$,
(H) $x + (y \dot{-} x) \approx y + (x \dot{-} y)$; see [BF00, Bos69].

Hence, the class $\mathcal{HO}$ of all hoops is a variety.

Among the easiest yet most relevant examples of hoops one has

Example 2.5. Let $\mathbf{G} = \langle G; +, -, 0, \vee, \wedge\rangle$ be a lattice-ordered Abelian group, or Abelian ℓ-group, for short, and $P(\mathbf{G})$ its positive cone, i.e, $P(\mathbf{G}) = \{x \in G : x \geq 0\}$. On $P(\mathbf{G})$ define the operation

$$x \dot{-} y = (x - y) \vee 0.$$

Then $\mathbf{P}(\mathbf{G}) = \langle P(\mathbf{G}); +, \dot{-}, 0\rangle$ is a hoop. If we consider the ℓ-group of the integers, $\mathbf{Z} = \langle \mathbb{Z}; +, -, 0, \vee, \wedge\rangle$ and its positive cone, the natural numbers, we obtain the infinite hoop $\mathbf{C}_\infty = \langle \{0, 1, 2, \dots\}; +, \dot{-}, 0\rangle$.

Example 2.6. Given an Abelian ℓ-group $\mathbf{G} = \langle G; +, -, 0, \vee, \wedge\rangle$ and an arbitrary element $u \in G$, $u > 0$, define on the set $G[u] = \{x \in G : 0 \leq x \leq u\}$ the following operations:

$$a +_u b = (a + b) \wedge u$$

$$a \dot{-} b = (a - b) \vee 0.$$

Then $\langle G[u]; +_u, \dot{-}, 0\rangle$ is a hoop.

In particular, as in Example 2.5, if we consider the Abelian ℓ-group of the integers $\mathbf{Z} = \langle \mathbb{Z}; +, -, 0, \vee, \wedge\rangle$ and an arbitrary positive integer m, we obtain the finite hoop $\mathbf{C}_m = \langle \{0, 1, 2, \dots, m\}; +_m, \dot{-}, 0\rangle$.

3. Commutative BCK-algebras

The order relation defined on BCK-algebras has no interesting properties in general. In fact, every poset with a least element 0 can be given the structure of a (Hilbert or positive implicative) BCK-algebra, by defining $a \dot{-} b = 0$ if $a \leq b$ and $a \dot{-} b = a$ if $a \not\leq b$—see [Cor82, Die66]. However, given a BCK-algebra, the term operation $x \dot{-} (x \dot{-} y)$ defines a lower bound of both x and y (cf. (6) and (7)).

It was shown by S. Tanaka [IT78, Tan75] that, given any BCK-algebra $\mathbf{A}$, the binary term $x \dot{-} (x \dot{-} y)$ defines a meet-semilattice operation on its universe if and only if it is a commutative term operation, i.e., if $\mathbf{A}$ satisfies the identity

$$x \dot{-} (x \dot{-} y) \approx y \dot{-} (y \dot{-} x). \tag{T}$$

BCK-algebras satisfying (T) are therefore called *commutative* BCK-*algebras* [Yut77]. It is well-known that the class of commutative BCK-algebras is a variety; indeed quasi-identity (4) follows from (T).

We say that a commutative BCK-algebra $\mathbf{A}$ is *upwards directed* if every pair of elements $a, b \in A$ has an upper bound; $\mathbf{A}$ is called *bounded* if it has a greatest element.

Lemma 3.1. *Every upwards directed commutative BCK-algebra can be embedded into a bounded commutative BCK-algebra.*

Proof. Let $\mathbf{A}$ be an upwards directed commutative BCK-algebra. For each $z \in A$, let $A[z] = \{x \in A : 0 \leq x \leq z\}$ be the order ideal below z. $A[z]$ is a subuniverse of $\mathbf{A}$ with largest element z and therefore $\mathbf{A}[z] = \langle A[z]; \dot{-}, 0\rangle$ is a bounded commutative BCK-algebra. We claim that $\mathbf{A} \in \mathsf{ISP_u}(\mathbf{A}[z] : z \in A)$.

In order to prove our claim, consider the family of order filters of A given by $F_a = \{z \in A : a \leq z\}$, $a \in A$. Each F_a is non-empty. Since $\mathbf{A}$ is upwards directed, for every $a, b \in A$ there exists $c \in A$, such that $F_c \subseteq F_a \cap F_b$. Thus the family $\{F_a : a \in A\}$ has the finite intersection property and therefore there exists an ultrafilter U over A containing all the members of the family. Fix such an ultrafilter and define the map $\psi : \mathbf{A} \longrightarrow \prod_{z\in A} \mathbf{A}[z]/U$ by $\psi(A) = (a \wedge z)_{z\in A}/U$.

Let $a, b \in A$, $a \neq b$. Without loss of generality assume $a \not\leq b$. Then $F_a \subseteq \{z \in A : a \wedge z \neq b \wedge z\}$. Since $F_a \in U$ then $\{z \in A : a \wedge z \neq b \wedge z\} \in U$, as well and therefore $\psi(A) \neq \psi(b)$. Thus ψ is one-one.

For arbitrary $a, b \in A$ we have $\psi(A) \dot{-} \psi(b) = \psi(a \dot{-} b)$ if and only if $F = \{z \in A : z \wedge (a \dot{-} b) = (z \wedge a) \dot{-} (z \wedge b)\} \in U$. Choose $c \in A$ such that $a, b \leq c$. Then for $z \in F_c$, $a, b \leq c \leq z$ implies that $a \dot{-} b \leq z$ (by (6)) and therefore $z \wedge (a \dot{-} b) = a \dot{-} b = (z \wedge a) \dot{-} (z \wedge b)$. Hence, $F_c \subseteq F$ and therefore $F \in U$. This concludes the proof that ψ is an embedding of $\mathbf{A}$ into the bounded commutative BCK-algebra $\prod_{z\in A} \mathbf{A}[z]/U$. □

Recall from Example 2.2, that in a BCK-algebra (L) does not imply (11). However, when we restrict ouselves to commutative BCK-algebras, (L) and (11) are equivalent. This result was established by Cornish, Sturm and Traczyk by means of a rather technical proof [CST84, Lemma 2.4].

A commutative BCK-algebra is called a *Łukasiewicz algebra* (Ł-algebra, for short) if it satisfies (L) or, equivalently if it satisfies (11). Łukasiewicz algebras form a subvariety of the variety of commutative BCK-algebras. They have been studied by Komori, under the name *C algebras* [Kom78]. Because *C* algebras are the algebraic models of the implicational fragment of the Łukasiewicz propositional calculus, we follow Pałasiński and Wroński [PW86] in naming these algebras after Łukasiewicz.

All totally ordered commutative BCK-algebras are necessarily L-algebras. Traczyk [Tra79] has shown that bounded commutative BCK-algebras are subdirect products of totally ordered bounded commutative BCK-algebras; therefore, bounded commutative BCK-algebras are L-algebras. Hence, by Lemma 3.1, every upwards directed commutative BCK-algebra is an L-algebra. This seems to have been known to specialists in BCK-algebras. However, we have not seen its proof in print.

A hoop is called a *Wajsberg hoop* if it satisfies (T). Wajsberg hoops are crucial to the study of hoops in general, as seen in [BF00]. Clearly, $\{\dot{-}, 0\}$-subreducts of Wajsberg hoops are L-algebras. Moreover, every bounded L-algebra, with largest element 1, is necessarily a ($\{\dot{-}, 0\}$-reduct of a) Wajsberg hoop. To the author's knowledge, this result was first announced without proof by Bosbach [Bos74] and also referred to by Romanowska and Traczyk [RT80]. We include here a proof that may be inferred from Bosbach's work on cone algebras [Bos82].

Lemma 3.2. [Bos74, RT80] *If* $\mathbf{A} = \langle A; \dot{-}, 0\rangle$ *is an* L-*algebra with largest element* 1*, then it is the* $\{\dot{-}, 0\}$*-reduct of a (bounded) Wajsberg hoop, where the monoid operation* $+$ *is defined by* $x + y := 1 \dot{-} (1 \dot{-} x \dot{-} y)$ *for all* $x, y \in A$.

Proof. We will show that $\langle A; +, 0\rangle$ is a commutative monoid which, when enriched with $\dot{-}$, satisfies (12), (5) and (H). Hence $\langle A; +, \dot{-}, 0\rangle$ is a hoop.

Observe that for arbitrary $a \in A$, $a = a \wedge 1 = 1 \dot{-} (1 \dot{-} a)$. Now we proceed to show that $+$ is commutative. Given $a, b \in A$, $a + b = 1 \dot{-} (1 \dot{-} a \dot{-} b) = 1 \dot{-} (1 \dot{-} b \dot{-} a) = b + a$. Also, $0 + a = 1 \dot{-} (1 \dot{-} 0 \dot{-} a) = 1 \dot{-} (1 \dot{-} a) = a \wedge 1 = a$ by (2) and (T). Hence we have proved that $\langle A, +, 0\rangle$ is a commutative groupoid with identity 0.

Before showing that $+$ is associative, we will prove that (H) and (12) hold in $\mathbf{A}$. As for (H), let $a, b \in A$. Then

$$\begin{aligned} a + (b \dot{-} a) &= 1 \dot{-} (1 \dot{-} a \dot{-} (b \dot{-} a)) \\ &= 1 \dot{-} (1 \dot{-} b \dot{-} (a \dot{-} b)) \quad \text{by (11)} \\ &= b + (a \dot{-} b). \end{aligned}$$

Next we will check that (12) also holds. Here we will use repeatedly our previous remark that for arbitrary $a \in A$, $a = a \wedge 1 = 1 \dot{-} (1 \dot{-} a)$. First note that $1 \dot{-} (a + b) = 1 \dot{-} (1 \dot{-} (1 \dot{-} a \dot{-} b)) = 1 \wedge (1 \dot{-} a \dot{-} b) = 1 \dot{-} a \dot{-} b$. Hence, for all $a, b, c \in A$,

$$\begin{aligned} a \dot{-} (b + c) &= 1 \dot{-} (1 \dot{-} a) \dot{-} (b + c) \\ &= 1 \dot{-} (b + c) \dot{-} (1 \dot{-} a) \quad \text{by (9)} \\ &= 1 \dot{-} b \dot{-} c \dot{-} (1 \dot{-} a) \\ &= 1 \dot{-} (1 \dot{-} a) \dot{-} b \dot{-} c \quad \text{by (9) twice} \\ &= a \dot{-} b \dot{-} c. \end{aligned}$$

Associativity of $+$ will now be derived as a consequence of (12). First observe that for all $a, b, c, d \in A$

$$d \dot{-} (a + (b + c)) = d \dot{-} a \dot{-} (b + c) = d \dot{-} a \dot{-} b \dot{-} c = d \dot{-} (a + b) \dot{-} c = d \dot{-} ((a + b) + c),$$

by several applications of (12). In particular, $((a + b) + c) \dot{-} (a + (b + c)) = ((a + b) + c) \dot{-} ((a+b)+c) = 0$. Thus $(a+b)+c \leq a+(b+c)$. Similarly, one obtains $a+(b+c) \leq (a+b)+c$.

Thus the operation $+$ endows A with the structure of a commutative monoid satisfying (12), (5) and (H). Therefore $\langle A; +, \dot{-}, 0\rangle$ is a hoop. Since it satisfies (T) and has a top element, $\langle A; +, \dot{-}, 0\rangle$ is a (bounded) Wajsberg hoop, as claimed. □

Every totally ordered L-algebra is trivially upwards directed and therefore may be embedded into a bounded L-algebra, by Lemma 3.1. Since every L-algebra is a subdirect product of totally ordered L-algebras, we obtain

Theorem 3.3. *Every L-algebra is a $\{\dot{-}, 0\}$-subreduct of a (bounded) Wajsberg hoop.*

We conclude this section with another consequence of Lemma 3.1.

Proposition 3.4. *If* $\mathbf{A}$ *is an upwards directed* L*-algebra then the quasivariety* $\mathcal{Q}(\mathbf{A}) = \mathsf{ISPP_u}(\mathbf{A})$ *is generated by its bounded members.*

Proof. As observed in the proof of Lemma 3.1, for each $z \in A$, $\mathbf{A}[z]$ is a (bounded) subalgebra of $\mathbf{A}$. Hence, $\mathbf{A}[z] \in \mathcal{Q}(\mathbf{A})$. Since $\mathbf{A} \in \mathsf{ISP_u}(\mathbf{A}[z] : z \in A)$, then $\mathcal{Q}(\mathbf{A}) = \mathsf{ISPP_u}(\mathbf{A}[z] : z \in A)$. □

4. Subdirectly irreducible HBCK-algebras

Let $\mathcal{Q}$ be a quasivariety of algebras of type τ and $\mathbf{A}$ an algebra of type τ. A *$\mathcal{Q}$-congruence on* $\mathbf{A}$ is any congruence ρ on $\mathbf{A}$ such that $\mathbf{A}/\rho \in \mathcal{Q}$. $\mathcal{Q}$-congruences on $\mathbf{A}$ are closed under arbitrary intersection and the set $\mathrm{Con}_{\mathcal{Q}}\mathbf{A}$, of all $\mathcal{Q}$-congruences on $\mathbf{A}$, is an algebraic lattice.

Given a quasivariety of algebras $\mathcal{Q}$, we say that $\mathbf{A} \in \mathcal{Q}$ is *subdirectly irreducible relative to* $\mathcal{Q}$ if the lattice $\mathbf{Con}_{\mathcal{Q}}\mathbf{A}$ has a unique minimal nontrivial congruence. We say that $\mathbf{A} \in \mathcal{Q}$ is *simple relative to* $\mathcal{Q}$ if $\mathbf{Con}_{\mathcal{Q}}\mathbf{A}$ is the two-element lattice. If no confusion arises, $\mathbf{A} \in \mathcal{Q}$ is simply called *relatively subdirectly irreducible* for short (respectively, *relatively simple*), whenever $\mathbf{Con}_{\mathcal{Q}}\mathbf{A}$ has a unique minimal nontrivial congruence (respectively, $\mathbf{Con}_{\mathcal{Q}}\mathbf{A}$ is the two element lattice).

If a class $\mathcal{K}$ of algebras is a relative subvariety of a quasivariety $\mathcal{Q}$ (i.e., $\mathcal{K} = \mathcal{Q} \cap \mathcal{V}$ for some variety $\mathcal{V}$) then it is easy to see that for $\mathbf{A} \in \mathcal{K}$, $\mathbf{Con}_{\mathcal{K}}\mathbf{A} = \mathbf{Con}_{\mathcal{Q}}\mathbf{A}$ and therefore $\mathbf{A} \in \mathcal{K}$ is subdirectly irreducible (respectively simple) relative to $\mathcal{K}$ if and only if $\mathbf{A}$ is subdirectly irreducible (respectively simple) relative to $\mathcal{Q}$. Moreover, if $\mathcal{K}$ itself is a variety and $\mathbf{A} \in \mathcal{K}$ then $\mathbf{Con}\ \mathbf{A} = \mathbf{Con}_{\mathcal{Q}}\mathbf{A}$ and $\mathbf{A}$ is subdirectly irreducible (respectively simple) relative to $\mathcal{Q}$ if and only if $\mathbf{A}$ is subdirectly irreducible (respectively simple) in the usual sense.

The discussion above applies to the quasivariety $\mathcal{BCK}$ of BCK-algebras and its relative subvarieties. In particular, in view of Theorem 2.3, for any HBCK-algebra $\mathbf{A}$, the lattice of congruences of $\mathbf{A}$ relative to $\mathcal{BCK}$ coincides with the full lattice of congruences of $\mathbf{A}$.

As in many algebraic structures, in particular those arising from Logic, (relative) congruences on BCK-algebras are in close association with certain distinguished subsets, namely ideals. To be more precise, given a BCK-algebra $\mathbf{A}$, and a subset $I \subseteq A$, we say that I is an *ideal* of $\mathbf{A}$ if $0 \in I$ and whenever $a \in I$ and $b \dot{-} a \in I$ then $b \in I$.

Every ideal of a BCK-algebra is an order ideal, since if $a \in I$ and $b \leq a$ then $b \dot{-} a = 0 \in I$ and so $b \in I$. Moreover, every ideal is also a subuniverse, since $b \dot{-} a \leq b$ in any BCK-algebra.

If $\mathbf{A}$ is a BCK-algebra and $X \subseteq A$, then the ideal generated by X is the set $I(X) = \{b \in A : \ \exists n \in \mathbb{N}\ \exists a_1, a_2, \dots, a_n \in X\ b \dot{-} a_1 \dot{-} a_2 \dot{-} \dots \dot{-} a_n = 0\}$. If, in particular, $X = \{a\}$ then $I(a) = \{b \in A : b \dot{-} na = 0 \text{ for some } n \in \mathbb{N}\}$. Ideals of a BCK-algebra form an algebraic lattice denoted by $\mathbf{Id}(\mathbf{A})$, where $I \wedge J = I \cap J$ and $I \vee J = \{a \in A : a \dot{-} g \dot{-} f = 0, \text{for some } f \in I, g \in J\}$.

If $\mathbf{A}$ is a BCK-algebra and ρ is a congruence on $\mathbf{A}$, the coset $0/\rho = \{a \in A : (a, 0) \in \rho\}$ is an ideal, frequently called the *kernel of* ρ. Conversely, whenever I is an ideal of $\mathbf{A}$ the binary relation Φ_I defined on $\mathbf{A}$ by $(a, b) \in \Phi_I$ if and only if $a \dot{-} b, b \dot{-} a \in I$ is not only a congruence on $\mathbf{A}$ but also $\mathbf{A}/\Phi_I$ is a BCK-algebra and therefore Φ_I is a congruence relative to $\mathcal{BCK}$. Indeed Φ_I is the largest congruence on $\mathbf{A}$ with I as its kernel.

The correspondence $\mathbf{Con}_{\mathcal{BCK}}\mathbf{A} \longrightarrow \mathbf{Id}(\mathbf{A})$ defined by $\rho \mapsto 0/\rho$ is an order isomorphism, with inverse map given by $I \mapsto \Phi_I$ (see [BR95]).

Since the relative subvariety $\mathcal{HBCK}$ is itself a variety (recall Theorem 2.3) when we consider an HBCK-algebra $\mathbf{A}$ we can say that the lattices $\mathbf{Id}(\mathbf{A})$ and $\mathbf{Con}\,\mathbf{A}$ are isomorphic. This observation, together with a simple argument on ideals, implies that the variety $\mathcal{HBCK}$ has the congruence extension property, (CEP, for short)—see, for instance [BR95], Proposition 1 (ii) for details.

In the remainder of this section, we will study in detail the structure of subdirectly irreducible HBCK-algebras. We start by describing simple BCK and HBCK-algebras.

The only congruence on any given BCK-algebra $\mathbf{A}$ having A as its kernel is the universal congruence A^2 [RRS91, Remarks 2.7a]. This implies that if $\mathbf{A}$ has only two ideals it is necessarily simple and conversely.

Lemma 4.1. [1] *Let* $\mathbf{A}$ *be an BCK-algebra.*

(i) $\mathbf{A}$ *is simple if and only if it satisfies*

(13) *For all* a, b, *if* $a \neq 0$ *then there exists* $n \in \mathbb{N}$ *such that* $b \dot{-} na = 0$.

(ii) *If* $\mathbf{A}$ *is simple, it satisfies*

(14) *For all* a, b $a \dot{-} b = a$ *implies* $a = 0$ *or* $b = 0$.

Proof. (i) Recall that, for every $a \in A$, $I(a) = \{b \in A : b \dot{-} na = 0 \text{ for some } n \in \mathbb{N}\}$; hence (13) states that $I(a) = A$ for every nonzero $a \in A$. Therefore (13) holds if and only if $\mathbf{A}$ has exactly two ideals, $\{0\}$ and A. By the remark above, this is equivalent to saying that $\mathbf{A}$ is simple.

(ii) Assume $\mathbf{A}$ is simple and let $a, b \in A$ be such that $a \dot{-} b = a$. Then for all $n \in \mathbb{N}$, $a \dot{-} nb = a$. Since $\mathbf{A}$ satisfies (13), if $b \neq 0$ then $a \dot{-} n_0 b = 0$, for some $n_0 \in \mathbb{N}$, and so $a = 0$. □

Observe that any totally ordered L-algebra satisfies (14). Conversely, we have:

Proposition 4.2. *Let* $\mathbf{A}$ *be an HBCK-algebra satisfying*

(14) *For all* a, b $a \dot{-} b = a$ *implies* $a = 0$ *or* $b = 0$.

Then $\mathbf{A}$ *is a totally ordered L-algebra.*

Proof. The fact that $\mathbf{A}$ is totally ordered is a consequence of (L) (see [Bos69]). Indeed, for any $a, b \in A$ $a \dot{-} b \dot{-} (b \dot{-} a) = a \dot{-} b$. Hence, by (14), $a \dot{-} b = 0$ or $b \dot{-} a = 0$ and therefore, $a \leq b$ or $b \leq a$.

To see that $\mathbf{A}$ satisfies (T), let $a, b \in A$ and assume without loss of generality that $b < a$; since $b \dot{-} (b \dot{-} a) = b \dot{-} 0 = b$, it suffices to show that $b = a \dot{-} (a \dot{-} b)$.

$$
\begin{aligned}
&a \dot{-} b \dot{-} (b \dot{-} (a \dot{-} (a \dot{-} b))) = \\
&\quad = \quad a \dot{-} (a \dot{-} (a \dot{-} b)) \dot{-} (b \dot{-} (a \dot{-} (a \dot{-} b))) \text{ by (8)} \\
&\quad = \quad a \dot{-} b \dot{-} (a \dot{-} (a \dot{-} b) \dot{-} b) \text{ by (11)} \\
&\quad = \quad a \dot{-} b \dot{-} 0 \text{ by (7)} \\
&\quad = \quad a \dot{-} b \text{ by (2).}
\end{aligned}
$$

Hence, by (14), either $a \dot{-} b = 0$ or $b \dot{-} (a \dot{-} (a \dot{-} b)) = 0$. Since the assumption was that $b < a$, we have $a \dot{-} b \neq 0$. Therefore $b \dot{-} (a \dot{-} (a \dot{-} b)) = 0$ and so $b = a \dot{-} (a \dot{-} b)$, as claimed. □

[1] The author is indebted to a Referee, who pointed out that Lemma 4.1(i) holds for arbitrary BCK-algebras and not only for HBCK-algebras, as originally stated.

Thus we conclude

Corollary 4.3. *Every simple HBCK-algebra is a totally ordered L-algebra.*

Example 4.4. The finite chains $\mathbf{C}_m{}^{\{\dot{-},0\}}, m \in \mathbb{N}$, as well as $\mathbf{C}_\infty{}^{\{\dot{-},0\}}$ are simple L-algebras.

We now turn our attention to subdirectly irreducible HBCK-algebras. Recall that an algebra $\mathbf{A}$ is subdirectly irreducible if and only if $\mathbf{A}$ has a congruence $\mu \neq \Delta$ that is contained in every nonzero congruence of $\mathbf{A}$. We call μ the *monolith* of $\mathbf{A}$. Observe that if $\mathbf{A}$ is a subdirectly irreducible HBCK-algebra, $U = 0/\mu$ is the unique minimal ideal of $\mathbf{A}$ distinct from $\{0\}$. U is contained in every ideal of $\mathbf{A}$ not equal to $\{0\}$. In the sequel, we will always denote this ideal by U.

Proposition 4.5. *Let $\mathbf{A}$ be a subdirectly irreducible HBCK-algebra with monolith μ. Then U is a subuniverse of $\mathbf{A}$ and $\langle U; \dot{-}, 0\rangle$ is a simple totally ordered L-algebra.*

Proof. We observed earlier that every ideal is a subuniverse. By CEP and Corollary 4.3, $\langle U; \dot{-}, 0\rangle$ is necessarily a simple, totally ordered L-algebra. □

Given a subdirectly irreducible HBCK-algebra $\mathbf{A}$ with least nonzero ideal U, we say that $a \in A$ is *fixed* if for all $u \in U, a \dot{-} u = a$. The set of fixed elements of $\mathbf{A}$ is denoted by F. The set $S = (A \setminus F) \cup \{0\}$ is called the *support of* U.

Proposition 4.6. *Let $\mathbf{A}$ be a subdirectly irreducible HBCK-algebra with monolith μ , $U = 0/\mu$ and set of fixed elements F. Then*

(i) *For all $a \in A$ if $a \neq 0$ then $u \neq 0$ and $u \leq a$ for some $u \in U$,*
(ii) *$U \cap F = \{0\}$,*
(iii) *An element $a \in A$ is fixed if and only if $a \dot{-} x = a$ for some $x \in U \setminus \{0\}$.*

Proof. (i) Let $a \in A,\ a \neq 0$ and let $I(A) = \{x \in A:\ x \dot{-} na = 0 \text{ for some } n \in \mathbb{N}\}$ be the ideal of $\mathbf{A}$ generated by a. Since $I(A) \neq \{0\}$, it follows that $U \subseteq I(A)$. Let $x \in U \setminus \{0\}$; then there exists $m \in \mathbb{N}$ such that $x \dot{-} ma = 0$. Choose m minimal with respect to the condition $x \dot{-} ma = 0$ and let $u = x \dot{-} (m-1)a$. Then $u \leq x$ and $u \neq 0$ and so $u \in U \setminus \{0\}$. Moreover, $u \dot{-} a = x \dot{-} (m-1)a \dot{-} a = x \dot{-} ma = 0$ and therefore $u \leq a$.

(ii) Let $a \in U \cap F$ and $u \in U,\ u \neq 0$. Since a is fixed, $a \dot{-} u = a$. By Proposition 4.5, U is the universe of a simple HBCK-algebra, and hence U satisfies (14) by Lemma 4.1(ii); since $a, u \in U$ we have $a = 0$, as claimed.

(iii) Assume $a \neq 0$ and $a \dot{-} x = a$ for some $x \in U,\ x \neq 0$. Let $u \in U$ be arbitrary; in order to show that $a \in F$, we verify that $a \dot{-} u = a$. Since $a \dot{-} x = a$, we have $a \dot{-} nx = a$ for every $n \in \mathbb{N}$. On the other hand, $a \dot{-} (a \dot{-} u) \leq u$ and so $a \dot{-} (a \dot{-} u) \in U$. Since U satisfies (13) and $x \neq 0$, $a \dot{-} (a \dot{-} u) \dot{-} n_0 x = 0$ for some $n_0 \in \mathbb{N}$. Hence, $0 = a \dot{-} (a \dot{-} u) \dot{-} n_0 x = (a \dot{-} n_0 x) \dot{-} (a \dot{-} u) = a \dot{-} (a \dot{-} u)$; thus $a \dot{-} u = a$.

The converse is immediate, since $U \neq \{0\}$. □

Lemma 4.7. *Let $\mathbf{A}$ be a subdirectly irreducible HBCK-algebra, U its least ideal distinct from $\{0\}$, F its set of fixed elements and S the support of U. Let $a \in F,\ a \neq 0$. Then*

(i) *for all $u \in U\ \ u \leq a$,*
(ii) *for all $b \in A$ if $a \leq b$ then $b \in F$,*
(iii) *for all $b \in A\ \ a \dot{-} b,\ b \dot{-} a \in F$,*
(iv) *for all $b \in S\ \ b \leq a,\ a \dot{-} b = a$.*

Proof. (i) Since $a \neq 0$, and U is the least ideal distinct from $\{0\}$, $U \subseteq I(A)$. Thus, given $u \in U$, $u \dot{-} ma = 0$ for some $m \in \mathbb{N}$. On the other hand,

$$\begin{array}{rcll} u \dot{-} a & = & u \dot{-} a \dot{-} (a \dot{-} u) & \text{by (L)} \\ & = & u \dot{-} a \dot{-} a & \text{since } a \text{ is fixed} \\ & = & u \dot{-} 2a & \end{array}$$

By induction, one derives $u \dot{-} na = u \dot{-} a$, for every $n \in \mathbb{N}$. In particular, $0 = u \dot{-} ma = u \dot{-} a$ and so $u \leq a$.

(ii) Let $a \leq b$. Let $u \in U$ be arbitrary. Then $a = a \dot{-} u \leq b \dot{-} u$ and so $a \dot{-} (b \dot{-} u) = 0$. Now,

$$\begin{array}{rcll} b \dot{-} (b \dot{-} u) & = & b \dot{-} (b \dot{-} u) \dot{-} 0 & (2) \\ & = & b \dot{-} (b \dot{-} u) \dot{-} (a \dot{-} (b \dot{-} u)) & \\ & = & b \dot{-} a \dot{-} (b \dot{-} u \dot{-} a) & \text{by (11)} \\ & = & b \dot{-} a \dot{-} (b \dot{-} u \dot{-} (a \dot{-} u)) & \text{since } a \text{ is fixed} \\ & = & b \dot{-} a \dot{-} (b \dot{-} a \dot{-} (u \dot{-} a)) & \text{by (11)} \\ & = & b \dot{-} a \dot{-} (b \dot{-} a \dot{-} 0) & \text{since } u \leq a \\ & = & b \dot{-} a \dot{-} (b \dot{-} a) & \text{by (2)} \\ & = & 0 & \text{by (5).} \end{array}$$

So $b \leq b \dot{-} u$ and $b \dot{-} u \leq b$ (by (6)) and therefore $b \dot{-} u = b$, yielding $b \in F$.

(iii) Let $b \in A$. We want to show that both $a \dot{-} b$ and $b \dot{-} a$ are in F. Let $u \in U$ be arbitrary. With respect to $a \dot{-} b$, observe that $a \dot{-} b \dot{-} u = a \dot{-} u \dot{-} b = a \dot{-} b$ and therefore $a \dot{-} b \in F$.

As for $b \dot{-} a$, we have

$$\begin{array}{rcll} b \dot{-} a & = & b \dot{-} a \dot{-} 0 & \text{by (2)} \\ & = & b \dot{-} a \dot{-} (u \dot{-} a) & \text{by (i)} \\ & = & b \dot{-} u \dot{-} (a \dot{-} u) & \text{by (11)} \\ & = & b \dot{-} u \dot{-} a & \text{since } a \text{ is fixed} \\ & = & b \dot{-} a \dot{-} u & \text{by (9).} \end{array}$$

Hence, $b \dot{-} a \in F$.

(iv) Let $b \in S$. If $b = 0$, all statements are trivially true. Hence we may assume without loss of generality that $b \neq 0$, i.e., $b \in A \setminus F$. By (iii), $b \dot{-} a \in F$. Since $b \dot{-} a \leq b$ and $b \notin F$, it follows from (ii) that $b \dot{-} a = 0$. Thus $b \leq a$. On the other hand, to show $a \dot{-} b = a$ it suffices to show that $a \dot{-} (a \dot{-} b) = 0$. Let $u \in U$, $u \neq 0$. Then $a \dot{-} u = a$ and

$$a \dot{-} (a \dot{-} b) \dot{-} u = a \dot{-} u \dot{-} (a \dot{-} b) = a \dot{-} (a \dot{-} b).$$

So $a \dot{-} (a \dot{-} b)$ is fixed. By (ii), the fact that $a \dot{-} (a \dot{-} b) \leq b$ and $b \notin F$, one gets $a \dot{-} (a \dot{-} b) = 0$ and so $a \dot{-} b = a$. □

Ordinal sums. In order to present the structure of subdirectly irreducible HBCK-algebras, we recall the construction of the *ordinal sum* of two BCK-algebras, $\mathbf{A}$ and $\mathbf{B}$, denoted $\mathbf{A} \oplus \mathbf{B}$. If, for simplicity, we assume $A \cap B \subseteq \{0^{\mathbf{A}}\}$, define $\mathbf{A} \oplus \mathbf{B}$ to be the algebra with universe $A \cup (B \setminus \{0\})$, and operations $0^{\mathbf{A} \oplus \mathbf{B}} = 0^{\mathbf{A}} = 0^{\mathbf{B}}$,

$$x \dot{-} y = \begin{cases} x \dot{-}^{\mathbf{A}} y & \text{for } x, y \in A, \\ x \dot{-}^{\mathbf{B}} y & \text{for } x, y \in B \setminus \{0\} \text{ and } x \not\leq y, \\ x & \text{for } x \in B \setminus \{0\}, y \in A, \\ 0 & \text{otherwise,} \end{cases}$$

Observe that this definition places all elements of A below all elements of $B \setminus \{0\}$.

If $A \cap B \not\subseteq \{0^{\mathbf{A}}\}$ replace $\mathbf{B}$ with an isomorphic copy $\mathbf{B}'$ such that $A \cap B' \subseteq \{0^{\mathbf{A}}\}$ and define their ordinal sum as above. This construction was introduced, in the study of BCK-algebras, by Iséki and Yutani [IY80]. For any BCK-algebras $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \oplus \mathbf{B}$ is a BCK-algebra and $\mathbf{A}$ and $\mathbf{B}$ are among its subalgebras; Cornish [Cor82] showed that if $\mathbf{A}$ and $\mathbf{B}$ are HBCK-algebras, so is $\mathbf{A} \oplus \mathbf{B}$.

Lemma 4.8. *Let $\mathbf{A} \oplus \mathbf{B}$ be the ordinal sum of two BCK-algebras and assume $|A| > 1$. If $\mathbf{A} \oplus \mathbf{B}$ is n-generated for some $n < \omega$ then $\mathbf{B}$ is k-generated for some $k < n$.*

Proof. We may assume that $A \cap B = \{0\}$. Since $\mathbf{B}$ is isomorphic to $\mathbf{A} \oplus \mathbf{B}/A$, it is therefore generated by the images (under the natural homomorphism) of the n generators of $\mathbf{A} \oplus \mathbf{B}$. Since $\mathbf{A}$ is non-trivial, it must contain (at least) one of the generators of $\mathbf{A} \oplus \mathbf{B}$; its image is 0, a redundant generator. Thus $\mathbf{B}$ is generated by (at most) $n-1$ generators. □

If $\mathbf{A}$ and $\mathbf{B}$ are HBCK-algebras and furthermore $\mathbf{A}$ is subdirectly irreducible then $\mathbf{A} \oplus \mathbf{B}$ is also subdirectly irreducible. We are now ready to describe subdirectly irreducible HBCK-algebras.

Theorem 4.9. *Let $\mathbf{A}$ be a subdirectly irreducible HBCK-algebra, U its least ideal distinct from $\{0\}$, F its set of fixed elements and S the support of U.*

(i) *F and S are subuniverses of $\mathbf{A}$.*
(ii) *$U \subseteq S$, and $\mathbf{S} = \langle S; \dot{-}, 0 \rangle$ is a subdirectly irreducible L-algebra. In particular, $\mathbf{S}$ is totally ordered.*
(iii) $\mathbf{A} = \mathbf{S} \oplus \mathbf{F}$.

Proof. (i) 0 is fixed, hence $0 \in F$. To see that F is closed under the binary operation, let $a, b \in F$. If $a = 0$ then $b \dot{-} a = b \dot{-} 0 = b \in F$. If $a \neq 0$ then $a \dot{-} b \in F$, by Lemma 4.7(iii). This concludes the proof that F is a subuniverse of $\mathbf{A}$.

To show that S is also a subuniverse of $\mathbf{A}$, let $a, b \in S$. Since $b \dot{-} a \leq b$, then $b \dot{-} a \in S$, by Lemma 4.7 (ii). Thus S is a subuniverse of $\mathbf{A}$ as claimed.

(ii) $U \subseteq S$ by Proposition 4.6 (ii) and the definition of S. Since U is the least ideal of $\mathbf{A}$ not equal to $\{0\}$ it follows from the congruence extension property that U is also the least ideal of $\mathbf{S}$ not equal to $\{0\}$. Hence $\mathbf{S}$ is subdirectly irreducible.

To prove that $\mathbf{S}$ is a totally ordered L-algebra, it suffices, by Proposition 4.2, to verify that it satisfies (14). Let $a, b \in S$ and assume that $a \dot{-} b = a$. If $b \neq 0$ then by Proposition 4.6 (i) there exists $u \in U$ such that $u \leq b$ and $u \neq 0$. Then $a = a \dot{-} b \leq a \dot{-} u \leq a$, so a is fixed, by Proposition 4.6(iii). Thus $a \in F \cap S$ and hence $a = 0$.

(iii) This follows immediately from (i) together with Lemma 4.7 (iv). □

A typical subdirectly irreducible HBCK-algebra may be depicted as in Figure 1 below.

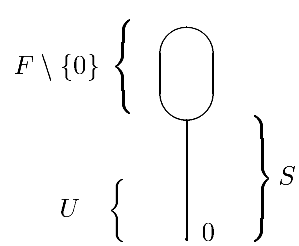

FIGURE 1. A subdirectly irreducible HBCK-algebra

Observe that under the conditions of Theorem 4.9 S is an ideal of $\mathbf{A}$ and $\mathbf{F} \simeq \mathbf{A}/S$.

Let $n < \omega$ and let $\mathcal{E}_n$ denote the class of all BCK-algebras which satisfy the identity

(ε_n) $x \dot{-} ny \approx x \dot{-} (n+1)y$.

For each $n < \omega$, $\mathcal{E}_n$ is a variety [Cor80b]. In particular, $\mathcal{E}_1$ is the well-known class of Hilbert algebras, first studied by Diego [Die66]. For these algebras, the description given in Theorem 4.9 is well-known; the subdirectly irreducible Hilbert algebras are precisely the algebras of the form $\mathbf{C}_1{}^{\{\dot{-},0\}} \oplus \mathbf{B}$, where $\mathbf{C}_1{}^{\{\dot{-},0\}}$ is the 2-element Hilbert algebra and $\mathbf{B}$ is any Hilbert algebra. More generally we have

Corollary 4.10. *If* $\mathbf{A}$ *is any HBCK-algebra in* $\mathcal{E}_n$*, then* $\mathbf{A}$ *is subdirectly irreducible if and only if* $\mathbf{A} = \mathbf{C}_m^{\{\dot{-},0\}} \oplus \mathbf{B}$ *for some* m*,* $1 \leq m \leq n$*, and some HBCK-algebra* $\mathbf{B}$ *in* $\mathcal{E}_n$*.*

Proof. One needs only to observe that if $\mathbf{A}$ is a subdirectly irreducible HBCK-algebra in $\mathcal{E}_n$, the support S of its minimal ideal U coincides with U. In fact, if $s \in S \setminus U$ and u is a nonzero element of U then $a := s \dot{-} nu$ is not in U (as U is an ideal) but $a \in S$. Since $a - u = a$ we have that $a \in F \cap S$, i.e. $a = 0$, a contradiction. Hence S is the universe of a subdirectly irreducible, totally ordered L-algebra $\mathbf{S}$ in $\mathcal{E}_n$, by Theorem 4.9(ii). It follows that $\mathbf{S}$ is a simple L-algebra in $\mathcal{E}_n$ and therefore is isomorphic to some $\mathbf{C}_m^{\{\dot{-},0\}}$, $1 \leq m \leq n$; see [Cor80a] for a detailed explanation. $\mathbf{B}$ is in $\mathcal{E}_n$ since it is a subalgebra of $\mathbf{A}$. $\square$

For L-algebras the decomposition given in Theorem 4.9(iii) becomes trivial. Indeed, if $\mathbf{A}$ is any subdirectly irreducible L-algebra we have $U \subseteq S = A$ and $F = \{0\}$, and if in addition $\mathbf{A}$ is simple then $U = S = A$ and $F = \{0\}$.

5. Embedding Theorems

The key to proving Wroński's conjecture lies in the observation that many properties of hoops do not depend on its monoid operation but solely on its residuation operation. One example is the characterization of subdirectly irreducible HBCK-algebras, which resembles closely the characterization of subdirectly irreducible hoops. Therefore, once the description of subdirectly irreducible HBCK-algebras has been established, we can prove Wroński's conjecture.

In order to show that $\mathcal{HBCK} = \mathsf{S}\mathcal{H}\mathsf{O}^{\{\dot{-},0\}}$ one would like to find for each HBCK-algebra $\mathbf{A}$, a hoop $\mathbf{B}$ such that $\mathbf{A}$ embeds into (the residuation reduct of) $\mathbf{B}$. For arbitrary BCK-algebras and pocrims, this was achieved by taking the universe of $\mathbf{B}$ to be the set of order filters of a certain set of "ideal-like" subsets of $\mathbf{A}$; the reader is referred to [BR97, Theorem 2.1] for a detailed account on this construction, due to Fleischer [Fle88] and Ono and Komori [OK85]. This method does not quite work for HBCK-algebras. However, we may reduce the general embedding problem by using the fact that every algebra is a subalgebra of an ultraproduct of its finitely generated subalgebras. Therefore it suffices to show that every finitely generated HBCK-algebra embeds into a hoop.

Theorem 5.1. *A BCK-algebra is isomorphic to a* $\{\dot{-},0\}$*-subreduct of a hoop if and only if it is an HBCK-algebra.*

Proof. Let $\mathbf{A}$ be an m-generated subdirectly irreducible HBCK-algebra. We prove the theorem by induction on the number of generators.

If $m = 1$ then $\mathbf{A}$ is the 2-element (H)BCK-algebra, which is the $\{\dot{-},0\}$-reduct of the 2-element hoop $\mathbf{C}_1$. Assume by induction that every k-generated HBCK-algebra with $k < m$, is isomorphic to a $\{\dot{-},0\}$-subreduct of a hoop. Let $\mathbf{A}$ be m-generated and subdirectly irreducible. Then $A \simeq \mathbf{B} \oplus \mathbf{C}$, where $\mathbf{B}$ is a subdirectly irreducible L-algebra. By Lemma 4.8,

$\mathbf{C}$ is k-generated for some $k < m$. Therefore, by induction, $\mathbf{C}$ is a $\{\dot{-}, 0\}$-subreduct of some hoop $\mathbf{C}^*$. By Theorem 3.3, $\mathbf{B}$ is a $\{\dot{-}, 0\}$-subreduct of a bounded Wajsberg hoop $\mathbf{B}^*$. It follows from the definition of the fundamental operations on ordinal sums that A is isomorphic to a $\{\dot{-}, 0\}$-subreduct of the hoop $\mathbf{B}^* \oplus \mathbf{C}^*$. □

A hoop is called *n-potent* if it satisfies the identity (δ_n) $x^n \approx x^{n+1}$ or equivalently (ε_n) $x \dot{-} ny \approx x \dot{-} (n+1)y$. In [BP94] Blok and Pigozzi asked whether every HBCK-algebra in $\mathcal{E}_n$ is a subreduct of a n-potent hoop. Lemma 4.8 and Corollary 4.10 give a positive answer to this question. Let $\mathcal{H}O_n$ denote the variety of n-potent hoops and $\mathcal{HBCK}_n$ denote $\mathcal{HBCK} \cap \mathcal{E}_n$.

Theorem 5.2. $\mathsf{S}\mathcal{H}\mathsf{O_n}^{\{\dot{-},0\}} = \mathcal{HBCK}_n$.

Proof. The proof is similar to that of Theorem 5.1. We need only to make the following observation. If $\mathbf{A} \cong \mathbf{B} \oplus \mathbf{C}$ is a subdirectly irreducible m-generated member of $\mathcal{HBCK}_n$ then $\mathbf{B} \cong \mathbf{C}_k^{\{\dot{-},0\}}$ for some $k \leq n$, by Corollary 4.10 and Lemma 4.8; and $\mathbf{C}$ is a t-generated member of $\mathcal{HBCK}_n$, for some $t < m$. By the induction hypothesis, $\mathbf{C}$ embeds into the $\{\dot{-}, 0\}$-reduct of some n-potent hoop $\mathbf{C}^*$. Hence $\mathbf{A}$ is isomorphic to a residuation subreduct of the hoop $\mathbf{C}_k \oplus \mathbf{C}^*$. It is easy to verify that $\mathbf{C}_k \oplus \mathbf{C}^*$ is n-potent since both $\mathbf{C}_k$ and $\mathbf{C}^*$ are n-potent. □

Concluding remarks. Some of the tools used to obtain the embedding theorems above, ordinal sum constructions among them, have found further use in studying certain quasi-varieties of BCK-algebras. As an example, the reader is referred to Blok and Raftery [BR97]. These authors have investigated conditions under which the residuation subreducts of a variety of pocrims form a variety of BCK-algebras and have found several examples of such varieties.

When applied to certain subvarieties of hoops, the techniques used to prove Wroński's conjecture are also useful. To mention one application, one may consider the variety $\mathcal{BH}$ of basic hoops (generated by totally ordered hoops), its subvarieties and respective varieties of residuation subreducts [AFM]. The structure of basic hoops seems relevant in understanding the algebraic semantics of (the positive fragment of) Propositional Basic Logic, which was introduced by Hájek [Haj98].

References

[AFM] P. Agliano, I.M.A. Ferreirim and F. Montagna, *Basic hoops: an algebraic study of continuous t-norms*, preprint.

[BF93] W.J. Blok and I.M.A. Ferreirim, *Hoops and their implicational reducts (abstract)*, Algebraic Methods in Logic and Computer Science, Banach Center Publications **28** (1993), 219-230.

[BF00] W.J. Blok and I.M.A. Ferreirim, *On the structure of hoops*, Algebra Universalis **43** (2000), 233-257.

[BP89] W.J. Blok and D. Pigozzi, *Algebraizable logics*, Mem. Amer. Math. Soc. **396** (1989).

[BP94] W.J. Blok and D. Pigozzi, *On the structure of varieties with equationally definable principal congruences III*, Algebra Universalis **32** (1994), 545-608.

[BR95] W.J. Blok and J.G. Raftery, *On the quasivariety of BCK-algebras and its subvarieties*, Algebra Universalis **33** (1995), 68-90.

[BR97] W.J. Blok and J.G. Raftery, *Varieties of commutative residuated integral pomonoids and their residuation subreducts*, J. Algebra **190** (1997), 280-328.

[Bos69] B. Bosbach, *Komplementäre halbgruppen. Axiomatik und arithmetik*, Fund. Math. **64** (1969), 257-287.

[Bos74] B. Bosbach, *Residuation groupoids*, Bull. Academie Polonaise Sc, Sér. des Sciences Math., Astr. et Phys. **22** (1974), 103-104.

[Bos82] B. Bosbach, *Concerning cone algebras*, Algebra Universalis **15** (1982), 38-66.

[BO] J.R. Büchi and T.M. Owens, *Complemented monoids and hoops*, unpublished manuscript.

[Cor80a] W.H. Cornish, *3-permutability and quasicommutative BCK-algebras*, Math. Japonica **25** (1980), 477-496.

[Cor80b] W.H. Cornish, *Varieties generated by finite BCK-algebras*, Bull. Austral. Math. Soc. **22** (1980), 411-430.

[Cor81] W.H. Cornish, *A large variety of BCK-algebras*, Math. Japonica **26** (1981), 339-344.

[Cor82] W.H. Cornish, *On Iséki's BCK-algebras*, Lect. Notes in Pure and Appl. Math. **74** (1982) 101-122, Marcel Dekker, New York.

[CST84] W.H. Cornish, T. Sturm and T. Traczyk, *Embedding of commutative BCK-algebras into distributive lattice BCK-algebras*, Math. Japonica **29** (1984) 309-320.

[Die66] A. Diego, *Sur les algèbres de Hilbert*, Collection Logique Mathematique, Series A, No. 21 Gauthiers-Villars, Paris, 1966.

[Fer92] I.M.A. Ferreirim, *On Varieties and quasivarieties of hoops and their reducts*, Ph. D. Thesis, University of Illinois at Chicago, 1992.

[Fle88] I. Fleischer, *Every BCK-algebra is a set of residuables in an integral pomonoid*, J. Algebra **119** (1988), 360-365.

[Gra87] G. Grätzer, *General lattice theory*, Birkhäuser Verlag, Basel und Stuttgart, 1987.

[Haj98] P. Hájek, *Metamathematics of Fuzzy Logic*, Trends in Logic, Kluwer Academic Publ., Dordrecht, Boston, London, 1998.

[Hig84] D. Higgs, *Dually residuated commutative monoids with identity element do not form an equational class*, Math. Japon. **29** (1984), 69-75.

[Ise66] K. Iséki, *An algebra related with a propositional calculus*, Proc. Japan Acad., **42** (1966), 26-29.

[IT78] K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japonica **23** (1978), 1-26.

[IY80] K. Iséki and H. Yutani, *Ordinal unions of BCK-algebras*, Math Seminar Notes, Kobe Univ., **8** (1980), 307-308.

[Kom78] Y. Komori, *Super-Lukasiewicz implicational logics*, Nagoya Math. J. **72** (1978), 127-133.

[Kow94] T. Kowalski, *A syntactic proof of a conjecture of Andrzej Wroński*, Rep. Math. Logic **28** (1994), 81-86.

[OK85] H. Ono and Y. Komori, *Logics without the contraction rule*, J. Symb. Logic, **50** (1985), 169-201.

[Pal82] M. Pałasiński, *An embedding theorem for BCK-algebras*, Math. Seminar Notes, Kobe Univ.**10** (1982), 749-751.

[PW86] M. Pałasiński and A. Wroński, *Eight simple questions concerning BCK-algebras*, Rep. Math. Logic **20** (1986), 87-91.

[RRS91] J.G. Raftery, I.G. Rosenberg and T. Sturm, *Tolerance relations and BCK-algebras*, Math. Japonica **36** (1991), 399-410.

[RT80] A. Romanowska and T. Traczyk, *On Commutative BCK-algebras*, Math. Japonica **25** (1980), 567-583.

[RT82] A. Romanowska and T. Traczyk, *Commutative BCK-algebras. Subdirectly irreducible algebras and varieties*, Math. Japonica **27** (1982), 35-48.

[Tan75] S. Tanaka, *On $\wedge$-commutative algebras*, Math. Seminar Notes, Kobe Univ. **3** (1975), 59-64.

[Tra79] T. Traczyk, *On the variety of bounded commutative BCK-algebras*, Math. Japonica **24** (1979), 283-292.

[Wro83] A. Wroński, *BCK-algebras do not form a variety*, Math. Japonica **28** (1983) , 211-213.

[Wro85] A. Wroński, *An algebraic motivation for BCK-algebras*, Math. Japonica **30** (1985) , 187-193.

[Yut77] H. Yutani, *On a system of axioms of a commutative BCK-algebra*, Math. Seminar Notes, Kobe Univ. **5** (1977), 255-256.

Centro de Álgebra da Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal

E-mail address: `mimafer@ptmat.lmc.fc.ul.pt`