A REPRESENTATION OF RING HOMOMORPHISMS ON COMMUTATIVE BANACH ALGEBRAS

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Abstract. We give a partial representation of ring homomorphisms between two commutative Banach algebras. To this end, we characterize non-zero ring homomorphisms whose kernels are regular maximal ideals.

1. Introduction

We say that a map between two algebras is a ring homomorphism, if the map preserves both addition and multiplication. By definition, ring homomorphisms need not preserve scalar multiplication. Homomorphisms are ring homomorphisms which also preserve scalar multiplication.

In this paper, we consider ring homomorphisms between two commutative Banach algebras (not necessarily unital). Let A and B be unital commutative Banach algebras, M_A and M_B the maximal ideal spaces of A and B, respectively. It is well-known that each homomorphism φ on A into B is induced by a continuous map between two maximal ideal spaces: there exist a closed and open subset M_0 of M_B and a continuous map Φ on $M_B \setminus M_0$ into M_A so that $\varphi(f)\hat{} = 0$ on M_0 and $\varphi(f)\hat{} = \hat{f} \circ \Phi$ on $M_B \setminus M_0$ for every $f \in A$, where $\hat{\cdot}$ denotes the Gelfand transform (cf. [2, 4, 12]). In this paper, we will use the same symbol $\hat{\cdot}$ for the Gelfand transform on A and B. It seems natural to predict that a similar result holds for ring homomorphisms between unital commutative Banach algebras, while in the simplest case where $A = B = \mathbb{C}$, the complex number field, ring homomorphisms on $\mathbb{C}$ into $\mathbb{C}$ are very complicated. For ring homomorphisms on $\mathbb{C}$ into $\mathbb{C}$, we simply say ring homomorphisms on $\mathbb{C}$. Typical examples of ring homomorphisms on $\mathbb{C}$ are $\rho(z) = 0$, $\rho(z) = z$ and $\rho(z) = \bar{z}$ for every $z \in \mathbb{C}$, where $\bar{\cdot}$ denotes the complex conjugate. We call them trivial ring homomorphisms on $\mathbb{C}$, or simply trivial. Other ring homomorphisms on $\mathbb{C}$ are called non-trivial. Indeed, there exists a non-trivial ring homomorphism on $\mathbb{C}$ (cf. [7]) and it is well-known that the cardinal number of the set of all automorphisms of $\mathbb{C}$ is $2^{\mathfrak{c}}$, where $\mathfrak{c}$ denotes the cardinal number of continuum. In fact, Charnow [3] proved that every algebraically closed field F has $2^{|F|}$ automorphisms, where $|F|$ denotes the cardinal number of the set F. On the other hand, with some additional condition ring homomorphisms happen to be linear or conjugate linear. Indeed, Arnold [1] proved that a ring isomorphism between two Banach algebras of all bounded operators on infinite dimensional Banach spaces is linear or conjugate linear. It is generalized by Kaplansky [6] as follows: if ρ is a ring isomorphism from one semisimple Banach algebra A onto another, then A is a direct sum $A_1 \oplus A_2 \oplus A_3$ with A_3 finite-dimensional, ρ linear on A_1 and ρ conjugate linear on A_2. Therefore, we are interested in ring homomorphisms which need not be bijective. One of such examples is a $*$-ring homomorphism on an involutive Banach algebra into another. The author [8] proved that if ρ is a $*$-ring homomorphism on an involutive commutative Banach algebra A into a

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symmetrically involutive commutative Banach algebra B, then there exist a decomposition $\{M_{-1}, M_0, M_1\}$ of M_B, the maximal ideal space of B, and a continuous map Φ on $M_{-1} \cup M_1$ into M_A such that $\rho(f)\hat{} = \overline{\hat{f} \circ \Phi}$ on M_{-1}, $\rho(f)\hat{} = 0$ on M_0 and $\rho(f)\hat{} = \hat{f} \circ \Phi$ on M_1 for every $f \in A$ (cf. [10]).

Takahasi and Hatori [11] proved the following result for a ring homomorphism ρ on a regular commutative Banach algebra A into a commutative Banach algebra B. Let M_A and M_B be the maximal ideal spaces of A and B, respectively. If $\{\rho(f)\hat{}(\varphi) : f \in A\} = \mathbb{C}$ holds for every $\varphi \in M_B$, then there exist a decomposition $\{M_{-1}, M_1, M_d\}$ of M_B and a continuous map Φ on M_B into M_A with the following properties: (i) $\rho(f)\hat{} = \overline{\hat{f} \circ \Phi}$ on M_{-1} and $\rho(f)\hat{} = \hat{f} \circ \Phi$ on M_1 for every $f \in A$. (ii) For each $\varphi \in M_d$ there corresponds a non-trivial ring homomorphism τ_φ on $\mathbb{C}$ so that $\rho(f)\hat{}(\varphi) = \tau_\varphi(\hat{f}(\Phi(\varphi)))$ for every $f \in A$ (cf. [9]).

In this paper, we consider a ring homomorphism between two commutative Banach algebras, which satisfies a certain condition, say (m). Many ring homomorphisms satisfy the condition (m), for instance, $*$-ring homomorphisms between involutive algebras and a ring homomorphism $\rho : A \to B$ satisfying $\rho(A)\hat{}(\varphi) = \mathbb{C}$ for every $\varphi \in M_B$. Applying the methods used in [5], we show that if ρ is a ring homomorphism between two commutative Banach algebras, then ρ is induced by a continuous map between the maximal ideal spaces. As a corollary, Theorem 2.1 in [8] and Theorem 1 in [11] are proved. Moreover if we consider a ring isomorphism, then two maximal ideal spaces are homeomorphic.

Finally we note that if ρ is a ring homomorphism on $\mathbb{C}$, then the following are equivalent: (i) ρ is non-trivial. (ii) ρ is unbounded. (iii) ρ is discontinuous. (iv) There exists a sequence $\{w_n\}_{n=1}^\infty \subset \mathbb{C}$ so that w_n converges to 0, while $|\rho(w_n)|$ tends to infinity as $n \to \infty$.

2. Main results

Let A be a commutative Banach algebra. We say that A is a radical algebra, if there is no non-zero complex-valued homomorphism on A. Then we define the radical of A to be A. Unless A is a radical algebra, we say that A is non-radical for the convenience, then M_A denotes the maximal ideal space of A. In this case, we define the radical of A to be the intersection of all the regular maximal ideals in A.

It is well-known that the kernels of non-zero complex homomorphisms on a non-radical commutative Banach algebra are regular maximal ideals. On the other hand, the kernels of complex ring homomorphisms need not be maximal (cf. [10, Example 5.4]). We give a characterization of ring homomorphisms whose kernels are regular maximal ideals.

Lemma 2.1. *Let A be a non-radical commutative Banach algebra, B a commutative Banach algebra and ρ a non-zero ring homomorphism on A into B. Then the following conditions are equivalent.*

(i) *The kernel* $\ker \rho = \{f \in A : \rho(f) = 0\}$ *is a regular maximal ideal in A.*
(ii) *There exists a ring homomorphism $\tilde{\rho}$ on A_e into B such that $\tilde{\rho}|_A = \rho$ and $\tilde{\rho}(\mathbb{C}e) = \rho(A)$, where A_e denotes the commutative Banach algebra obtained by adjunction of a unit e to A.*
(iii) *There exist a unique ring isomorphism τ on $\mathbb{C}$ onto $\rho(A)$ and a unique $\psi \in M_A$ such that $\rho = \tau \circ \psi$.*

Proof. (i) $\Rightarrow$ (ii) There exists a $\varphi \in M_A$ such that $\ker \rho = \ker \varphi$, by hypothesis. Since $\varphi(A) = \mathbb{C}$, for every $\lambda \in \mathbb{C}$ there exists a $g_\lambda \in A$ such that $\lambda = \varphi(g_\lambda)$. We define $\tilde{\rho}$ on A_e into B as

$$\tilde{\rho}((f, \lambda)) = \rho(f) + \rho(g_\lambda), \quad ((f, \lambda) \in A_e).$$

Then $\tilde{\rho}$ is well-defined. In fact, let g_λ and h_λ be elements of A so that $\varphi(g_\lambda) = \lambda = \varphi(h_\lambda)$, hence $g_\lambda - h_\lambda \in \ker\varphi$. Since $\ker\rho = \ker\varphi$, we have $\rho(g_\lambda) = \rho(h_\lambda)$ and this implies that $\tilde{\rho}$ is well-defined. By definition $\tilde{\rho}$ is an extension of ρ. We show that the map $\tilde{\rho}$ is a ring homomorphism on A_e into B. In fact, let (f_j, λ_j) be any element of A_e and g_j an element of A so that $\varphi(g_j) = \lambda_j$ for $j = 1, 2$. By a simple calculation we have

$$\tilde{\rho}((f_1, \lambda_1) + (f_2, \lambda_2)) = \tilde{\rho}((f_1, \lambda_1)) + \tilde{\rho}((f_2, \lambda_2)).$$

Next we show that $\tilde{\rho}$ is multiplicative. To do this, note that the equality

$$\rho(\lambda_2 f_1) = \rho(g_2 f_1) = \rho(f_1)\rho(g_2)$$

holds, since $\lambda_2 f_1 - g_2 f_1 \in \ker\varphi = \ker\rho$. Therefore,

$$\begin{aligned}
\tilde{\rho}((f_1, \lambda_1)(f_2, \lambda_2)) &= \tilde{\rho}((f_1 f_2 + \lambda_2 f_1 + \lambda_1 f_2, \lambda_1\lambda_2)) \\
&= \rho(f_1)\rho(f_2) + \rho(\lambda_2 f_1) \\
&\qquad + \rho(\lambda_1 f_2) + \rho(g_1)\rho(g_2) \\
&= \{\rho(f_1) + \rho(g_1)\}\{\rho(f_2) + \rho(g_2)\} \\
&= \tilde{\rho}((f_1, \lambda_1))\,\tilde{\rho}((f_2, \lambda_2)).
\end{aligned}$$

That is, $\tilde{\rho}$ is a ring homomorphism on A_e into B. Finally, we show that $\tilde{\rho}(\mathbb{C}e) = \rho(A)$. It is easy to see that $\tilde{\rho}(\mathbb{C}e) = \tilde{\rho}((0, \mathbb{C})) \subset \rho(A)$, by the definition of $\tilde{\rho}$. Conversely, for every $f \in A$

$$\rho(f) = \tilde{\rho}(0, \varphi(f)) = \tilde{\rho}(\varphi(f)e) \in \tilde{\rho}(\mathbb{C}e).$$

Thus, we proved that $\tilde{\rho}(\mathbb{C}e) = \rho(A)$.

(ii) $\Rightarrow$ (iii) Let $\tilde{\rho}$ be a ring homomorphism on A_e into B so that $\tilde{\rho}|_A = \rho$ and $\tilde{\rho}(\mathbb{C}e) = \rho(A)$. Let τ be a restriction of $\tilde{\rho}$ to $\mathbb{C}e$. That is,

$$\tau(\lambda) = \tilde{\rho}(\lambda e), \quad (\lambda \in \mathbb{C}).$$

Then we show that τ is a ring isomorphism on $\mathbb{C}$ onto $\rho(A)$. In fact, τ is surjective, since $\tilde{\rho}(\mathbb{C}e) = \rho(A)$. Suppose that τ is not injective. Then there exist $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $\lambda_1 \neq \lambda_2$ and $\tau(\lambda_1) = \tau(\lambda_2)$. Put $\lambda_3 = \lambda_1 - \lambda_2$. Since $\tilde{\rho}$ is an extension of ρ, we have

$$\rho(f) = \tau(\lambda_3)\,\rho\left(\frac{f}{\lambda_3}\right) = 0$$

for every $f \in A$. Since ρ is non-zero, we arrived at a contradiction. That is, we proved that τ is a ring isomorphism on $\mathbb{C}$ onto $\rho(A)$. Therefore, τ^{-1} is a ring isomorphism on $\rho(A)$ onto $\mathbb{C}$. Put $\Psi = \tau^{-1} \circ \tilde{\rho}$, then it is easy to see that Ψ is a non-zero complex homomorphism on A_e. In this case, $\tilde{\rho} = \tau \circ \Psi$ holds. Put $\psi = \Psi|_A$, then ψ is a non-zero complex homomorphism on A since ρ is non-zero. Hence, $\psi \in M_A$ and $\rho = \tau \circ \psi$ holds. Finally we show that both τ and ψ are unique. In fact, suppose that $\tau_1 \circ \psi_1 = \rho = \tau_2 \circ \psi_2$ holds for ring isomorphisms τ_j on $\mathbb{C}$ onto $\rho(A)$ and $\psi_j \in M_A$ for $j = 1, 2$. Since both τ_1 and τ_2 are injective, it follows that $\ker\psi_1 = \ker\psi_2$. By a simple calculation we see that $\psi_1 = \psi_2$, then $\tau_1 = \tau_2$ is trivial since $\psi_j(A) = \mathbb{C}$.

(iii) $\Rightarrow$ (i) If τ is a ring isomorphism on $\mathbb{C}$ onto $\rho(A)$ and ψ is an element of M_A such that $\rho = \tau \circ \psi$, then $\ker\rho = \ker\psi$. Hence, $\ker\rho$ is a regular maximal ideal in A. This completes the proof. □

Definition 2.1. *Let A be a commutative Banach algebra, B a non-radical commutative Banach algebra and ρ a ring homomorphism on A into B. For every element φ of M_B we define the induced ring homomorphism ρ_φ on A into $\mathbb{C}$ as*

$$\rho_\varphi(f) = \rho(f)\hat{\ }(\varphi), \quad (f \in A).$$

Definition 2.2. *Let A be a commutative Banach algebra, B a non-radical commutative Banach algebra and ρ a ring homomorphism on A into B. We say that ρ satisfies the condition* (m), *if* $\ker \rho_\varphi$ *is a regular maximal ideal in A or* $\ker \rho_\varphi = A$ *for every* $\varphi \in M_B$.

Definition 2.3. *Let A be a commutative Banach algebra, B a non-radical commutative Banach algebra and ρ a ring homomorphism on A into B, which satisfies the condition* (m). *We denote*

$$M_0 = \{\varphi \in M_B : \ker \rho_\varphi = A\}.$$

If A is non-radical, for every $\varphi \in M_B \setminus M_0$ we can write $\rho_\varphi = \tau_\varphi \circ \psi_\varphi$ for a unique ring homomorphism τ_φ on $\mathbb{C}$ and a unique $\psi_\varphi \in M_A$, by Lemma 2.1. *Then we define the subsets M_{-1}, M_1 and M_d of M_B as*

$$\begin{aligned} M_{-1} &= \{\varphi \in M_B \setminus M_0 : \tau_\varphi(z) = \bar{z}, \quad (z \in \mathbb{C})\}, \\ M_1 &= \{\varphi \in M_B \setminus M_0 : \tau_\varphi(z) = z, \quad (z \in \mathbb{C})\}, \\ M_d &= \{\varphi \in M_B \setminus M_0 : \tau_\varphi \textit{ is non-trivial}\}. \end{aligned}$$

It is easy to see that M_{-1}, M_0, M_1 and M_d are mutually disjoint and $M_B = M_{-1} \cup M_0 \cup M_1 \cup M_d$ holds. Thus, $\{M_{-1}, M_0, M_1, M_d\}$ is a decomposition of M_B.

Definition 2.4. *Let $\{M_{-1}, M_0, M_1, M_d\}$ be the decomposition of M_B as in Definition* 2.3. *We define the map Φ on $M_B \setminus M_0$ into M_A as*

$$\Phi(\varphi) = \psi_\varphi, \quad (\varphi \in M_B \setminus M_0),$$

where ψ_φ is a unique element of M_A so that $\rho_\varphi = \tau_\varphi \circ \psi_\varphi$ for a unique ring homomorphism τ_φ on $\mathbb{C}$.

Note that for every $\varphi \in M_B \setminus M_0$ we have

$$\rho(f)\hat{}(\varphi) = (\tau_\varphi \circ \psi_\varphi)(f) = \tau_\varphi(\hat{f}(\Phi(\varphi)))$$

for every $f \in A$. Under the assumptions above, we show the following lemmas on topological structures of M_{-1}, M_0, M_1 and M_d.

Lemma 2.2. *M_0 is a closed subset of M_B.*

Proof. Let $\{\varphi_\alpha\}$ be any net in M_0 converging to φ. By definition $\rho(f)\hat{}(\varphi_\alpha) = 0$ holds for every $f \in A$. Since $\rho(f)\hat{}$ is continuous on M_B, we have $\rho(f)\hat{}(\varphi) = 0$ for every $f \in A$. This implies $\varphi \in M_0$, hence M_0 is a closed subset of M_B. □

Lemma 2.3. *$M_{-1} \cup M_0$ and $M_0 \cup M_1$ are closed subsets of M_B.*

Proof. Since M_0 is closed, it is enough to show that $\bar{M_j} \subset M_0 \cup M_j$ for $j = -1, 1$, where $\bar{\cdot}$ denotes the closure in M_B. For this end, let φ be any point of $\bar{M_j}$ and $\{\varphi_\alpha\}$ a net in M_j converging to φ. We show that φ belongs to $M_0 \cup M_j$. Since M_{-1}, M_0, M_1 and M_d are mutually disjoint, it sufficies to show that $\varphi \notin M_{-j} \cup M_d$. Suppose that φ is an element of M_d, then there exist a non-trivial ring homomorphism τ_φ on $\mathbb{C}$ and a $\Phi(\varphi) \in M_A$ such that $\rho(f)\hat{}(\varphi) = \tau_\varphi(\hat{f}(\Phi(\varphi)))$ holds for every $f \in A$. Choose an element $f_0 \in A$ with $\hat{f}_0(\Phi(\varphi)) = 1$, and since τ_φ is non-trivial, there exists a non-zero sequence $\{\lambda_n\}$ in $\mathbb{C}$ such that $|\lambda_n| < 1/n$ and $|\tau_\varphi(\lambda_n)| > n$ for every $n \in \mathbb{N}$, the space of all natural numbers. On one hand

$$|\rho(\lambda_n f_0)\hat{}(\varphi_\alpha)| = |\lambda_n \hat{f}_0(\Phi(\varphi_\alpha))| < \|\hat{f}_0\|_\infty / n,$$

since $\varphi_\alpha \in M_j$, where $\|\cdot\|_\infty$ denotes the supremum norm on M_A. On the other hand, we have

$$|\rho(\lambda_n f_0)\hat{}(\varphi)| = |\tau_\varphi(\lambda_n)| > n$$

for every $n \in \mathbb{N}$. This contradicts with the continuity of the function $\rho(\lambda_n f_0)\hat{}$ on M_B, for a sufficiently large $n \in \mathbb{N}$. Therefore, φ does not belong to M_d.

Suppose that φ is an element of M_{-j}. As a first step, we consider the case where $\Phi(\varphi_\alpha)$ converges to $\Phi(\varphi)$. In this case $\hat{f}(\Phi(\varphi_\alpha))$ converges to $\hat{f}(\Phi(\varphi))$ for every $f \in A$, since $\hat{f}$ is continuous on M_A. Choose an element f_1 of A so that $\hat{f}_1(\Phi(\varphi)) = i$, then $\hat{f}_1(\Phi(\varphi_\alpha))$ converges to i, since $\Phi(\varphi_\alpha) \to \Phi(\varphi)$. Therefore, $\rho(f_1)\hat{}(\varphi_\alpha)$ converges to ji. On the other hand, $\rho(f_1)\hat{}(\varphi_\alpha)$ converges to $-ji$, since $\rho(f_1)\hat{}$ is continuous and since $\varphi \in M_{-j}$. We arrived at a contradiction, hence we proved that φ does not belong to M_{-j}, in case where $\Phi(\varphi_\alpha)$ converges to $\Phi(\varphi)$.

Next we consider the case where $\Phi(\varphi_\alpha)$ does not converge to $\Phi(\varphi)$ (as we will prove later, such a case does not occur). Hence, there exists an $f_2 \in A$ such that $\hat{f}_2(\Phi(\varphi_\alpha))$ does not converge to $\hat{f}_2(\Phi(\varphi))$. In particular, $\hat{f}_2(\Phi(\varphi)) \neq \overline{\hat{f}_2(\Phi(\varphi))}$, since $\rho(f_2)\hat{}$ is continuous on M_B. Put

$$f_3 = \frac{\overline{\hat{f}_2(\Phi(\varphi))}}{|\hat{f}_2(\Phi(\varphi))|} f_2 \in A,$$

then we obtain $\hat{f}_3(\Phi(\varphi)) = \overline{\hat{f}_3(\Phi(\varphi))}$. Therefore, $\hat{f}_3(\Phi(\varphi_\alpha))$ converges to $\hat{f}_3(\Phi(\varphi))$, since $\rho(f_3)\hat{}$ is continuous on M_B. On the other hand, the equality

$$|\hat{f}_3(\Phi(\varphi_\alpha)) - \hat{f}_3(\Phi(\varphi))| = |\hat{f}_2(\Phi(\varphi_\alpha)) - \hat{f}_2(\Phi(\varphi))|$$

holds, and this contradicts with the assumption that $\hat{f}_2(\Phi(\varphi_\alpha))$ does not converge to $\hat{f}_2(\Phi(\varphi))$. Hence, we proved that φ does not belong to M_{-j} in case where $\Phi(\varphi_\alpha)$ does not converge to $\Phi(\varphi)$. This implies $\bar{M}_j \subset M_0 \cup M_j$ for $j = -1, 1$. □

Lemma 2.4. *The range $\Phi(M_d)$ is at most finite subset of M_A.*

Proof. Assume to the contrary that the range $\Phi(M_d)$ is not a finite set. Then $\Phi(M_d)$ has a countable subset $\{\psi_n\}_{n=1}^{\infty}$ so that $\psi_n \neq \psi_m$ if $n \neq m$. By definition, for every $n \in \mathbb{N}$ there exists a $\varphi_n \in M_d$ such that $\psi_n = \Phi(\varphi_n)$, then $\varphi_n \neq \varphi_m$ if $n \neq m$. Since φ_n is an element of M_d, there corresponds a non-trivial ring homomorphism τ_n on $\mathbb{C}$ such that

$$\rho(f)\hat{}(\varphi_n) = \tau_n(\hat{f}(\Phi(\varphi_n))) = \tau_n(\hat{f}(\psi_n))$$

holds for every $f \in A$. Since τ_1 is non-trivial, there exists an $f_1 \in A$ so that

$$\|f_1\| < 1/2, \ |\tau_1(\hat{f}_1(\psi_1))| > 2.$$

Inductively we can find an $f_n \in A$ such that

$$\|f_n\| < 2^{-n}, \ |\tau_n(\hat{f}_n(\psi_n))| > 2^n + \left| \tau_n \left(\sum_{k=1}^{n-1} \hat{f}_k(\psi_n) \right) \right|$$

and also

$$\hat{f}_n(\psi_1) = \hat{f}_n(\psi_2) = \cdots = \hat{f}_n(\psi_{n-1}) = 0.$$

Therefore, $\sum_{n=1}^{\infty} f_n$ converges to some element $f_0 \in A$. Note that, for every $k \in \mathbb{N}$, $\hat{f}_j(\psi_k) = 0$ if $j > k$, then $\hat{f}_0(\psi_k) = \sum_{n=1}^{k} \hat{f}_n(\psi_k)$, since the Banach norm on A dominates the supremum norm on M_A. Thus we have the inequality

$$|\rho(f_0)\hat{}(\varphi_k)| = |\tau_k(\hat{f}_0(\psi_k))| = \left| \tau_k \left(\sum_{n=1}^{k} \hat{f}_n(\psi_k) \right) \right| > 2^k,$$

and this implies that $\rho(f_0)\hat{}$ is unbounded on M_B. We arrived at a contradiction, hence we proved that the range $\Phi(M_d)$ is at most finite subset of M_A. □

Lemma 2.5. *Put* $\Phi(M_d) = \{\psi_1, \psi_2, \cdots, \psi_n\}$. *For every* $j \in \{1, 2, \cdots, n\}$ *the set* $M_{d,j} = \{\varphi \in M_d : \Phi(\varphi) = \psi_j\}$ *is open in* M_B.

Proof. For each $j \in \{1, 2, \cdots, n\}$ we can find an $f_j \in A$ such that

$$\hat{f}_j(\psi_j) = 1,\ \hat{f}_j(\psi_k) = 0, \quad (k \neq j).$$

Suppose that $M_{d,j}$ is not an open subset of M_B, then there exist an element φ_j of $M_{d,j}$ and a net $\{\varphi_\alpha\}$ in $M_B \setminus M_{d,j}$ such that φ_α converges to φ_j. Since $M_{-1} \cup M_0 \cup M_1$ is closed in M_B, by Lemma 2.3, $M_d = M_B \setminus (M_{-1} \cup M_0 \cup M_1)$ is an open subset of M_B. Therefore, without loss of generality we may assume that the net $\{\varphi_\alpha\}$ consists of elements of $M_d \setminus M_{d,j}$. Then $\Phi(\varphi_\alpha) \neq \psi_j$, hence we have $\hat{f}_j(\Phi(\varphi_\alpha)) = 0$ by definition. On the other hand, we have $\rho(f_j)\hat{}(\varphi_j) = \tau_{\varphi_j}(\hat{f}_j(\Phi(\varphi_j))) = 1$ and $\rho(f_j)\hat{}(\varphi_\alpha) = \tau_{\varphi_\alpha}(\hat{f}_j(\Phi(\varphi_\alpha))) = 0$, where τ_η denotes the non-trivial ring homomorphism on $\mathbb{C}$ corresponding to $\eta \in M_d$. This is a contradiction, since $\rho(f_j)\hat{}$ is continuous on M_B. This completes the proof. ☐

Theorem 2.6. *Let* A *be a commutative Banach algebra,* B *a non-radical commutative Banach algebra and* ρ *a ring homomorphism on* A *into* B*, which satisfies the condition* (m). *Then the radical of* A *is mapped into the radical of* B. *Moreover if* A *is non-radical, let* $\{M_{-1}, M_0, M_1, M_d\}$ *be the decomposition of* M_B *as in Definition* 2.3. *Then the map* Φ *is continuous on* $M_B \setminus M_0$ *into* M_A *with the following property: for every* $\varphi \in M_d$ *there corresponds a non-trivial ring homomorphism on* $\mathbb{C}$ *so that the equality*

$$\rho(f)\hat{}(\varphi) = \begin{cases} \overline{\hat{f}(\Phi(\varphi))}, & \varphi \in M_{-1}, \\ 0, & \varphi \in M_0, \\ \hat{f}(\Phi(\varphi)), & \varphi \in M_1, \\ \tau_\varphi(\hat{f}(\Phi(\varphi))), & \varphi \in M_d \end{cases}$$

holds for every $f \in A$.

Proof. If A is a radical algebra, we have $M_B = M_0$ by the condition (m). Therefore, ρ_φ is identically zero for every $\varphi \in M_B$. By definition, the radical of A is mapped into the radical of B, if A is a radical algebra.

If A is non-radical, we have the equality

$$\begin{aligned} \rho(f)\hat{}(\varphi) &= \begin{cases} 0, & \varphi \in M_0, \\ \tau_\varphi(\hat{f}(\Phi(\varphi))), & \varphi \in M_B \setminus M_0 \end{cases} \\ &= \begin{cases} \overline{\hat{f}(\Phi(\varphi))}, & \varphi \in M_{-1}, \\ 0, & \varphi \in M_0, \\ \hat{f}(\Phi(\varphi)), & \varphi \in M_1, \\ \tau_\varphi(\hat{f}(\Phi(\varphi))), & \varphi \in M_d \end{cases} \end{aligned}$$

for every $f \in A$. In particular, for every $f \in \operatorname{rad} A$ we have $\rho(f)\hat{}(\varphi) = 0$ for every $\varphi \in M_B$. That is, we proved that the radical of A is mapped into the radical of B.

We show that the map Φ on $M_B \setminus M_0$ into M_A is continuous. By Lemma 2.4 we can write $\Phi(M_d) = \{\psi_1, \psi_2, \cdots, \psi_n\}$. As a first step, we show that Φ is continuous at each point of M_d. For every $\varphi_0 \in M_d$ there exists a $\psi_j \in \Phi(M_d)$ such that $\Phi(\varphi_0) = \psi_j$. Since $M_{d,j} = \{\varphi \in M_d : \Phi(\varphi) = \psi_j\}$ is open in M_B, by Lemma 2.5, we see that Φ is continuous at $\varphi_0 \in M_d$.

Next we show that Φ is continuous on M_j for $j = -1, 1$. Let φ_j be any point of M_j and $\{\varphi_\alpha\}_{\alpha \in I}$ any net in $M_B \setminus M_0$ converging to φ_j. Since $M_0 \cup M_{-j}$ is closed in M_B, by Lemma 2.3, we see that $M_j \cup M_d = M_B \setminus (M_0 \cup M_{-j})$ is an open subset of M_B. Hence, without loss of generality we may assume that the net $\{\varphi_\alpha\}_{\alpha\in I}$ consists of elements of $M_j \cup M_d$. Then we show that there exists an $\alpha_0 \in I$ such that φ_α belongs to $M_j \cup \{\varphi \in M_d : \Phi(\varphi) = \Phi(\varphi_j)\}$ for every $\alpha \in I$ with $\alpha \geq \alpha_0$. In fact, since $\Phi(M_d)$ is at most finite, we can find an element f_0 of A so that $\hat{f}_0(\Phi(\varphi_j)) = 1$ and $\hat{f}_0(\psi_k) = 0$ for every element ψ_k of $\Phi(M_d) \setminus \{\Phi(\varphi_j)\}$. By the continuity of $\rho(f_0)\hat{}$ there exists an $\alpha_0 \in I$ such that $|\rho(f_0)\hat{}(\varphi_\alpha) - 1| < 1/2$ holds for every element α of I with $\alpha \geq \alpha_0$. In particular we have $\hat{f}_0(\Phi(\varphi_\alpha)) \neq 0$, hence $\Phi(\varphi_\alpha)$ does not belong to $\Phi(M_d) \setminus \{\Phi(\varphi_j)\}$ if $\alpha \geq \alpha_0$, since $\hat{f}_0 = 0$ on $\Phi(M_d) \setminus \{\Phi(\varphi_j)\}$. Therefore, we proved that φ_α is an element of $M_j \cup \{\varphi \in M_d : \Phi(\varphi) = \Phi(\varphi_j)\}$ for every $\alpha \in I$ with $\alpha \geq \alpha_0$. Hence, we have the inequality

$$|\hat{f}(\Phi(\varphi_\alpha)) - \hat{f}(\Phi(\varphi_j))| \leq |\rho(f)\hat{}(\varphi_\alpha) - \rho(f)\hat{}(\varphi_j)|$$

for every element f of A, if $\alpha \geq \alpha_0$. We conclude that $\Phi(\varphi_\alpha)$ converges to $\Phi(\varphi_j)$, hence Φ is continuous on M_j for $j = -1, 1$. Thus we proved that the map Φ is continuous on $M_B \setminus M_0$ and this completes the proof. □

As a corollary, we have the following results.

Corollary 2.7. [8, Theorem 2.1] *Let A be a commutative Banach algebra with an involution $*$, B a non-radical commutative Banach algebra with a symmetric involution $\star$. If ρ is a $*$-ring homomorphism on A into B, then the radical of A is mapped into the radical of B. Therefore*

$$\rho(f)\hat{} = 0 \quad (f \in A)$$

holds on M_B, if A is a radical algebra. If A is non-radical, there exist a decomposition $\{M_{-1}, M_0, M_1\}$ of M_B and a continuous map Φ on $M_{-1} \cup M_1$ into M_A such that the equality

$$\rho(f)\hat{}(\varphi) = \begin{cases} \overline{\hat{f}(\Phi(\varphi))}, & \varphi \in M_{-1}, \\ 0, & \varphi \in M_0, \\ \hat{f}(\Phi(\varphi)), & \varphi \in M_1 \end{cases}$$

holds for every $f \in A$.

Proof. We consider the case where A is non-radical. If A is unital, then we define the ring homomorphism $\rho_{\varphi,e}$ on $\mathbb{C}$ as

$$\rho_{\varphi,e}(\lambda) = \rho_\varphi(\lambda e), \quad (\lambda \in \mathbb{C}),$$

for each $\varphi \in M_B$. Since ρ preserves the involution, we see that $\rho_{\varphi,e}$ is trivial. Thus, $\rho_\varphi \in M_A$ or $\overline{\rho_\varphi} \in M_A$ or $\rho_\varphi = 0$.

If A has no unit, then we consider the commutative Banach algebra A_e obtained by adjunction of a unit e to A. Unless ρ_φ is identically zero, there exists a $g \in A$ so that $\rho_\varphi(g) \neq 0$. Then we define $\tilde{\rho}_\varphi$ on A_e to $\mathbb{C}$ by

$$\tilde{\rho}_\varphi((f,\lambda)) = \rho_\varphi(f) + \frac{\rho_\varphi(\lambda g)}{\rho_\varphi(g)}, \quad ((f,\lambda) \in A_e).$$

Then it is easy to see that $\tilde{\rho}_\varphi$ is a $*$-ring homomorphism on A_e with respect to the involution $(f,\lambda) \mapsto (f^*, \bar{\lambda})$ on A_e. Thus, we have the conclusion by Theorem 2.6. □

Takahasi and Hatori [11] proved the following result in case where A is regular and satisfies a certain condition, while we can prove the result without such assumptions.

Corollary 2.8. *Let A and B be non-radical commutative Banach algebras, ρ a ring homomorphism on A into B so that*

$$\{\rho(f)\hat{}(\varphi) : f \in A\} = \mathbb{C},$$

for every $\varphi \in M_B$. Then there exist a decomposition $\{M_{-1}, M_1, M_d\}$ of M_B and a continuous map Φ on M_B into M_A with the following property: for every $\varphi \in M_d$ there exists a non-trivial ring homomorphism τ_φ on $\mathbb{C}$ such that

$$\rho(f)\hat{}(\varphi) = \begin{cases} \overline{\hat{f}(\Phi(\varphi))}, & \varphi \in M_{-1}, \\ \hat{f}(\Phi(\varphi)), & \varphi \in M_1, \\ \tau_\varphi\big(\hat{f}\big(\Phi(\varphi)\big)\big), & \varphi \in M_d \end{cases}$$

holds for every $f \in A$.

Proof. By Theorem 2.6, it is enough to show that $\ker \rho_\varphi$ is a regular maximal ideal in A for every $\varphi \in M_B$. As a first step, we consider the case where A has a unit element e. Since $\ker \rho_\varphi$ is a proper algebra ideal, there exists a $\psi \in M_A$ so that $\ker \rho_\varphi \subset \ker \psi$. Suppose that g does not belong to $\ker \rho_\varphi$, then there corresponds an $h \in A$ such that $\rho_\varphi(g)\,\rho_\varphi(h) = 1$ since $\{\rho(f)\hat{}(\varphi) : f \in A\} = \mathbb{C}$. Therefore $\rho_\varphi(gh - e) = 0$. Since $\ker \rho_\varphi$ is contained in $\ker \psi$, we have $\psi(gh) = 1$ hence $\psi(g) \neq 0$. Thus, we proved that $\ker \rho_\varphi$ is a maximal ideal in A.

Next we consider the case where A does not have a unit element. Let A_e be the commutative Banach algebra obtained by adjunction of a unit e to A. Since $\{\rho(f)\hat{}(\varphi) : f \in A\} = \mathbb{C}$, there exists a $g_\varphi \in A$ such that $\rho_\varphi(g_\varphi) = 1$. Define the map $\tilde{\rho}_\varphi$ on A_e to $\mathbb{C}$ by

$$\tilde{\rho}_\varphi((f, \lambda)) = \rho_\varphi(f) + \rho_\varphi(\lambda g_\varphi), \quad ((f, \lambda) \in A_e).$$

Then it is easy to see that $\tilde{\rho}_\varphi$ is a ring homomorphism on A_e onto $\mathbb{C}$. As proved above, $\ker \tilde{\rho}_\varphi$ is a maximal ideal in A_e. Since $\tilde{\rho}_\varphi$ is an extension of ρ_φ, we have that ρ_φ is a regular maximal ideal in A. □

Corollary 2.9. *Let A and B be non-radical commutative Banach algebras with the maximal ideal spaces M_A and M_B, respectively. If ρ is a ring isomorphism on A onto B, then M_A is homeomorphic to M_B.*

Proof. Since ρ is surjective, $\{\rho(f)\hat{}(\varphi) : f \in A\} = \mathbb{C}$ holds for every $\varphi \in M_B$. By Corollary 2.8, there exists a continuous map Φ on M_B into M_A with the following property: for every $\varphi \in M_B$ there corresponds a non-zero ring homomorphism τ_φ on $\mathbb{C}$ so that $\rho(f)\hat{}(\varphi) = \tau_\varphi(\hat{f}(\Phi(\varphi)))$ for every $f \in A$. Since ρ is a ring isomorphism, we can write $\rho^{-1}(x)\hat{}(\psi) = \eta_\psi(\hat{x}(\Psi(\psi)))$ for every $x \in B$ and every $\psi \in M_A$, where Ψ is the continuous map on M_A into M_B and η_ψ is a non-zero ring homomorphism on $\mathbb{C}$. Put $x = \rho(f)$ for each $f \in A$ and $\psi = \Phi(\varphi)$ for each $\varphi \in M_B$. Then we have the equality

$$\begin{aligned} \hat{x}(\varphi) &= \rho(f)\hat{}(\varphi) = \tau_\varphi(\hat{f}(\Phi(\varphi))) \\ &= \tau_\varphi(\rho^{-1}(x)\hat{}(\psi)) \\ &= \tau_\varphi(\eta_\psi(\hat{x}(\Psi(\psi)))). \end{aligned}$$

If $\hat{x}(\Psi(\psi)) = 0$, we have $\hat{x}(\varphi) = 0$. Unless $\hat{x}(\Psi(\psi)) = 0$, put $y = x/\hat{x}(\Psi(\psi))$. Then we obtain the equality

$$\hat{y}(\varphi) = \tau_\varphi\eta_\psi(\hat{y}(\Psi(\psi))) = 1,$$

that is, $\hat{x}(\varphi) = \hat{x}(\Psi(\psi))$. Therefore, $\varphi = \Psi(\psi) = \Psi \circ \Phi(\varphi)$ holds for every $\varphi \in M_B$. In a way similar to the above, we have $\psi = \Phi \circ \Psi(\psi)$ holds for every $\psi \in M_A$. Hence M_A is homeomorphic to M_B and this completes the proof. □

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