SOME RESULTS ON IDEALS OF BCK–ALGEBRAS

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Abstract. We introduce a special set in a BCK-algebra, and we give an example in which the special set is not an ideal. we state a condition for this special set to be an ideal. Using this special set, we establish an equivalent condition of an ideal. We prove that an ideal can be represented by the union of such special sets.

1. Introduction

The notion of BCK-algebras was proposed by Imai and Iséki in 1966. For the general development of BCK-algebras, the ideal theory plays an important role. In this paper, we introduce a special set in a BCK-algebra, and we give an example in which the special set is not an ideal. we state a condition for this special set to be an ideal. Using this special set, we establish an equivalent condition of an ideal. We prove that an ideal can be represented by the union of such special sets.

2. Preliminaries

By a *BCK-algebra* we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the following conditions:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $0 * x = 0$,
(V) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y, z \in X$. We can define a partial ordering "$\leq$" on X by $x \leq y$ if and only if $x * y = 0$.

In any BCK-algebra X, the following hold:

(P1) $x * 0 = x$,
(P2) $x * y \leq x$,
(P3) $(x * y) * z = (x * z) * y$,
(P4) $(x * z) * (y * z) \leq x * y$,
(P5) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

A BCK-algebra X is said to be *positive implicative* if $(x * z) * (y * z) = (x * y) * z$ for all $x, y, z \in X$. A non-empty subset S of a BCK-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$.

A non-empty subset I of a BCK-algebra X is called an *ideal* of X if

(I1) $0 \in I$,
(I2) $x * y \in I$ and $y \in I$ imply $x \in I$.

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For any $a \in X$ let $A(a)$ denote the set of all elements of X which are less than or equal to a, i.e.,

$$A(a) := \{x \in X \mid x \leq a\}.$$

Note that $0 \in A(a)$, and $A(a)$ is not an ideal of X. Iséki and Tanaka gave a condition for the set $A(a)$ to be an ideal of X as follows:

Proposition 2.1 (Iséki and Tanaka [2, Proposition 2]). *Any set $A(a)$ is an ideal of X if and only if $x * y \leq z$ and $y \leq z$ imply $x \leq z$ for all $x, y, z \in X$.*

3. Main Results

In what follows let X and $\mathbb{N}$ denote a BCK-algebra and the set of all positive integers, respectively, unless otherwise specified. For any x and y of a BCK-algebra X, let $x * y^k$ denote $(\cdots((x * y) * y) * \cdots) * y$ in which y occurs k times, where $k \in \mathbb{N}$.

Definition 3.1. For any $a, b \in X$ and $k \in \mathbb{N}$ we define

$$[a; b^k] := \{x \in X \mid (x * a) * b^k = 0\}.$$

Obviously, $0, a, b \in [a; b^k]$ for all $a, b \in X$ and $k \in \mathbb{N}$.

Proposition 3.2. *Let $a, b \in X$ and $k \in \mathbb{N}$. If $x \in [a; b^k]$, then $x * y \in [a; b^k]$ for all $y \in X$, and so $[a; b^k]$ is a subalgebra of X.*

Proof. Assume that $x \in [a; b^k]$. Then

$$((x * y) * a) * b^k = ((x * a) * y) * b^k = ((x * a) * b^k) * y = 0 * y = 0$$

for all $y \in X$. Hence $x * y \in [a; b^k]$ for all $y \in X$. $\square$

Using (P3) and (IV) we have the following proposition.

Proposition 3.3. *Let $b \in X$ satisfy the equality $x * b = 0$ for all $x \in X$. Then $[a; b^k] = X = [b; a^k]$ for all $a \in X$ and $k \in \mathbb{N}$.*

Note that $[a; 0^k] = A(a)$ for every $a \in X$ and $k \in \mathbb{N}$. If X is positive implicative, then $[a; 0^k]$ is an ideal of X for every $a \in X$ and $k \in \mathbb{N}$ (see Iséki [1]).

The following example shows that there exist $a, b \in X$ and $k \in \mathbb{N}$ such that $[a; b^k]$ is not an ideal of X.

Example 3.4. Consider a BCK-algebra $X = \{0, a, b, c, d\}$ with the following Cayley table.

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	b	0
c	c	a	c	0	a
d	d	d	d	d	0

Then $[d; b^2] = \{0, a, b, d\}$ is not an ideal of X because $c * d = a \in [d; b^2]$ and $d \in [d; b^2]$, but $c \notin [d; b^2]$.

We state a condition for a set $[a; b^k]$ to be an ideal.

Theorem 3.5. *If X is positive implicative, then $[a; b^k]$ is an ideal of X for all $a, b \in X$ and $k \in \mathbb{N}$.*

Proof. Let $x, y \in X$ be such that $x * y \in [a; b^k]$ and $y \in [a; b^k]$. Then

$$\begin{aligned} 0 &= ((x * y) * a) * b^k = (((x * a) * (y * a)) * b) * b^{k-1} \\ &= (((x * a) * b) * ((y * a) * b)) * b^{k-1} \\ &\cdots\cdots\cdots\cdots \\ &= ((x * a) * b^k) * ((y * a) * b^k) \\ &= ((x * a) * b^k) * 0 = (x * a) * b^k, \end{aligned}$$

and so $x \in [a; b^k]$. Therefore $[a; b^k]$ is an ideal of X. □

Using the set $[a; b^k]$ we establish a condition for a subset I of X to be an ideal of X.

Theorem 3.6. *Let I be a non-empty subset of X. Then I is an ideal of X if and only if $[a; b^k] \subseteq I$ for every $a, b \in I$ and $k \in \mathbb{N}$.*

Proof. Assume that I is an ideal of X and let $a, b \in I$ and $k \in \mathbb{N}$. If $x \in [a; b^k]$, then $(x * a) * b^k = 0 \in I$. Since $a, b \in I$, by using (I2) repeatedly we get $x \in I$. Hence $[a; b^k] \subseteq I$.

Conversely, suppose that $[a; b^k] \subseteq I$ for all $a, b \in I$ and $k \in \mathbb{N}$. Note that $0 \in [a; b^k] \subseteq I$. Let $x, y \in X$ be such that $x * y \in I$ and $y \in I$. Then

$$(x * (x * y)) * y^k = ((x * (x * y)) * y) * y^{k-1} = 0 * y^{k-1} = 0,$$

and thus $x \in [x * y; y^k] \subseteq I$. Hence I is an ideal of X □

Since every ideal in a BCK-algebra is a subalgebra, we have the following corollary.

Corollary 3.7. *Every non-empty subset I of X containing $[a; b^k]$ for all $a, b \in I$ and $k \in \mathbb{N}$ is a subalgebra of X.*

Theorem 3.8. *If I is an ideal of X, then $I = \bigcup_{a,b \in I} [a; b^k]$ for every $k \in \mathbb{N}$.*

Proof. Let I be an ideal of X. Then the inclusion $\bigcup_{a,b \in I} [a; b^k] \subseteq I$ follows from Theorem 3.6. Let $x \in I$. Since $x \in [x; 0^k]$, it follows that

$$I \subseteq \bigcup_{x \in I} [x; 0^k] \subseteq \bigcup_{a,b \in I} [a; b^k].$$

This completes the proof. □

We use the notation $x \wedge y$ instead of $y * (y * x)$ for all $x, y \in X$.

Definition 3.9 (Sakuragi [5]). A non-empty subset I of X is called a *quasi-left* (resp. *quasi-right*) *ideal* of X if

(I1) $0 \in I$,

(I3) $x \in I$ and $y \in X$ imply $y \wedge x \in I$ (resp. $x \wedge y \in I$).

If an ideal I is both quasi-left and quasi-right, we say that I is a *quasi-ideal*.

Theorem 3.10. *For every $a, b \in X$ and $k \in \mathbb{N}$, the set $[a; b^k]$ is a quasi-ideal of X.*

Proof. Note that $0 \in [a; b^k]$. Let $x \in [a; b^k]$ and $y \in X$. Then

$$\begin{aligned}((y \wedge x) * a) * b^k &= ((x * (x * y)) * a) * b^k \\ &= ((x * a) * (x * y)) * b^k \\ &= ((x * a) * b^k) * (x * y) \\ &= 0 * (x * y) = 0,\end{aligned}$$

and so $y \wedge x \in [a; b^k]$. Therefore $[a; b^k]$ is a quasi-left ideal of X. Now we get

$$\begin{aligned}((x \wedge y) * a) * b^k &= ((y * (y * x)) * a) * b^k \\ &= ((y * a) * (y * x)) * b^k \\ &\leq (x * a) * b^k = 0,\end{aligned}$$

and hence $((x \wedge y) * a) * b^k = 0$ which shows that $x \wedge y \in [a; b^k]$, i.e., $[a; b^k]$ is a quasi-right ideal of X. This completes the proof. □

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