OPTIMAL FEEDBACK CONTROLS FOR KELLER–SEGEL EQUATIONS

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Abstract. We continue the optimal control problem governed by the Keller-Segel equations. In the previous paper [16], we have proved existence of the optimal controls and the first order necessary condition. Here we will show that the optimal controls satisfy the feedback law expressed by the value function and the value function is a solution to the Hamilton-Jacobi equation in a weak sense.

1. Introduction

This paper is concerned with the optimal feedback control for the following problem:

(P) $$\underset{u}{\text{Minimize}} \quad J(u)$$

with the cost functional $J(u)$ of the form

$$J(u) = \frac{1}{2}\int_0^T \|y(u) - y_d\|^2_{L^2(\Omega)} dt + \gamma \int_0^T \|u(t)\|^2_E dt,\ u(t) \in L^2(0,T;E),$$

where $y = y(u)$ is governed by the Keller-Segel equations

(K–S) $$\begin{cases} \dfrac{\partial y}{\partial t} = a\Delta y - b\nabla\{y\nabla\rho\} & \text{in } \Omega \times (0,T], \\ \dfrac{\partial \rho}{\partial t} = d\Delta\rho + fy - g\rho + \nu u & \text{in } \Omega \times (0,T], \\ \dfrac{\partial y}{\partial n} = \dfrac{\partial \rho}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T], \\ y(x,0) = y_0(x), \qquad \rho(x,0) = \rho_0(x) & \text{in } \Omega. \end{cases}$$

Here, Ω is a bounded region in $\mathbb{R}^2$ of $\mathcal{C}^3$ class. a, b, d, f, $g > 0$ are given positive numbers and γ, $\nu \geq 0$ are given non negative numbers. $u(t) \geq 0$ is a control function in some bounded convex subset E_{ad} of $E = H^{1+\varepsilon}(\Omega)$, $0 < \varepsilon < \frac{1}{2}$. $n = n(x)$ is the outer normal vector at a boundary point $x \in \partial\Omega$ and $\frac{\partial}{\partial n}$ denotes differentiation along a vector n. $y_0(x)$, $\rho_0(x) \geq 0$ are non negative initial functions in $L^2(\Omega)$ and in $H^{1+\varepsilon}(\Omega)$, respectively. y, ρ are unknown functions of the Cauchy problem (K–S).

The Keller-Segel equations were introduced in [11] to describe the aggregation process of the cellular slime mold by the chemical attraction. $y = y(x,t)$ denotes the concentration of amoebae in Ω at the time t, and $\rho = \rho(x,t)$ the concentration of chemical substance in

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Ω at the time t. The chemotactic term $-b\nabla \cdot \{y\nabla\rho\}$ indicates that the cells are sensitive to chemicals and are attracted by them, and the production term fy indicates that the chemical substance is itself emitted by cells. (K–S) is then a strongly coupled reaction diffusion system.

Several authors have already been interested in the equations, the existence and uniqueness of solution and the asymptotic behavior of solution were studied by them in the case when (K–S) has no control term, $u \equiv 0$ (see e.g. [10, 15, 19]). In the previous paper [16] of the present authors, we have already shown the existence and uniqueness of non negative local weak solutions by using the Galerkin method and the classical compact method (see [12, Chap. 1] and [7, Chap. III]), and have studied the existence of optimal controls and the necessary conditions of optimality for (P) with a sufficiently small $T > 0$.

Aggregation of cellular slime mold is known as a model of the self-organization by cell interaction mediated by the chemical substance called cAMP. In this paper, we are concerned with the question of whether the optimal control at the time t can be determined by the concentration of amoebae and the concentration of cAMP at the same time t or not. For simplicity we shall show that the optimal control is a feedback optimal control expressed by the value function and the value function is a solution to the Hamilton-Jacobi equation (H-J equation).

According to the classical Hamilton-Jacobi theory, if the value function is smooth, then it is a solution of the H-J equation, see e.g. [8, 13]. In general, however, this is not the case and therefore the value function can not satisfy the H-J equation in the classical sense. In the control theory, several devices have been developed to overcome this difficulty, see [1, 2, 3, 5, 6]. In this paper, we will follow the method developed in Barbu [1] (see [2]) in which semilinear equations of monotone type were dealt. Although (K–S) contains an unbounded non-monotone term, the present work may also be considered as a generalization of [1].

This paper is organized as follows. In section 3, we recall some known results on local solutions of (K–S) together with their regularity properties. Section 4 is devoted to presenting a pointwise necessary condition for optimality. In Section 5, it is proved that every optimal control u for the problem (P) is a feedback optimal control. Furthermore, it is shown that the value function is a solution of the H-J equation in a certain weak sense.

Notations. $\mathbb{N}$ and $\mathbb{R}$ denote the sets of natural numbers and real numbers respectively, and $\mathbb{R}_+ = \{x \in \mathbb{R}; x \geq 0\}$. For a region $\Omega \subset \mathbb{R}^2$, the usual L^p space of real valued functions in Ω is denoted by $L^p(\Omega)$, $1 \leq p \leq \infty$. The real Sobolev space in Ω with an exponent $s \geq 0$ is denoted by $H^s(\Omega)$. $\mathcal{C}(\overline{\Omega})$ denotes the space of continuous functions on $\overline{\Omega}$. Let I be an interval in $\mathbb{R}$. $L^p(I;\mathcal{H})$, $1 \leq p \leq \infty$, denotes the L^p space of measurable functions in I with values in a Hilber space $\mathcal{H}$. $\mathcal{C}(I;\mathcal{H})$ denotes the space of continuous functions in I with values in $\mathcal{H}$. For simplicity, we shall use a universal constant C to denote various constants which are determined in each occurrence in a specific way by $\Omega, a, b, d, f, g, \varepsilon, \nu, \delta, M$, and so forth. In a case when C depends also on some parameter, say θ, it will be denoted by C_θ.

2. Preliminary

We shall state some well known results on the Sobolev spaces and on the fractional powers of Laplacian which will be used in this paper. For the proof, we refer the reader to Brezis [4], Friedman [9], Lions & Magenes [14], and Triebel [18].

Let $0 \leq s_0 < s_1 < \infty$. For $s_0 < s < s_1$, $H^s(\Omega) = [H^{s_0}(\Omega), H^{s_1}(\Omega)]_\theta$ with $s = (1-\theta)s_0 + \theta s_1$, and the following estimate holds:

$$\|\cdot\|_{H^s} \leq C_{s_0,s_1}\|\cdot\|_{H^{s_0}}^{1-\theta}\|\cdot\|_{H^{s_1}}^{\theta}. \tag{2.1}$$

When $0 < s < 1$, $H^s(\Omega) \subset L^p(\Omega)$ for $\frac{1}{p} = \frac{1-s}{2}$ with the estimate

$$\|\cdot\|_{L^p} \le C_s \|\cdot\|_{H^s}. \tag{2.2}$$

When $s = 1$, $H^1(\Omega) \subset L^q(\Omega)$ for any finite $1 \le q < \infty$ with the estimate

$$\|\cdot\|_{L^q} \le C_{q,p} \|\cdot\|_{H^1}^{1-p/q} \|\cdot\|_{L^p}^{p/q}, \tag{2.3}$$

where $1 \le p < q$. When $s > 1$, $H^s(\Omega) \subset \mathcal{C}(\overline{\Omega})$ with the estimate

$$\|\cdot\|_{\mathcal{C}} \le C_s \|\cdot\|_{H^s}. \tag{2.4}$$

Furthermore, we shall use the following estimates which are easily obtained by utilitizing (2.1) $\sim$ (2.4).

Let $0 < \varepsilon \le 1$. For any $0 \le \theta \le 1$,

$$\begin{cases} \|uv\|_{H^\theta} \le C_\varepsilon \|u\|_{H^{1+\varepsilon}} \|v\|_{H^\theta}, & u \in H^{1+\varepsilon}(\Omega),\ v \in H^\theta(\Omega), \\ \|uv\|_{H^\theta} \le C_\varepsilon \|u\|_{H^1} \|v\|_{H^{\varepsilon+\theta}}, & u \in H^1(\Omega),\ v \in H^{\varepsilon+\theta}(\Omega). \end{cases} \tag{2.5}$$

In fact, when $\theta = 0$ or $\theta = 1$, these estimates are verified directly from (2.2), (2.3) and (2.4). For $0 < \theta < 1$, the estimates are then obtained by the interpolation theorem applied to the operator of multiplication $v \mapsto uv$ and by (2.1). In particular, it follows that

$$\begin{cases} \|uv\|_{H^\varepsilon} \le C_\varepsilon \|u\|_{H^{1+\varepsilon}} \|v\|_{H^\varepsilon}, & u \in H^{1+\varepsilon}(\Omega),\ v \in H^\varepsilon(\Omega), \\ \|uv\|_{H^\varepsilon} \le C_\varepsilon \|u\|_{H^1} \|v\|_{H^{2\varepsilon}}, & u \in H^1(\Omega),\ v \in H^{2\varepsilon}(\Omega). \end{cases} \tag{2.6}$$

Applying these we observe also that

$$\begin{cases} \|\nabla\{u\nabla\rho\}\|_{L^2} \le C_\varepsilon \|u\|_{H^{1+\varepsilon}} \|\rho\|_{H^2}, & u \in H^{1+\varepsilon}(\Omega),\ \rho \in H^2(\Omega), \\ \|\nabla\{u\nabla\rho\}\|_{L^2} \le C_\varepsilon \|u\|_{H^1} \|\rho\|_{H^{2+\varepsilon}}, & u \in H^1(\Omega),\ \rho \in H^{2+\varepsilon}(\Omega), \\ \|\nabla\{u\nabla\rho\}\|_{H^1} \le C \|u\|_{H^2} \|\rho\|_{H^3}, & u \in H^2(\Omega),\ \rho \in H^3(\Omega). \end{cases} \tag{2.7}$$

Let $\overline{L} = -\Delta + 1$ be the Laplace operator equipped with the Neumann boundary conditions, $\overline{L}$ is an isomorphism from $H^1(\Omega)$ to $(H^1(\Omega))'$. The part of $\overline{L}$ in $L^2(\Omega)$ is denoted by L, L is a positive definite self-adjoint operator in $L^2(\Omega)$ with the domain $\mathcal{D}(L) = \{y \in H^2(\Omega); \frac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega\}$. It is known that

$$\begin{cases} \mathcal{D}(L^\theta) = H^{2\theta}(\Omega), & 0 \le \theta < \dfrac{3}{4}, \\ \mathcal{D}(L^\theta) = H_n^{2\theta}(\Omega) = \{y \in H^{2\theta}(\Omega); \dfrac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega\}, & \dfrac{3}{4} < \theta \le \dfrac{3}{2} \end{cases} \tag{2.8}$$

with norm equivalence. In fact, (2.8) is well known for $0 \le \theta \le 1$ (even for $\theta = \frac{3}{4}$, a characterization of $\mathcal{D}(L^{3/4})$ is known). Since Ω is of $\mathcal{C}^3$ class, $\mathcal{D}(L^{3/2}) = L^{-1}(H^1(\Omega)) = H_n^3(\Omega)$; then, for $1 \le \theta \le \frac{3}{2}$, (2.8) is verfied from the fact that $\mathcal{D}(L^\theta) = [\mathcal{D}(L), \mathcal{D}(L^{3/2})]_\mu$ with $\theta = 1 + \frac{\mu}{2}$. Clearly, L is a linear isomorphism from $\mathcal{D}(L^{\theta+1})$ to $\mathcal{D}(L^\theta)$ for $\theta \ge 0$.

3. The weak formulation of problem and the known results

Let us briefly recall the way how to formulate (K–S) as a semilinear abstract differential equation in a Hilbert space. Let $\overline{A}_1 = -a\Delta + a$ and $\overline{A}_2 = -d\Delta + g$ be the Laplace operators equipped with the Neumann boundary conditions, $\overline{A}_i (i = 1, 2)$ are linear isomorphisms from $H^1(\Omega)$ to $(H^1(\Omega))'$. The part of $\overline{A}_i$ in $L^2(\Omega)$ is denoted by A_i, $A_i (i = 1, 2)$ is a positive definite self-adjoint operator in $L^2(\Omega)$ with the domain $\mathcal{D}(A_i) = H^2_n(\Omega)$. As noticed in (2.8), $\mathcal{D}(A_i^\theta) = H^{2\theta}(\Omega)$ for $0 \le \theta < \frac{3}{4}$, and $\mathcal{D}(A_i^\theta) = H_n^{2\theta}(\Omega)$ for $\frac{3}{4} < \theta \le \frac{3}{2}$.

We introduce two product Hilbert spaces $\mathcal{V} \subset \mathcal{H}$ as

$$\mathcal{V} = H^1(\Omega) \times \mathcal{D}(A_2^{1+\varepsilon/2}) \quad \text{and} \quad \mathcal{H} = L^2(\Omega) \times \mathcal{D}(A_2^{(1+\varepsilon)/2}),$$

respectively, where ε is some fixed exponent $\varepsilon \in (0, \frac{1}{2})$. By the identification of $\mathcal{H}$ and its dual $\mathcal{H}'$, we have: $\mathcal{V} \subset \mathcal{H} = \mathcal{H}' \subset \mathcal{V}'$. It is then seen that

$$\mathcal{V}' = (H^1(\Omega))' \times \mathcal{D}(A_2^{\varepsilon/2})$$

with the duality product

$$\langle \Phi, Y \rangle_{\mathcal{V}' \times \mathcal{V}} = \langle \zeta, y \rangle_{(H^1)' \times H^1} + \big(A_2^{\varepsilon/2}\varphi, A_2^{1+\varepsilon/2}\rho\big)_{L^2}, \ \Phi = \begin{pmatrix} \zeta \\ \varphi \end{pmatrix} \in \mathcal{V}', \ Y = \begin{pmatrix} y \\ \rho \end{pmatrix} \in \mathcal{V}.$$

We denote the scalar product of $\mathcal{H}$ by $(\cdot, \cdot)$ and the norm by $|\cdot|$. The duality product between $\mathcal{V}'$ and $\mathcal{V}$ which coincides with the scalar product of $\mathcal{H}$ on $\mathcal{H} \times \mathcal{H}$ is denoted by $\langle \cdot, \cdot \rangle$, and the norms of $\mathcal{V}$ and $\mathcal{V}'$ by $\|\cdot\|$ and $\|\cdot\|_*$, respectively.

We set also a symmetric sesquilinear form on $\mathcal{V} \times \mathcal{V}$:

$$a(Y, \tilde{Y}) = \big(A_1^{1/2}y, A_1^{1/2}\tilde{y}\big)_{L^2} + \big(A_2^{1+\varepsilon/2}\rho, A_2^{1+\varepsilon/2}\tilde{\rho}\big)_{L^2}, \qquad Y = \begin{pmatrix} y \\ \rho \end{pmatrix}, \tilde{Y} = \begin{pmatrix} \tilde{y} \\ \tilde{\rho} \end{pmatrix} \in \mathcal{V}.$$

Obviously, the form satisfies:

(a.i) $$|a(Y, \tilde{Y})| \le M\|Y\|\|\tilde{Y}\|, \qquad Y, \tilde{Y} \in \mathcal{V},$$

(a.ii) $$a(Y, Y) \ge \delta\|Y\|^2, \qquad Y \in \mathcal{V}$$

with some constants $M \ge 0$ and $\delta > 0$. This form then defines a linear isomorphism $\overline{A} = \begin{pmatrix} \overline{A}_1 & 0 \\ 0 & A_2 \end{pmatrix}$ from $\mathcal{V}$ to $\mathcal{V}'$, and the part A of $\overline{A}$ in $\mathcal{H}$ is a positive definite self-adjoint operator in $\mathcal{H}$ with the domain $\mathcal{D}(A) = \mathcal{D}(A_1) \times \mathcal{D}(A_2^{(3+\varepsilon)/2})$.

(K–S) is, then, formulated as an abstract equation

(E) $$\begin{cases} \dfrac{dY}{dt} + \overline{A}Y = F(Y) + U(t), \qquad 0 < t \le T, \\ Y(0) = Y_0 \end{cases}$$

in the space $\mathcal{V}'$. Here, $F(\cdot) : \mathcal{V} \to \mathcal{V}'$ is the mapping

$$F(Y) = \begin{pmatrix} -b\nabla\{y\nabla\rho\} + ay \\ fy \end{pmatrix}, \qquad Y = \begin{pmatrix} y \\ \rho \end{pmatrix} \in \mathcal{V}.$$

$U(t)$ and Y_0 are defined by $U(t) = \binom{0}{\nu u(t)}$ and $Y_0 = \binom{y_0}{\rho_0}$, respectively.

As verified in [16, Sec. 2], $F(\cdot)$ satisfies the following conditions.

(f.i)
$$\begin{cases} \|F(Y)\|_* \le \eta\|Y\| + \phi_\eta(|Y|), & Y \in \mathcal{V}, \\ |F(Y)| \le \eta\|Y\|_{\mathcal{D}(A)} + \phi_\eta(\|Y\|), & Y \in \mathcal{D}(A) \end{cases}$$

for each $\eta > 0$ with some increasing continuous function $\phi_\eta(\cdot)$.

(f.ii)
$$\begin{cases} \|F(\widetilde{Y}) - F(Y)\|_* \le \eta\|\widetilde{Y} - Y\| \\ \qquad + (\|\widetilde{Y}\| + \|Y\| + 1)\psi_\eta(|\widetilde{Y}| + |Y|)|\widetilde{Y} - Y|, \quad \widetilde{Y}, Y \in \mathcal{V}, \\ |F(\widetilde{Y}) - F(Y)| \le \eta\|\widetilde{Y} - Y\|_{\mathcal{D}(A)} \\ \qquad + (\|\widetilde{Y}\|_{\mathcal{D}(A)} + \|Y\|_{\mathcal{D}(A)} + 1)\psi_\eta(\|\widetilde{Y}\| + \|Y\|)\|\widetilde{Y} - Y\|, \quad \widetilde{Y}, Y \in \mathcal{D}(A) \end{cases}$$

for each $\eta > 0$ with some increasing continuous function $\psi_\eta(\cdot)$.

Furthermore, $F(Y)$ is Fréchet differentiable with the derivative

$$F'(Y)Z = \begin{pmatrix} -b\nabla\{y\nabla w\} - b\nabla\{z\nabla\rho\} + az \\ fz \end{pmatrix}, \quad Y = \begin{pmatrix} y \\ \rho \end{pmatrix}, \; Z = \begin{pmatrix} z \\ w \end{pmatrix} \in \mathcal{V}.$$

$F'(\cdot)$ satisfies the following estimates.

(f.iii)
$$|\langle F'(Y)Z, P\rangle| \le \begin{cases} \eta\|Z\|\|P\| + C_\eta(\|Y\| + 1)|Z|\|P\|, & Y, Z, P \in \mathcal{V}, \\ \eta\|Z\|\|P\| + C_\eta(\|Y\| + 1)\|Z\||P|, & Y, Z, P \in \mathcal{V} \end{cases}$$

for each $\eta > 0$ with some constant C_η, and

(f.iv)
$$\|F'(Y)Z\|_{(D(A))'} \le \eta|Z| + \mu_\eta(|Y|)\|Z\|_*, \qquad Y, Z \in \mathcal{V}$$

for each $\eta > 0$ with some increasing continuous function $\mu_\eta(\cdot)$. Furthermore,

(f.v)
$$\|F'(\tilde{Y})Z - F'(Y)Z\|_* \le C\|Z\||\tilde{Y} - Y|, \qquad \tilde{Y}, Y, Z \in \mathcal{V}.$$

Indeed, (f.iii) and (f.v) were also verified in [16, Sec. 5]. So we have only to verify (f.iv). Let $y \in H^1(\Omega)$ and $w \in H_n^{2+\varepsilon}(\Omega)$. Then, since $y\nabla w \in H^1(\Omega)$, it follows from (2.1) and (2.2) that

$$\begin{aligned} \|\nabla\{y\nabla w\}\|_{(\mathcal{D}(A_1))'} &= \sup_{\|v\|_{\mathcal{D}(A_1)} \le 1} \left|\int_\Omega y\nabla w\nabla v dx\right| \\ &\le \sup_{\|v\|_{\mathcal{D}(A_1)} \le 1} \|y\|_{L^2}\|\nabla w\|_{L^{4/(2-\varepsilon)}}\|\nabla v\|_{L^{4/\varepsilon}} \le C\|y\|_{L^2}\|w\|_{H^{\varepsilon/2+1}} \\ &\le C\|y\|_{L^2}\|w\|_{H^{\varepsilon+1}}^{1-\varepsilon/2}\|w\|_{H^\varepsilon}^{\varepsilon/2} \le \eta\|w\|_{H^{\varepsilon+1}} + C_\eta\|y\|_{L^2}^{2/\varepsilon}\|w\|_{H^\varepsilon}, \end{aligned}$$

where $\eta > 0$ is arbitrary. On the other hand, let $z \in H^1(\Omega)$ and $\rho \in H_n^{2+\varepsilon}(\Omega)$. Then, it follows from (2.5) and (2.6) that

$$\begin{aligned} \|\nabla\{z\nabla\rho\}\|_{(\mathcal{D}(A_1))'} &= \sup_{\|v\|_{\mathcal{D}(A_1)} \le 1} |\langle\nabla\{z\nabla\rho\}, v\rangle_{(\mathcal{D}(A_1))' \times \mathcal{D}(A_1)}| \\ &= \sup_{\|v\|_{\mathcal{D}(A_1)} \le 1} |\langle z, \nabla\rho\cdot\nabla v\rangle_{(\mathcal{D}(A_1^{\bar\varepsilon/2}))' \times \mathcal{D}(A_1^{\bar\varepsilon/2})}| \le \sup_{\|v\|_{\mathcal{D}(A_1)} \le 1} C\|z\|_{(\mathcal{D}(A_1^{\bar\varepsilon/2}))'}\|\nabla\rho\cdot\nabla v\|_{H^{\bar\varepsilon}} \\ &\le C\|z\|_{L^2}^{1-\bar\varepsilon}\|z\|_{(\mathcal{D}(A_1^{1/2}))'}^{\bar\varepsilon}\|\rho\|_{H^{1+\bar\varepsilon+\delta}} \le \eta\|z\|_{L^2} + C_\eta\|\rho\|_{H^{1+\varepsilon}}^{1/\varepsilon}\|z\|_{\mathcal{D}(A_1^{1/2})'}, \end{aligned}$$

where $\eta > 0$ is arbitrary. Then, (f.iv) is an immediate consequence of these estimates.

According to [16, Theorem 2.1], for $U \in L^2(0,T;\mathcal{V}')$ and $Y_0 \in \mathcal{H}$, there exists a unique solution to (E) on an interval $[0, T(Y_0,U)]$ such that

$$Y \in L^2(0,T(Y_0,U);\mathcal{V}) \cap C([0,T(Y_0,U)];\mathcal{H}) \cap H^1(0,T(Y_0,U);\mathcal{V}'), \tag{3.1}$$

the number $T(Y_0,U) > 0$ being determined by the norms $\|U\|_{L^2(0,T;\mathcal{V}')}$ and $|Y_0|$.

Furthermore, if we handle (E) in the spaces $\mathcal{D}(A) \subset \mathcal{V} \subset \mathcal{H}$, then (f.i) and (f.ii) yield the following reguality result. For $Y_0 \in \mathcal{V}$ and $U \in L^2(0,T;\mathcal{H})$, there exists a unique solution to (E) on an interval $[0,T(Y_0,U)]$ such that

$$Y \in H^1(0,T(Y_0,U);\mathcal{H}) \cap C([0,T(Y_0,U)];\mathcal{V}) \cap L^2(0,T(Y_0,U);\mathcal{D}(A)), \tag{3.2}$$

the number $T(Y_0,U) > 0$ being determined by the norms $\|U\|_{L^2(0,T;\mathcal{H})}$ and $\|Y_0\|$.

Similarly, handling (E) in the spaces $\mathcal{D}(A^{\frac{3}{2}}) \subset \mathcal{D}(A) \subset \mathcal{V}$, we verify the following result. For $Y_0 \in \mathcal{D}(A)$ and $U \in L^2(0,T;\mathcal{V})$, there exists a unique solution to (E) on an interval $[0,T(Y_0,U)]$ such that

$$Y \in H^1(0,T(Y_0,U);\mathcal{V}) \cap C([0,T(Y_0,U)];\mathcal{D}(A)) \cap L^2(0,T(Y_0,U);\mathcal{D}(A^{3/2})), \tag{3.3}$$

the number $T(Y_0,U) > 0$ being determined by the norms $\|U\|_{L^2(0,T;\mathcal{V})}$ and $\|Y_0\|_{\mathcal{D}(A)}$.

Finally we present some uniform estimate for the solutions to (E) which was essentially established in the proof of [16, Theorem 2.1].

Theorem 3.1. *Let $r_0 > 0$ and $R > 0$, then there exists a number $S > 0$ and an increasing continuous function $r(t)$ defined on $[0,S]$ such that the following statement is true. If $Y_0 \in \mathcal{H}$ and $U \in L^2(0,T;\mathcal{V}')$ satisfy $|Y_0|^2 < r_0$ and $\operatorname*{ess.sup}_{0<t<T} \|U(t)\|_* < R$, respectively, then the number $T(Y_0,U)$ in (3.1) is larger than S and the estimate $|Y(t)|^2 < r(t)$ holds for every $t \in [0,S]$.*

Proof. Let $Y_0 \in \mathcal{H}$ and $U \in L^2(0,T;\mathcal{V}')$ satisfy $|Y_0|^2 < r_0$ and $\operatorname*{ess.sup}_{0<t<T} \|U(t)\|_* < R$, respectively, and let $Y(t)$ be the solution of (E). As shown in the proof of [16, Theorem 2.1], $Y(t)$ satisfies the following estimate

$$\frac{1}{2}\frac{d}{dt}|Y(t)|^2 \le \tilde{\phi}(|Y(t)|^2) + C\|U(t)\|_*^2$$

with some increasing, locally Lipschitz continuous function $\tilde{\phi}\colon [0,\infty) \to [0,\infty)$ determined by ϕ_η.

Then, let $r(\cdot)$ be the solution to the following differential equation:

$$\begin{cases} \dfrac{dr}{dt} = 2\tilde{\phi}(r) + 2CR, \qquad 0 < t \le T, \\ r(0) = r_0. \end{cases} \tag{3.4}$$

$r(\cdot)$ is then defined on an interval $[0,S]$, $S < T$, determined by r_0, $\tilde{\phi}$ and R. The comparision theorem then yields that the solution $Y(t)$ exists at least on $[0,S]$ and the estimate $|Y(t)|^2 < r(t)$ holds for $0 \le t \le S$. $\square$

4. Optimality conditions

Let $\mathcal{H}$ be the Hilbert space defined in Section 3. In $\mathcal{H}$, we consider an optimal control problem

$$(\overline{P}) \qquad \underset{U\in\mathcal{U}_{ad}}{\text{Minimize}} \;\; J(U)$$

with the cost functional

$$J(U) = \frac{1}{2}\int_0^S |Y(U) - Y_d|^2 dt + \gamma \int_0^S |U|^2 dt,$$

where $Y_0 \in \mathcal{H}$ and $Y_d = \binom{y_d}{0} \in L^2(0,T;\mathcal{H})$ are given and

$$U \in \mathcal{U}_{ad} = \left\{U \in L^2(0,T;\mathcal{H}); U(t) \in \mathcal{C} \text{ a.e.}\right\},$$

$\mathcal{C} \subset \mathcal{H}$ being a closed, bounded and convex set containing the orign, and where $Y(U)$, $U \in \mathcal{U}_{ad}$, is the solution to (E) on $[0,S]$. In view of Theorem 3.1, we know that there exists a solution $Y(t)$ of (E) on $[0,S]$ such that the estimate $|Y(t)|^2 < r(t)$ holds. Here, $r(t)$ is an increasing continuous solution $r(t)$ of (3.4) on $[0,S]$. We consider the increasing open set $B_t = \{\xi \in \mathcal{H}; |\xi|^2 < r(t),\ 0 \le t \le S\}$.

For each $0 \le t \le S$ and $\xi \in B_t$, we consider an auxiliary optimal control problem:

$$(\overline{P}_{t,\xi}) \qquad \underset{U\in\mathcal{U}_{ad}}{\text{Minimize}} \;\; J_{t,\xi}(U)$$

with the cost functional

$$J_{t,\xi}(U) = \frac{1}{2}\int_t^S |Y(s;t,\xi,U) - Y_d|^2 ds + \gamma \int_t^S |U|^2 ds,$$

where $Y(s;t,\xi,U)$ is the solution to the following equation

$$(E_{t,\xi}) \qquad \begin{cases} \dfrac{dY}{ds} + \overline{A}Y = F(Y) + U(s), & t < s \le S, \\ Y(t) = \xi. \end{cases}$$

As before, for $0 \le t \le S$ and $\xi \in B_t$, there exists the solution $r(s)$ on $[t,S]$ to (3.4) with initial condition $r(t) = r(t;0,r_0) > |\xi|^2$. Therefore, for $0 \le t < S$, $U \in \mathcal{U}_{ad}$ and $\xi \in B_t$, there exists a unique solution $Y(s) = Y(s;t,\xi,U)$ to $(E_{t,\xi})$ on $[t,S]$ such that $Y \in L^2(t,S;\mathcal{V}) \cap C([t,S];\mathcal{H}) \cap H^1(t,S;\mathcal{V}')$ and the estimate $|Y(s)| < r(s),\ s \in [t,S]$, holds. Moreover, in the same way as in [16, Sec. 5], the differntiability of $Y(U)$ with respect to U and the optimality condition are verified. Therefore, the following necessary condition is true.

Let $\overline{U}$ be an optimal control of $(\overline{P}_{t,\xi})$ and let $\overline{Y} \in L^2(t,S;\mathcal{V}) \cap \mathcal{C}([t,S];\mathcal{H}) \cap H^1(t,S;\mathcal{V}')$ be the optimal state, that is $\overline{Y}(s) = Y(s;t,\xi,\overline{U})$ is the solution to $(E_{t,\xi})$ with the control $\overline{U}$. Then, there exists a unique solution $P \in L^2(t,S;\mathcal{V}) \cap \mathcal{C}([t,S];\mathcal{H}) \cap H^1(t,S;\mathcal{V}')$ to the linear problem

$$(4.1) \qquad \begin{cases} \dfrac{dP}{ds} - \overline{A}P + F'(\overline{Y})^*P = \overline{Y} - Y_d, & t \le s < S, \\ P(S) = 0, \end{cases}$$

and $\overline{U}$ and P satisfy the inequality

$$(4.2) \qquad \int_t^S (-P + 2\gamma\overline{U}, V - \overline{U})dt \ge 0 \qquad \text{for all } V \in \mathcal{U}_{ad}.$$

Moreover, the following pointwise necessary condition for optimality is also deduced.

Theorem 4.1. *Let $\overline{U}$ be an optimal control of $(\overline{P}_{t,\xi})$ and let $\overline{Y} \in L^2(t,S;\mathcal{V}) \cap \mathcal{C}([t,S];\mathcal{H}) \cap H^1(t,S;\mathcal{V}')$ be the optimal state. Then,*

$$\overline{U}(s) = \mathbb{P}_{\mathcal{C}}(P(s)), \qquad \text{a.e. } s \in [t,S], \tag{4.3}$$

where $P(s) = P(s;t,\xi)$ is given by (4.1). Here, $\mathbb{P}_{\mathcal{C}} = (2\gamma 1 + \partial I_{\mathcal{C}})^{-1}$ denotes the projection of $\mathcal{H}$ onto $\mathcal{C}$ and $\partial I_{\mathcal{C}}$ is the subdifferential of the indicator function $I_{\mathcal{C}}$ of $\mathcal{C}$:

$$I_{\mathcal{C}}(W) = \begin{cases} 0 & \text{if } W \in \mathcal{C} \\ +\infty & \text{if } W \in \mathcal{C}^c. \end{cases}$$

Proof. Let $s \in I = [t,S],\ 0 < \varepsilon < S - s$. Let $W \in \mathcal{C}$ be arbitrary and define

$$V_\varepsilon(s) = \begin{cases} \overline{U}(\tau) & \text{if } \tau \in I - [s, s+\varepsilon], \\ W & \text{if } \tau \in (s, s+\varepsilon). \end{cases}$$

Clearly, $V_\varepsilon \in \mathcal{U}_{ad}$. Substituting V_ε for V in (4.2) and dividing the resulting inequality by ε, we see that

$$\frac{1}{\varepsilon}\int_s^{s+\varepsilon} (-P(\tau) + 2\gamma\overline{U}(\tau), W - \overline{U}(\tau))d\tau \geq 0. \tag{4.4}$$

Since all the integrands in (4.4) are Lebesgue integrable on I, the Lebesgue density theorem [17, p. 17] is available. Then, by letting $\varepsilon \to 0$ in (4.4), we have:

$$\big(-P(s) + 2\gamma\overline{U}(s), W - \overline{U}(s)\big) \geq 0, \qquad \text{a.e. } s \in I.$$

Since $W \in \mathcal{C}$ is arbitrary, we see that

$$P(s) - 2\gamma\overline{U}(s) \in \partial I_{\mathcal{C}}(\overline{U}(s)), \qquad \text{a.e. } s \in I,$$

that is, $P(s) \in (\partial I_{\mathcal{C}} + 2\gamma 1)(\overline{U}(s))$, a.e. $s \in I$. Hence, we prove the desired result (4.3). □

Remark 4.2. *Let $\mathcal{C} = \{U \in \mathcal{H} : |U| \leq R\}$ be a ball, where R is given constant. Then, we have:*

$$\mathbb{P}_{\mathcal{C}}(U) = \begin{cases} U & \text{if } |U| \leq R, \\ R\dfrac{U}{|U|} & \text{if } |U| > R. \end{cases}$$

5. Optimal feedback control

In this section we shall investigate properties of the value function and consider the feedback problem. The main result is that the value function ψ satisfies a Hamilton-Jacobi equation in a generalized sense. For simplicity, we shall assume in this section that $Y_d = 0$ and $\gamma = \frac{1}{2}$.

As noticed in Section 4, the value function $\psi(t,\xi)$ is defined for $(t,\xi) \in [0,S] \times B_t$ by

$$\psi(t,\xi) = \inf_{U \in \mathcal{U}_{ad}} J_{t,\xi}(U).$$

We begin with a series of lemmas showing regularity properties of the trajectory Y and the value function ψ.

Lemma 5.1. *Let $\xi, \zeta \in \mathcal{B}_t$ and $U \in \mathcal{U}_{ad}$. Then we have:*

$$|Y(s;t,\xi,U) - Y(s;t,\zeta,U)|^2 + \int_t^s \|Y(\tau;t,\xi,U) - Y(\tau;t,\zeta,U)\|^2 d\tau \le C|\xi - \zeta|^2, \qquad t \le s \le S. \tag{5.1}$$

Proof. Let Y and $\widetilde{Y}$ be solutions of $(E_{t,\xi})$ and $(E_{t,\zeta})$, respectively. Then it is seen that $W = \widetilde{Y} - Y$ satisfies:

$$\begin{cases} \dfrac{dW(s)}{ds} + \overline{A}W(s) = F(\widetilde{Y}(s)) - F(Y(s)), \qquad t < s \le S, \\ W(t) = \xi - \zeta. \end{cases} \tag{5.2}$$

Taking the scalar product of the equation of (5.2) with W, we have:

$$\frac{1}{2}\frac{d}{ds}|W(s)|^2 + \langle \overline{A}W(s), W(s)\rangle = \langle F(\widetilde{Y}(s)) - F(Y(s)), W(s)\rangle..$$

From (a.ii) and (f.ii), it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{ds}|W(s)|^2 + \delta\|W(s)\|^2 \\ &\quad \le \eta\|W(s)\|^2 + (\|\widetilde{Y}(s)\| + \|Y(s)\| + 1)\psi_\eta(|\widetilde{Y}(s)| + |Y(s)|)|W(s)|\|W(s)\| \\ &\qquad \le \frac{\delta}{2}\|W(s)\|^2 + C(\|\widetilde{Y}(s)\|^2 + \|Y(s)\|^2 + 1)\psi_{\frac{\delta}{4}}(|\widetilde{Y}(s)| + |Y(s)|)^2|W(s)|^2. \end{aligned} \tag{5.3}$$

Therefore, by Gronwall's lemma,

$$|W(s)|^2 \le |W(t)|^2 e^{\int_t^S C(\|\widetilde{Y}(s)\|^2 + \|Y(s)\|^2 + 1)\psi_{\frac{\delta}{4}}(|\widetilde{Y}(s)| + |Y(s)|)^2 ds}.$$

Using this result in (5.3) and integrating from t to s, we obtain the estimate for

$$\int_t^s \|Y(\tau;t,\xi,U) - Y(\tau;t,\zeta,U)\|^2 d\tau. \quad \square$$

Lemma 5.2. *For each $t \in [0, S]$, $\psi(t, \cdot)$ is Lipschitz continuous in $\xi \in B_t$.*

Proof. Let ξ_i $(i = 1, 2) \in B_t$ and let $\overline{U} \in \mathcal{U}_{ad}$ be an optimal control for $(\overline{P}_{t,\xi_2})$ such that

$$\psi(t, \xi_2) = \frac{1}{2}\int_t^S |Y(s;t,\xi_2,\overline{U})|^2 ds + \frac{1}{2}\int_t^S |\overline{U}|^2 ds.$$

By the definition of ψ,

$$\begin{aligned} &\psi(t,\xi_1) - \psi(t,\xi_2) \\ &\qquad \le \frac{1}{2}\int_t^S |Y(s;t,\xi_1,\overline{U})|^2 - |Y(s;t,\xi_2,\overline{U})|^2 ds \\ &\qquad = \frac{1}{2}\int_t^S (Y(s;t,\xi_1,\overline{U}) - Y(s;t,\xi_2,\overline{U}), Y(s;t,\xi_1,\overline{U}) + Y(s;t,\xi_2,\overline{U}))ds. \end{aligned}$$

By using (5.1), we have:

$$\begin{aligned} |\psi(t,\xi_1) - \psi(t,\xi_2)| &\le C\int_t^S |Y(s;t,\xi_1,\overline{U}) - Y(s;t,\xi_2,\overline{U})| ds \\ &\le C|\xi_1 - \xi_2|, \qquad \forall \xi_1,\ \xi_2 \in B_t. \quad \square \end{aligned}$$

Lemma 5.3. *For each $\xi \in \mathcal{D}(A) \cap B_t$, $\psi(\cdot, \xi)$ is Lipschitz continuous in $\tau \in [t, S]$.*

Proof. From (3.3), there exists some $\tau_2 \in (\tau_1, S]$ such that $Y(s; \tau_1, \xi, 0) \in C([\tau_1, \tau_2]; \mathcal{D}(A))$. Define a control $U(\cdot) \in \mathcal{U}_{ad}$ as

$$U(s) = \begin{cases} 0 & \text{if } s \in [\tau_1, \tau_2], \\ \overline{U}(s) & \text{if } s \in (\tau_2, S], \end{cases} \tag{5.4}$$

here, $\overline{U} = U(s; \tau_2, \xi)$ is an optimal control to $(\overline{P}_{\tau_2, \xi})$. Then, since $\overline{Y}(s; \tau_2, \xi, \overline{U})$ is the optimal state on $[\tau_2, S]$ and $Y(s; \tau_1, \xi, U)$ is not, we have:

$$\begin{aligned}
&|\psi(\tau_1, \xi) - \psi(\tau_2, \xi)| \\
&\quad \leq \frac{1}{2} \int_{\tau_2}^{S} \big|\, |Y(s; \tau_1, \xi, U)|^2 - |\overline{Y}(s; \tau_2, \xi, \overline{U})|^2 \,\big| ds + \frac{1}{2} \int_{\tau_1}^{\tau_2} |Y(s; \tau_1, \xi, 0)|^2 ds \\
&\quad \leq \frac{1}{2} \int_{\tau_2}^{S} |\langle Y(s; \tau_1, \xi, U) - \overline{Y}(s; \tau_2, \xi, \overline{U}), Y(s; \tau_1, \xi, U) + \overline{Y}(s; \tau_2, \xi, \overline{U})\rangle| ds \\
&\qquad + \frac{1}{2} \int_{\tau_1}^{\tau_2} |Y(s; \tau_1, \xi, 0)|^2 ds = I_1 + I_2.
\end{aligned}$$

From (3.3),

$$\begin{aligned}
Y(\tau_2; \tau_1, \xi, 0) - \xi &= \int_{\tau_1}^{\tau_2} \frac{dY(s; \tau_1, \xi, 0)}{ds} ds \\
&= \int_{\tau_1}^{\tau_2} -AY(s; \tau_1, \xi, 0) + F(Y(s; \tau_1, \xi, 0)) ds.
\end{aligned}$$

From (3.2), (3.3) and (f.i), it follows that

$$\begin{aligned}
|Y(\tau_2; \tau_1, \xi, 0) - \xi| &\leq \int_{\tau_1}^{\tau_2} |AY(s; \tau_1, \xi, 0)| + |F(Y(s; \tau_1, \xi, 0))| ds \\
&\leq \int_{\tau_1}^{\tau_2} (1 + \eta)|AY(s; \tau_1, \xi, 0)| + \psi_\eta(\|Y(s; \tau_1, \xi, 0)\|) ds \\
&\leq C_\xi(\tau_2 - \tau_1).
\end{aligned}$$

Therefore, by (5.1),

$$\begin{aligned}
I_1 &\leq C_\xi |Y(s; \tau_1, \xi, U) - \overline{Y}(s; \tau_2, \xi, \overline{U})| \\
&\leq C_\xi |Y(s; \tau_2, Y(\tau_2; \tau_1, \xi, 0), \overline{U}) - \overline{Y}(s; \tau_2, \xi, \overline{U})| \\
&\leq C_\xi |Y(\tau_2; \tau_1, \xi, 0) - \xi| \leq C_\xi(\tau_2 - \tau_1).
\end{aligned}$$

Since $Y(s; \tau_1, \xi, 0) \in C(\tau_1, S; \mathcal{H})$, we have:

$$I_2 \leq C_\xi(\tau_2 - \tau_1).$$

Hence,

$$|\psi(\tau_1, \xi) - \psi(\tau_2, \xi)| \leq C_\xi(\tau_2 - \tau_1), \qquad 0 \leq t \leq \tau_1 < \tau_2 \leq S.$$

On the other hand, if $\tau_1 > \tau_2$, then instead of (5.4) we define $U(s) = \overline{U}(s)$, $s \in (\tau_1, S]$ and repeating the same argument as above to obtain that

$$|\psi(\tau_2, \xi) - \psi(\tau_1, \xi)| \leq C_\xi(\tau_1 - \tau_2), \qquad 0 \leq t \leq \tau_2 < \tau_1 \leq S.$$

Hence we have proved the desired result. □

The following lemma is seen in Barbu [1, Lemma 6.1.3].

Lemma 5.4. *For $0 \le t \le \tau \le S$ and $\xi \in B_t$,*

$$\psi(t,\xi) = \inf_{U \in \mathcal{U}_{ad}} \left\{ \frac{1}{2} \int_t^\tau |Y(s;t,\xi,U)|^2 ds + \frac{1}{2} \int_t^\tau |U|^2 ds + \psi(\tau, Y(\tau;t,\xi,U)) \right\}.$$

Lemma 5.5. *For every $(t,\xi_0) \in [0,S] \times B_t$, we have:*

$$-P(t) \in D_\xi^+ \psi(t,\xi_0).$$

Here, the superdifferential $D_\xi^+ \psi$ is defined as follows (see [5]):

$$D_\xi^+ \psi(t,\xi_0) = \Big\{ P \in \mathcal{H}; \limsup_{\xi \to \xi_0} \frac{\psi(t,\xi) - \psi(t,\xi_0) - (\xi - \xi_0, P)}{|\xi - \xi_0|} \le 0 \Big\}.$$

Proof. By the definition of ψ,

$$\begin{aligned} \psi(t,\xi_1) - \psi(t,\xi_2) \le & \frac{1}{2} \int_t^S |Y(s;t,\xi_1,\overline{U})|^2 - |Y(s;t,\xi_2,\overline{U})|^2 ds \\ = & \frac{1}{2} \int_t^S |Y(s;t,\xi_1,\overline{U}) - Y(s;t,\xi_2,\overline{U})|^2 ds \\ & + \int_t^S (Y(s;t,\xi_1,\overline{U}) - Y(s;t,\xi_2,\overline{U}), Y(s;t,\xi_2,\overline{U})) ds, \end{aligned} \tag{5.5}$$

where $\overline{U} \in \mathcal{U}_{ad}$ is an optimal control to $(\overline{P}_{t,\xi_2})$ such that

$$\psi(t,\xi_2) = \frac{1}{2} \int_t^S |Y(s;t,\xi_2,\overline{U})|^2 ds + \frac{1}{2} \int_t^S |\overline{U}|^2 ds.$$

We have consider a Cauchy problem

$$\begin{cases} \dfrac{dW}{ds} + \overline{A}W - F'(Y(s;t,\xi_2,\overline{U}))W = 0, & t < s \le S, \\ W(t) = \xi_1 - \xi_2. \end{cases} \tag{5.6}$$

It is easily verified from (a.i), (a.ii), (f.i), (f.ii), (f.iii) and (f.v) that (5.6) possesses a unique weak solution $W \in H^1(t,S;\mathcal{V}') \cap C([t,S];\mathcal{H}) \cap L^2(t,S;\mathcal{V})$ on $[t,S]$. Then we will verify that

$$|Y(s;t,\xi_1,\overline{U}) - Y(s;t,\xi_2,\overline{U}) - W| \le o(|\xi_1 - \xi_2|), \qquad \forall s \in [t,S],$$

where $o(|\xi_2 - \xi_1|)/|\xi_2 - \xi_1| \to 0$ as $\xi_1 \to \xi_2$. In fact, let $Y(s;t,\xi_1,\overline{U})$ and $Y(s;t,\xi_2,\overline{U})$ be denoted by Y_1 and Y_2, respectively. First, we regard $Y_1 - Y_2 - W$ as a solution of the following problem:

$$\begin{cases} \dfrac{d(Y_1 - Y_2 - W)}{ds} + \overline{A}(Y_1 - Y_2 - W) \\ \qquad = F(Y_1) - F(Y_2) - F'(Y_2)W, \qquad t < s \le S, \\ (Y_1 - Y_2 - W)(t) = 0. \end{cases}$$

Taking the scalar product with $Y_1 - Y_2 - W$, we obtain that

$$\frac{1}{2}\frac{d}{ds}|(Y_1 - Y_2 - W)(s)|^2 + \langle \overline{A}(Y_1 - Y_2 - W)(s), (Y_1 - Y_2 - W)(s)\rangle$$
$$= \langle \int_0^1 F'\big(Y_2 + \theta(Y_2 - Y_1)\big)d\theta(Y_1 - Y_2 - W)(s), (Y_1 - Y_2 - W)(s)\rangle$$
$$+ \langle \int_0^1 \big\{F'\big(Y_2 + \theta(Y_2 - Y_1)\big) - F'(Y_2)\big\}d\theta W(s), (Y_1 - Y_2 - W)(s)\rangle.$$

From (a.ii), (f.iii), and (f.v),

$$\frac{1}{2}\frac{d}{ds}|(Y_1 - Y_2 - W)(s)|^2 + \delta\|(Y_1 - Y_2 - W)(s)\|^2$$
$$\le \frac{\delta}{2}\|(Y_1 - Y_2 - W)(s)\|^2 + C\big\{(\|Y_1(s)\|^2 + \|Y_2(s)\|^2 + 1)|(Y_1 - Y_2 - W)(s)|^2$$
$$+ |Y_1(s) - Y_2(s)|^2\|W(s)\|^2\big\}.$$

Therefore, integrating this inequality from t to s, $t < s < S$, we get

$$\frac{1}{2}|(Y_1 - Y_2 - W)(s)|^2 + \frac{\delta}{2}\int_t^s \|(Y_1 - Y_2 - W)(\tau)\|^2 d\tau$$
$$\le C\Big\{\int_t^s (\|Y_1(\tau)\|^2 + \|Y_2(\tau)\|^2 + 1)|(Y_1 - Y_2 - W)(\tau)|^2 d\tau$$
$$+ |Y_2 - Y_1|^2_{\mathcal{C}([t,S];\mathcal{H})}\int_t^S \|W(\tau)\|^2 ds\Big\}.$$

Since $\|W\|_{L^2(t,S;\mathcal{V})} \le C|\xi_1 - \xi_2|$ by the definition (5.6) of W,

$$|(Y_1 - Y_2 - W)(s)|^2 + \int_t^s \|(Y_1 - Y_2 - W)(\tau)\|^2 d\tau$$
$$\le C|Y_2 - Y_1|^2_{\mathcal{C}([t,S];\mathcal{H})}\|W\|^2_{L^2(t,S;\mathcal{V})} e^{\int_t^S C(\|Y_1(\tau)\|^2 + \|Y_2(\tau)\|^2 + 1)d\tau} \le C|\xi_1 - \xi_2|^4$$

for any $s \in [t, S]$. Therefore, from (5.5),

$$\psi(t, \xi_1) - \psi(t, \xi_2)$$
$$= \frac{1}{2}\int_t^S |Y_1 - Y_2|^2 ds + \int_t^S (Y_1 - Y_2 - W, Y_2)ds + \int_t^S (W, Y_2)ds$$
$$\le C|\xi_1 - \xi_2|^2 + \int_t^S (W, Y_2)ds.$$

In addition, using the adjoint equation (4.1), the integral is written as

$$\int_t^S (W, Y_2)ds = \int_t^S \langle W, \frac{dP}{ds} - \overline{A}P + F'(Y_2(s))^* P\rangle ds$$
$$= \int_t^S \frac{d}{ds}(W, P) + \int_t^S \langle -\frac{dW}{ds} - \overline{A}W + F'(Y_2(s))W, P\rangle ds$$
$$= (\xi_2 - \xi_1, P(t)).$$

Thus, we conclude that

$$\psi(t, \xi_1) - \psi(t, \xi_2) - (\xi_1 - \xi_2, -P(t)) \le o(|\xi_2 - \xi_1|). \quad \square$$

We can now prove the main result of the paper.

Theorem 5.6. *For $0 \le t \le S$ and $\xi \in \mathcal{D}(A) \cap B_t$, $\psi(\cdot,\xi)$ is Lipschitz continuous in $\tau \in [t,S]$ and $\psi(\tau,\xi)$ satisfies*

$$\psi_\tau(\tau,\xi) - \frac{1}{2}\big(|P(\tau)|^2 - |\mathbb{P}_{\mathcal{C}}(P(\tau)) - P(\tau)|^2\big) + (A\xi - F(\xi), P(\tau)) + \frac{1}{2}|\xi|^2 = 0$$

for a.e. $\tau \in [t,S)$. *For $0 \le t \le S$ and $\xi \in B_t$, $D_\xi^+\psi(t,\xi) \ne \varnothing$; in fact,*

$$-P(t) \in D_\xi^+\psi(t,\xi),$$

where $P(t)$ is given by (4.1).

Proof. Lemma 5.3 yields that, for $\xi \in \mathcal{D}(A) \cap B_t$, $\psi(\cdot,\xi)$ is Lipschitz continuous on $[t,S]$. Hence, $\psi(\cdot,\xi)$ is differentiable almost everywhere in $[t,S]$. Let $\tau \in [t,S]$ be such a point. Let $\overline{U}(\cdot)$ be an optimal control of $(\overline{P}_{\tau,\xi})$. From the definition of the value function,

$$\begin{aligned}\psi(\tau+\varepsilon,\xi) &- \psi(\tau,\xi)\\ &\le \int_{\tau+\varepsilon}^{S} (Y(s;\tau+\varepsilon,\xi,\overline{U}) - Y(s;\tau,\xi,\overline{U}), Y(s;\tau+\varepsilon,\xi,\overline{U}))ds\\ &\quad - \left\{\frac{1}{2}\int_\tau^{\tau+\varepsilon} |Y(s;\tau,\xi,\overline{U})|^2 ds + \frac{1}{2}\int_\tau^{\tau+\varepsilon} |\overline{U}(s)|^2 ds\right\}.\end{aligned} \tag{5.7}$$

Dividing (5.7) by ε, we observe that

$$\begin{aligned}&\frac{\psi(\tau+\varepsilon,\xi) - \psi(\tau,\xi)}{\varepsilon}\\ &\le \int_{\tau+\varepsilon}^{S} \Big(\frac{Y(s;\tau+\varepsilon,\xi,\overline{U}) - Y(s;\tau,\xi,\overline{U})}{\varepsilon} - Z(s), Y(s;\tau+\varepsilon,\xi,\overline{U})\Big)ds\\ &\quad + \int_{\tau+\varepsilon}^{S} (Z(s), Y(s;\tau+\varepsilon,\xi,\overline{U}))ds - \frac{1}{2\varepsilon}\int_\tau^{\tau+\varepsilon} |Y(s;\tau,\xi,\overline{U})|^2 ds\\ &\quad - \frac{1}{2\varepsilon}\int_\tau^{\tau+\varepsilon} |\overline{U}(s)|^2 ds,\end{aligned}$$

where Z is a solution of the Cauchy problem

$$\begin{cases} \dfrac{dZ}{ds} + \overline{A}Z - F'(Y(s,t;\xi;\overline{U}))Z = 0, & \tau < s \le S,\\ Z(\tau) = A\xi - F(\xi) - \overline{U}(\tau). \end{cases} \tag{5.8}$$

We can easily verify from (a.i), (a.ii) and Lemma 2.1 that (5.8) possesses a unique weak solution $Z \in H^1(\tau,S;\mathcal{V}') \cap C([\tau,S];\mathcal{H}) \cap L^2(\tau,S;\mathcal{V})$ on $[\tau,S]$. Then we have:

$$\lim_{\varepsilon\to 0} \frac{\psi(\tau+\varepsilon,\xi) - \psi(\tau,\xi)}{\varepsilon} \le \int_\tau^S (Z(s), Y(s;\tau,\xi,\overline{U}))ds - \left\{\frac{1}{2}|\xi|^2 + \frac{1}{2}|\overline{U}(\tau)|^2\right\}$$

with the aid of the Lemma 5.7 below. Hence, by the use of (4.1) and (5.8), we conclude that

$$\begin{aligned}\psi_\tau(\tau,\xi) &\le -(Z(\tau), P(\tau)) - \frac{1}{2}|\xi|^2 - \frac{1}{2}|\overline{U}(\tau)|^2\\ &= -(A\xi - F(\xi) - \overline{U}(\tau), P(\tau)) - \frac{1}{2}|\xi|^2 - \frac{1}{2}|\overline{U}(\tau)|^2.\end{aligned} \tag{5.9}$$

On the other hand, let $\overline{U}$ be the optimal control of $(\overline{P}_{\tau+\varepsilon,\xi})$. Then, setting

$$U(s) = \begin{cases} \overline{U}(s) & \text{if } s \in (\tau+\varepsilon, S], \\ \overline{U}(t+\varepsilon) & \text{if } s \in (\tau, \tau+\varepsilon], \end{cases}$$

we have:

$$\begin{aligned} \psi(\tau+\varepsilon,\xi)-&\psi(\tau,\xi) \\ &\geq \frac{1}{2}\int_{\tau+\varepsilon}^{S}\Big\{|Y(s;\tau+\varepsilon,\xi,\overline{U}(s)|^2 - |Y(s;\tau,\xi,U)|^2\Big\}ds \\ &- \left\{\frac{1}{2}\int_{\tau}^{\tau+\varepsilon}|Y(s;\tau,\xi,\overline{U}(\tau+\varepsilon))|^2 ds + \frac{1}{2}\int_{\tau}^{\tau+\varepsilon}|\overline{U}(\tau+\varepsilon)|^2 ds\right\} \\ &\geq \int_{\tau+\varepsilon}^{S}(Y(s;\tau+\varepsilon,\xi,\overline{U}) - Y(s;\tau,\xi,U), Y(s;\tau,\xi,U))ds \\ &- \left\{\frac{1}{2}\int_{\tau}^{\tau+\varepsilon}|Y(s;\tau,\xi,\overline{U}(\tau+\varepsilon))|^2 ds + \frac{1}{2}\int_{\tau}^{\tau+\varepsilon}|\overline{U}(\tau+\varepsilon)|^2 ds\right\}. \end{aligned}$$

By the similar arguments as above, we can observe that

$$\lim_{\varepsilon\to 0}\frac{\psi(\tau+\varepsilon,\xi)-\psi(\tau,\xi)}{\varepsilon} \geq \int_{\tau}^{S}(Z(s), Y(s;\tau,\xi,\overline{U}))ds - \frac{1}{2}\left\{|\xi|^2 + |\overline{U}(\tau)|^2\right\}.$$

Therefore,

$$\begin{aligned} \psi_\tau(\tau,\xi) &\geq -(Z(\tau), P(\tau)) - \frac{1}{2}|\xi|^2 - \frac{1}{2}|\overline{U}(\tau)|^2 \\ &= -(A\xi - F(\xi) - \overline{U}(\tau), P(\tau)) - \frac{1}{2}|\xi|^2 - \frac{1}{2}|\overline{U}(\tau)|^2. \end{aligned} \tag{5.10}$$

Hence, it follows from (5.9) and (5.10) that

$$\psi_\tau(\tau,\xi) + (A\xi - F(\xi), P(\tau)) - (\overline{U}(\tau), P(\tau)) + \frac{1}{2}|\overline{U}(\tau)|^2 + \frac{1}{2}|\xi|^2 = 0.$$

Using Lemma 5.5 and $\overline{U}(\tau) = \mathbb{P}_{\mathcal{C}}(P(\tau))$, we also verify that

$$\psi_\tau(\tau,\xi) - \frac{1}{2}\big(|P(\tau)|^2 - |\mathbb{P}_{\mathcal{C}}(P(\tau))|^2\big) + (A\xi - F(\xi), P(\tau)) + \frac{1}{2}|\xi|^2 = 0, \qquad \text{a.e. } \tau \in [t,S]$$

with $-P(\tau) \in D_\xi^+\psi(\tau,\xi)$. □

Lemma 5.7.

$$\lim_{\varepsilon\to 0}\int_{\tau+\varepsilon}^{S}|Y(s;\tau+\varepsilon,\xi,\overline{U}) - Y(s;\tau,\xi,\overline{U})|^2 ds = 0, \tag{5.11}$$

$$\lim_{\varepsilon\to 0}\int_{\tau+\varepsilon}^{S}\Big|\frac{Y(s;\tau+\varepsilon,\xi,\overline{U}) - Y(s;\tau,\xi,\overline{U})}{\varepsilon} - Z\Big|^2 ds = 0. \tag{5.12}$$

Proof. Let $Y(s;\tau+\varepsilon,\xi,\overline{U})$ and $Y(s;\tau,\xi,\overline{U})$ be denoted by $Y_{\tau+\varepsilon}$ and Y_τ. Arguing in a similar way as in Lemma 5.1, we have:

$$|Y(s;\tau+\varepsilon,\xi,\overline{U})-Y(s;\tau,\xi,\overline{U})|^2\le C|\xi-Y(\tau+\varepsilon;\tau,\xi,U)|^2.$$

Since $Y(s;\tau,\xi,\overline{U})\in C([\tau,S];\mathcal{H})$, we get (5.11). On the other hand, let $\widetilde{W}=\frac{Y_{\tau+\varepsilon}-Y_\tau}{\varepsilon}-Z$. We regard $\widetilde{W}$ as a solution of the problem:

$$\text{(5.13)}\qquad \begin{cases}\dfrac{d}{ds}\widetilde{W}+\overline{A}\widetilde{W}=\dfrac{F(Y_{\tau+\varepsilon})-F(Y_\tau)}{\varepsilon}-F'(Y_{\tau+\varepsilon})Z, & \tau+\varepsilon<s\le S,\\ \widetilde{W}(\tau+\varepsilon)=\dfrac{\xi-Y(\tau+\varepsilon;\tau,\xi,\overline{U})}{\varepsilon}-Z(\tau+\varepsilon) & \end{cases}$$

in the space $(\mathcal{D}(A))'$. Taking the scalar product of the equation in (5.13) with $A^{-1}\widetilde{W}$ in $\mathcal{H}$ and using $F(Y_{\tau+\varepsilon})-F(Y_\tau)=\int_0^1F'\big(Y_\tau+\theta(Y_\tau-Y_{\tau+\varepsilon})\big)d\theta(Y_{\tau+\varepsilon}-Y_\tau)$, we obtain that

$$\begin{aligned}\frac12\frac{d}{ds}|A^{-1/2}\widetilde{W}(s)|^2&+(A\widetilde{W}(s),A^{-1}\widetilde{W}(s))\\ &=\Big(\int_0^1F'\big(Y_\tau+\theta(Y_\tau-Y_{\tau+\varepsilon})\big)d\theta\widetilde{W}(s),A^{-1}\widetilde{W}(s)\Big)\\ &\quad+\Big(\int_0^1\big\{F'\big(Y_\tau+\theta(Y_\tau-Y_{\tau+\varepsilon})\big)-F'(Y_\tau)\big\}d\theta Z(s),A^{-1}\widetilde{W}(s)\Big).\end{aligned}$$

From (f.iv) and (f.v), it follows that

$$\text{(5.14)}\qquad\begin{aligned}\frac12\frac{d}{ds}\|\widetilde{W}(s)\|_*^2&+|\widetilde{W}(s)|^2\\ &\le\frac{\delta}{2}|\widetilde{W}(s)|^2+C\mu(|Y_{\tau+\varepsilon}(s)|+|Y_\tau(s)|)^2\|\widetilde{W}(s)\|_*^2\\ &\qquad+|Y_{\tau+\varepsilon}(s)-Y_\tau(s)|^2\|Z(s)\|^2\big\}.\end{aligned}$$

Integrating (5.14) from $\tau+\varepsilon$ to s, we obtain that

$$\begin{aligned}\frac12\|\widetilde{W}(s)\|_*^2&+(1-\frac{\delta}{2})\int_{\tau+\varepsilon}^s|\widetilde{W}(\sigma)|^2d\sigma\\ &\le C\Big\{\int_{\tau+\varepsilon}^s\mu(|Y_{\tau+\varepsilon}(\sigma)|+|Y_\tau(\sigma)|)^2\|\widetilde{W}(\sigma)\|_*^2d\sigma\\ &\quad+|Y_{\tau+\varepsilon}-Y_\tau|^2_{\mathcal{C}([\tau+\varepsilon,S];\mathcal{H})}\int_{\tau+\varepsilon}^S\|Z(\sigma)\|^2d\sigma\Big\}+\frac12\|\widetilde{W}(\tau+\varepsilon)\|_*^2.\end{aligned}$$

Therefore,

$$\text{(5.15)}\qquad\begin{aligned}\|\widetilde{W}(s)\|_*^2&+\int_{\tau+\varepsilon}^s|\widetilde{W}(\sigma)|^2d\sigma\\ &\le C\Big\{|Y_{\tau+\varepsilon}-Y_\tau|^2_{\mathcal{C}([\tau+\varepsilon,S];\mathcal{H})}+\|\widetilde{W}(\tau+\varepsilon)\|_*^2\Big\}e^{\int_{\tau+\varepsilon}^S C\mu(|Y_{\tau+\varepsilon}(\sigma)|+|Y_\tau(\sigma)|)^2d\sigma}\end{aligned}$$

for all $s\in[\tau+\varepsilon,S]$. Since $\xi\in\mathcal{D}(A)$, there exists $\varepsilon>0$ such that $Y(s;\tau,\xi,\overline{U})\in C([\tau,\tau+\varepsilon];\mathcal{V})$. Since $\overline{A}\in\mathcal{L}(\mathcal{V},\mathcal{V}')$ and $\overline{U}(s)\in C([\tau,S];\mathcal{H})$, we see that $Y(s;\tau,\xi,\overline{U})\in C^1([\tau,\tau+\varepsilon];\mathcal{V}')$ and

$$\frac{Y_{\tau+\varepsilon}(\tau+\varepsilon)-Y_\tau(\tau+\varepsilon)}{\varepsilon}\to Z(\tau)\qquad\text{in }\mathcal{V}'\qquad\text{as }\varepsilon\to0.$$

Moreover, since $Z(s) \in C([\tau, S]; \mathcal{H})$, we obtain also that

$$Z(\tau+\varepsilon) \to Z(\tau) \qquad \text{in } \mathcal{V}' \qquad \text{as } \varepsilon \to 0.$$

Thus, it follows from the definition (5.13) of $\widetilde{W}$ that

$$\|\widetilde{W}(\tau+\varepsilon)\|_*^2 \to 0 \qquad \text{as } \varepsilon \to 0.$$

Since $Y(s, \tau; \xi; \overline{U}) \in C([\tau, S; \mathcal{H}])$, it is also clear that

$$|Y_\tau - Y_{\tau+\varepsilon}|^2_{\mathcal{C}([\tau+\varepsilon, S]; \mathcal{H})} \to 0 \qquad \text{as } \varepsilon \to 0.$$

Then, letting ε tend to zero in (5.15), we conclude (5.12). □

Concerning the optimal feedback controllers, we verify the following result.

Theorem 5.8. *Every optimal control $\overline{U}$ of $(\overline{P})$ is expressed as a function of the optimal state $\overline{Y}$ by the feedback law*

$$\overline{U}(t) \in \mathbb{P}_{\mathcal{C}}(-D_Y^+\psi(t, \overline{Y}(t))), \qquad \forall t \in [0, S].$$

Proof. Let $(\overline{Y}, \overline{U})$ be any optimal pair of the problem $(\overline{P})$. Then, by Lemma 5.4, we see that, for every $t \in (0, S)$, $(\overline{Y}, \overline{U})$ is also optimal for the problem

$$\inf_{U \in \mathcal{U}_{ad}} \left\{ \frac{1}{2}\int_t^S |Y(U)|^2 ds + \frac{1}{2}\int_t^S |U|^2 ds \right\},$$

where $Y(U)$ is a solution of $(E_{t, \overline{Y}(t)})$. From Theorem 4.1, we have:

$$\overline{U}(s) = \mathbb{P}_{\mathcal{C}}(P(s)), \qquad \text{a.e. } s \in [t, S],$$

where P is a solution to the system (4.1) with $Y = \overline{Y}$. Then, by Lemma 5.5, we conclude that $\overline{U}(t) \in \mathbb{P}_{\mathcal{C}}(-D_Y^+\psi(t, \overline{Y}(t)))$, $\forall t \in [0, S]$, as claimed. □

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